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On a numerical method for the regularization of descriptor systems

We discuss a numerically stable algorithm, originally proposed in [3], that computes a regularizing output feedback for descriptor systems. In [3], the system is transformed to a condensed form via a sequence of singular value decompositions (SVDs). We show that the expensive SVDs can be replaced by reliable, more efficient orthogonal decompositions. Moreover, we demonstrate how the feedback is obtained from our slightly different condensed form.

1. Introduction

Consider the linear, time-invariant descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x^0, \qquad y(t) = Cx(t), \qquad A, E \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, \tag{1}$$

where x, y, and u are the states, outputs, and inputs (controls), respectively, of the system.

The solution of optimal control problems with quadratic performance criterion for descriptor systems of the form (1) is considered, e.g., in [3,4]. In [3] it is shown that essentially, the classical theory from the linear-quadratic optimal control problem with E invertible carries over to the descriptor case (E singular) if the pencil $A - \lambda E$ is regular and has index at most 1. If the system is strongly stabilizable and strongly detectable it can then be shown (see [1,3]) that an output feedback $K \in \mathbb{R}^{m\times p}$ exists such that the matrix pencil $(A + BKC) - \lambda E$ has the required properties. As output feedback represents a generalized state-space transformation, the solution of the original problem is equivalent to the solution of the transformed system. Here we will discuss a stable numerical method for computing the required regularizing output feedback.

2. Output regularization

In [3] a numerically stable algorithm is given that computes an output feedback for a strongly stabilizable/detectable system such that the transformed system is regular and has index at most 1. A similar algorithm for systems with the more common properties of strong or complete controllability/observability is considered in [1]. As fewer generalized state-space transformations are required in the algorithm from [3], we demonstrate our ideas here for this algorithm due to space limitations. But the algorithm from [1] can be modified analogously.

In the algorithms in [1,3], SVDs are chosen for reducing the system to a condensed form as these reductions involve several critical rank decisions. Unfortunately, SVDs are quite expensive. We show here that cheaper orthogonal decompositions can be employed that also reveal the rank of a matrix reliably. A *rank revealing QR decomposition* (RRQR) of a matrix $A \in \mathbb{R}^{n\times m}$ is given by $A = URP^T = U\begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix}P^T$, where $U \in \mathbb{R}^{n\times n}$ is orthogonal, $P \in \mathbb{R}^{m\times m}$ is a permutation matrix, $R_1 \in \mathbb{R}^{k\times k}$ is upper triangular, and $\|R_3\|_2 < tol \cdot \sigma_{max}(A)$, for a given tolerance threshold tol. The size $k \leq m$ of R_1 is the *numerical rank* of A. An RRQR is computed via a QR decomposition with column pivoting followed by some iterative post-processing in order to determine the rank correctly, see [2] and the references therein. A *UTV (decomposition)* is given by a factorization $A = UTV^T = U\begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}V^T$, where $U \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{m\times m}$ are orthogonal, $T_1 \in \mathbb{R}^{k\times k}$ is a numerically non-singular, triangular matrix, tol is as above, and $\|T_j\|_2 < tol \cdot \sigma_{max}(A)$, $j = 2, 3$. Again, the numerical rank of A is determined by the size k of T_1. Stable algorithms for computing a UTV with T_1 upper or lower triangular have been proposed; see [2] and the references therein. The SVD is a special UTV with T_1 diagonal. Here we will use *URVs*, i.e., UTVs with T_1 upper triangular. The complexity of computing these decompositions and accumulating the orthogonal transformations (which is needed here) for a nonsingular $n \times n$ matrix is $21n^3$ flops for the SVD, $\frac{8}{3}n^3$ flops for the RRQR and the URV (the latter two decompositions being identical in this case). If the rank of the matrix is $r < n$, the cost of the SVD remains the same while the costs for RRQRs and URVs become $8r(n^2 - nr + \frac{1}{3}r^2)$ and $2r(6n^2 - 5nr + \frac{5}{3}r^2)$ flops, respectively. As we replace two of the four SVDs needed in the regularization method from [3] by RRQRs and the other two by URVs, it is clear that a significant acceleration will be obtained. Moreover, if the left and right orthogonal factors of such a decomposition have to be applied to other matrices (as is the case in the regularization procedure considered in the sequel), an RRQR is very advantageous as its right orthogonal factor is a permutation matrix such that its application to other matrices can be performed without floating point operations.

Assume now a strongly stabilizable/detectable system as in (1) is given. The modified regularization method can then be formulated as follows.

1. Compute a URV of E: $U_E^T E V_E = \begin{bmatrix} R_E & 0 \\ 0 & 0 \end{bmatrix} =: E_1$, where $R_E \in \mathbb{R}^{n_E \times n_E}$. Apply U_E, V_E to A, B, C:

$$A_1 := U_E^T A V_E = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B_1 := U_E^T B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \qquad C_1 := C V_E = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix},$$

where the partitioning of A_1, B_1, C_1 is analogous to that of E_1. If $n_E = n$, then we are done.

