Nonlinear Model Order Reduction via Quadratic Bilinear Differential Algebraic Equations

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Nowadays many technical and industrial processes require accurate and systematic analysis and simulation of the underlying mathematical models. However, the demand on the models of being as exact as possible frequently leads to very large-scale control systems which prevent efficient numerical treatment. In this talk, we will discuss the problem of model order reduction of state-nonlinear single-input and single-output control systems, i.e. systems of the form

$$\begin{aligned} x(t) &= f(x(t)) + bu(t) \\ y(t) &= cx(t), \quad x(0) = x_0, \end{aligned} \tag{1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ nonlinear and $b, c^T \in \mathbb{R}^n$. While most existing reduction techniques like TPWL and POD require specific training inputs and snapshots of a state trajectory, respectively, we want to focus on a numerically efficient Krylov-based approach which will additionally allow constructing input independent reduced order models. As has recently been shown in [1], for a large class of nonlinear systems, this can be achieved by transforming the original system (1) into an equivalent, though increased, quadratic bilinear differential algebraic system

$$\begin{aligned} E\dot{\tilde{x}}(t) &= A_1\tilde{x}(t) + A_2\tilde{x}(t) \otimes \tilde{x}(t) + N\tilde{x}(t)u(t) + Bu(t) \\ y(t) &= C\tilde{x}(t), \quad \tilde{x}(0) = \tilde{x}_0, \end{aligned}$$

with $E, A_1, N \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, $A_2 \in \mathbb{R}^{\tilde{n}\times\tilde{n}^2}$, $B, C^T \in \mathbb{R}^{\tilde{n}}$. As the above model combines the class of quadratic and bilinear control systems its output can be characterized by means of generalized transfer functions obtained from nonlinear system analysis concepts described in [2]. Finally, expanding these transfer functions at certain frequencies then leads to the idea of multimoment-matching. We will discuss the choice of interpolation points, possible two-sided projection techniques as well as difficulties arising from large-scale Kronecker products. The effectiveness of the new approach is evaluated with the help of some numerical examples resulting from semi-discretized nonlinear PDEs.

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GMA-Fachausschuss 1.30
Salzburg, 23. September 2010

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Outline

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Nonlinear Model Order Reduction

Motivation

Consider a large-scale state-nonlinear control system of the form

$$\Sigma : \quad \begin{cases} \dot{x}(t) = f(x(t)) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ nonlinear, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$.

MOR

$$\hat{\Sigma} : \quad \begin{cases} \dot{\hat{x}}(t) = \hat{f}(\hat{x}(t)) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t), \quad \hat{x}(0) = \hat{x}_0, \end{cases}$$

with $\hat{f} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}}$, $\hat{B} \in \mathbb{R}^{\hat{n} \times m}$, $\hat{C} \in \mathbb{R}^{p \times \hat{n}}$, $x \in \mathbb{R}^{\hat{n}}$, $u \in \mathbb{R}^m$, $\hat{y} \in \mathbb{R}^p$, $\hat{n} \ll n$.

Goal

$\hat{y} \approx y$ for all admissible u.

Nonlinear Model Order Reduction

Common Reduction Techniques

Proper Orthogonal Decomposition (POD)

- Take computed or experimental 'snapshots' of full model: $[x(t_1), x(t_2), \ldots, x(t_N)] =: X$,
- perform SVD of snapshot matrix: $X = VSW^T \approx V_{\hat{n}} S_{\hat{n}} W_{\hat{n}}^T$.
- Reduction by POD-Galerkin projection: $\dot{\hat{x}} = V_{\hat{n}}^T f(V_{\hat{n}}\hat{x}) + V_{\hat{n}}^T Bu$.
- Requires evaluation of f
  $\rightsquigarrow$ discrete empirical interpolation [Sorensen/Chaturantabut '10].
- Input dependency due to 'snapshots'!

Trajectory Piecewise Linear (TPWL)

- Linearize f along trajectory,
- reduce resulting linear systems,
- construct reduced model by weighting sum of linear systems.
- Requires simulation of original model and several linear reduction steps, many heuristics.

We will consider quadratic-bilinear SISO systems of the form

$$\begin{aligned} E\dot{x}(t) &= A_1x(t) + A_2x(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \\ y(t) &= Cx(t), \quad x(0) = x_0, \end{aligned}$$

where $E, A_1, N \in \mathbb{R}^{n\times n}, A_2 \in \mathbb{R}^{n\times n^2}, B, C^T \in \mathbb{R}^n$.

- A large class of nonlinear control-affine systems can be transformed into the above type of control system.
- The transformation is exact, but a slight increase of the state dimension has to be accepted.
- Input-output behavior can be characterized by generalized transfer functions $\rightsquigarrow$ enables us to use Krylov-based reduction techniques.

Theorem [Gu'09]

Assume that the state equation of a nonlinear system Σ is given by

$$\dot{x} = a_0x + a_1g_1(x) + \ldots + a_kg_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking Lie derivatives and adding algebraic equations, respectively, Σ can be transformed into a system of quadratic-bilinear DAEs of dimension $N > n$.

Example (Taking Lie Derivatives)

- $\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2+1}, \quad \dot{x}_2 = \sin x_2 + u.$
- $z_1 := \exp(-x_2), \quad z_2 := \sqrt{x_1^2+1}, \quad z_3 := \sin x_2, \quad z_4 := \cos x_2.$
- $\dot{x}_1 = z_1 \cdot z_2, \quad \dot{x}_2 = z_3 + u, \quad \dot{z}_1 = -z_1 \cdot (z_3 + u),$
  $\dot{z}_2 \overset{z_2>0\forall t}{=} \frac{2\cdot x_1 \cdot z_1 \cdot z_2}{2\cdot z_2} = x_1 \cdot z_1, \quad \dot{z}_3 = z_4 \cdot (z_3+u), \quad \dot{z}_4 = -z_3 \cdot (z_3+u)$

Theorem [Gu'09]

Assume that the state equation of a nonlinear system Σ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking Lie derivatives and adding algebraic equations, respectively, Σ can be transformed into a system of quadratic-bilinear DAEs of dimension $N > n$.

Example (Adding Algebraic Equations)

Accepting DAEs, it might be advantageous adding algebraic equations

- $\dot{x} = -x^4, \quad y := x^2,$
  $\rightsquigarrow \dot{x} = -y^2, \quad y - x^2 = 0,$
  $\rightsquigarrow \dot{x} = -y^2, \quad \dot{y} = -2 \cdot x \cdot y^2.$

Analysis of nonlinear systems by variational equation approach:

- consider input of the form $\alpha u(t)$,
- nonlinear system is assumed to be a series of homogeneous nonlinear subsystems, i.e. response should be of the form

$$x(t) = \alpha x_1(t) + \alpha^2 x_2(t) + \alpha^3 x_3(t) + \ldots.$$

- comparison of terms $\alpha^i, i = 1, 2, \ldots$ leads to series of systems

$$\begin{aligned} E\dot{x}_1 &= A_1 x_1 + Bu, \\ E\dot{x}_2 &= A_1 x_2 + A_2 x_1 \otimes x_1 + N x_1 u, \\ E\dot{x}_3 &= A_1 x_3 + A_2 (x_1 \otimes x_2 + x_2 \otimes x_1) + N x_2 u \\ &\vdots \end{aligned}$$

- although i-th subsystem is coupled nonlinearly to preceding systems, linear systems are obtained if terms x_j, $j < i$, are interpreted as pseudo-inputs.

Quadratic-Bilinear DAEs
Generalized Transfer Functions

In a similar way, a series of generalized symmetric transfer functions can be obtained via the growing exponentials approach:

$$H_1(s_1) = C\,\underbrace{(s_1E - A_1)^{-1}B}_{G_1(s_1)},$$

$$\begin{aligned}
H_2(s_1,s_2) = \frac{1}{2!}C\left((s_1+s_2)E - A_1\right)^{-1}&\left[N\left(G_1(s_1)+G_1(s_2)\right)\right.\\
&\left.+A_2\left(G_1(s_1)\otimes G_1(s_2)+G_1(s_2)\otimes G_1(s_1)\right)\right],
\end{aligned}$$

$$\begin{aligned}
H_3(s_1,s_2,s_3) = \frac{1}{3!}C\left((s_1+s_2+s_3)E - A_1\right)^{-1}&\\
&\Big[N(G_2(s_1,s_2)+G_2(s_2,s_3)+G_2(s_1,s_3))\\
&+A_2\Big(G_1(s_1)\otimes G_2(s_2,s_3)+G_1(s_2)\otimes G_2(s_1,s_3)\\
&\quad+G_1(s_3)\otimes G_2(s_1,s_3)+G_2(s_2,s_3)\otimes G_1(s_1)\\
&\quad+G_2(s_1,s_3)\otimes G_1(s_2)+G_2(s_1,s_2)\otimes G_1(s_3)\Big)\Big].
\end{aligned}$$

Quadratic-Bilinear DAEs
Characterization via Multimoments

For simplicity, focus on the first two transfer functions. For $H_1(s_1)$, choosing σ and making use of the Neumann lemma leads to

$$H_1(s_1) = \sum_{i=0}^{\infty} C\,\underbrace{\left((A_1-\sigma E)^{-1}E\right)^i (A_1-\sigma E)^{-1}B\,(s_1-\sigma)^i}_{m^i_{s_1,\sigma}}.$$

Similarly, specifying an expansion point (τ,ξ) yields

$$H_2(s_1,s_2) = \frac{1}{2}\sum_{i=0}^{\infty} C\left((A_1-(\tau+\xi)E)^{-1}E\right)^i (A_1-(\tau+\xi)E)^{-1}\,(s_1+s_2-\tau-\xi)^i\cdot$$

$$\left[A_2\left(\sum_{j=0}^{\infty} m^j_{s_1,\tau}\otimes\sum_{k=0}^{\infty} m^k_{s_2,\xi}+\sum_{k=0}^{\infty} m^k_{s_2,\xi}\otimes\sum_{j=0}^{\infty} m^j_{s_1,\tau}\right)+N\left(\sum_{p=0}^{\infty} m^p_{s_1,\tau}+\sum_{p=0}^{\infty} m^q_{s_2,\xi}\right)\right]$$