2. Compute an RRQR of B_{21}: $\hat{U}_B^T B_{21} P_B = \begin{bmatrix} R_B & B_{22} \\ 0 & 0 \end{bmatrix}$, where $R_B \in \mathbb{R}^{n_B \times n_B}$ and $B_{22} \in \mathbb{R}^{n_B \times (m - n_B)}$.

Let $U_B := \begin{bmatrix} I_{n_E} & 0 \\ 0 & \hat{U}_B \end{bmatrix}$ and set $E_2 := E_1$, $C_2 := C_1$,

$$B_2 := U_B^T B_1 P_B = \begin{bmatrix} B_{11} & B_{12} \\ R_B & B_{22} \\ 0 & 0 \end{bmatrix}, \qquad A_2 := U_B^T A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}.$$

3. Compute a URV of A_{32}: $\hat{U}_A^T A_{32} \hat{V}_A = \begin{bmatrix} R_A & 0 \\ 0 & 0 \end{bmatrix}$, where $R_A \in \mathbb{R}^{n_A \times n_A}$.

Let $U_A := \begin{bmatrix} I_{n_E + n_B} & 0 \\ 0 & \hat{U}_A \end{bmatrix}$, $V_A := \begin{bmatrix} I_{n_E} & 0 \\ 0 & \hat{V}_A \end{bmatrix}$, and set $E_3 := E_2$, $C_3 := C_2 V_A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \end{bmatrix}$,

$$A_3 := U_A^T A_2 V_A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & R_A & 0 \\ A_{41} & 0 & 0 \end{bmatrix}, \qquad B_3 := U_A^T B_2 = \begin{bmatrix} B_{11} & B_{12} \\ R_B & B_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

4. Compute an RRQR of C_{13}: $U_C^T C_{13} P_C = \begin{bmatrix} R_C & C_{14} \\ 0 & 0 \end{bmatrix}$, where $R_C \in \mathbb{R}^{n_C \times n_C}$, $C_{14} \in \mathbb{R}^{n_C \times (n - n_E - n_A - n_C)}$.

Let $V_C := \begin{bmatrix} I_{n_E + n_A} & 0 \\ 0 & P_C \end{bmatrix}$ and set $E_4 := E_3$, $B_4 := B_3$,

$$A_4 := A_3 V_C = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & R_A & 0 & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix}, \qquad C_4 := U_C^T C_3 V_C = \begin{bmatrix} C_{11} & C_{12} & R_C & C_{14} \\ C_{21} & C_{22} & 0 & 0 \end{bmatrix}.$$

The system (E_4, A_4, B_4, C_4) is strongly stabilizable/detectable as orthogonal changes of basis preserve these properties. Hence, $n_E + n_B + n_A = n = n_E + n_A + n_C$, which implies $n_B = n_C$. Thus, the new system has the form

$$E_4 := \begin{bmatrix} R_E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_4 := \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & R_A & 0 \end{bmatrix}, \; B_4 := \begin{bmatrix} B_{11} & B_{12} \\ R_B & B_{22} \\ 0 & 0 \end{bmatrix}, \; C_4 := \begin{bmatrix} C_{11} & C_{12} & R_C \\ C_{21} & C_{22} & 0 \end{bmatrix}.$$

5. Now let $\Sigma_A := \sigma_{max}(R_A) I_{n_B}$. The output feedback is $K := \begin{bmatrix} K_{11} & K_{12} \\ 0 & 0 \end{bmatrix}$, where $K_{11} = R_B^{-1} (\Sigma_A - A_{23}) R_C^{-1}$ and $K_{12} = -R_B^{-1} \left(A_{22} + (\Sigma_A - A_{23}) R_C^{-1} C_{12} \right) C_{22}^{+}$. (Note that $\sigma_{max}(R_A)$ is a by-product of the URV in Step 3.)

The regularized system is then given by $\hat{A} := A_4 + B_4 \hat{K} C_4 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & (A_{22} + (\Sigma_A - A_{23})) R_C^{-1} C_{12} (I - C_{22}^{+} C_{22}) & \Sigma_A \\ A_{13} & R_A & 0 \end{bmatrix}$, $\hat{E} := E_4$, $\hat{B} := B_4$, and $\hat{C} := C_4$. The system $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ has the desired properties, i.e., the matrix pencil $\hat{A} - \lambda \hat{E}$ is regular and has index at most 1 because R_E, R_A, and Σ_A are nonsingular. Moreover, the regularized system is still strongly stabilizable/detectable as output feedback is a generalized state-space transformation.

3. References

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