For derivatives around $\sigma=\tau=\xi$ up to order $q-1$, construct the Krylov spaces:

$$
\begin{aligned}
&U = \mathcal{K}_q\left((A_1-\sigma E)^{-1}E,(A_1-\sigma E)^{-1}B\right)\\
&\textbf{for } i=1:q\\
&\quad W_i = \mathcal{K}_{q-i+1}\left((A_1-2\sigma E)^{-1}E,(A_1-2\sigma E)^{-1}NU_i\right),\\
&\qquad \textbf{for } j=1:\min(q-i+1,i)\\
&\qquad\quad Z_i = \mathcal{K}_{q-i-j+2}\left((A_1-2\sigma E)^{-1}E,(A_1-2\sigma E)^{-1}A_2(U_i\otimes U_j+U_j\otimes U_i)\right),
\end{aligned}
$$

U_i denoting the i-*th* column of U.

Set $V=\text{orth}([U,W,Z])$ and construct $\hat{\Sigma}$ by the Galerkin-Projection $P=VV^T$:

$$
\hat{A}_1 = V^TA_1V\in\mathbb{R}^{\hat{n}\times\hat{n}},\quad \hat{A}_2=V^TA_2V\otimes V\in\mathbb{R}^{\hat{n}\times\hat{n}^2},
$$

$$
\hat{N}=V^TNV\in\mathbb{R}^{\hat{n}\times\hat{n}},\quad \hat{B}=V^TB\in\mathbb{R}^{\hat{n}},\quad \hat{C}^T=V^TC\in\mathbb{R}^{\hat{n}}.
$$

Let us consider a simple nonlinear PDE

$$
\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\nu\frac{\partial^2 v}{\partial x^2},\quad (x,t)\in(0,1)\times(0,T),
$$

subject to initial and boundary conditions

$$
v(x,0)=0,\quad x\in[0,1],\qquad v(0,t)=u(t),\quad v(1,t)=0,\quad t\geq 0.
$$

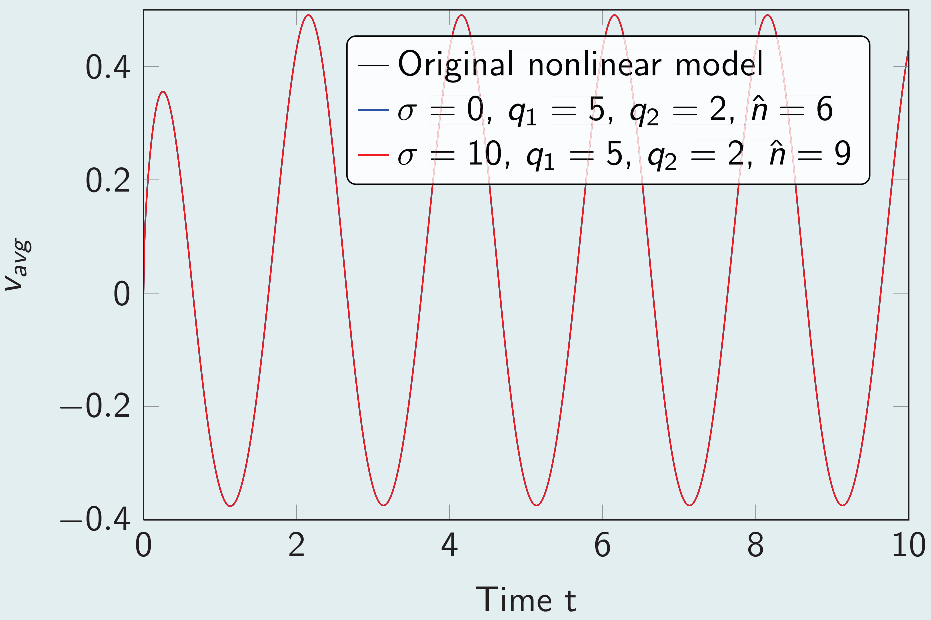
A spatial discretization by e.g. finite differences yields a system of QBODEs

$$
\begin{aligned}
\dot{v}_1 &= \frac{\nu}{h^2}(v_2-2v_1)-\frac{v_1\cdot v_2}{2h}+\frac{v_1}{2h}u(t)+\frac{\nu}{h^2}u(t),\\
\dot{v}_i &= \frac{\nu}{h^2}(v_{i+1}-2v_i+v_{i-1})-\frac{v_i\cdot(v_{i+1}-v_{i-1})}{2h},\quad 1<i<k,\\
\dot{v}_k &= \frac{\nu}{h^2}(-2v_k+v_{k-1})+\frac{v_{k-1}\cdot v_k}{2h}.
\end{aligned}
$$

Output: $v_{avg}=\frac{1}{k}\begin{bmatrix}1 & \dots & 1\end{bmatrix}.$

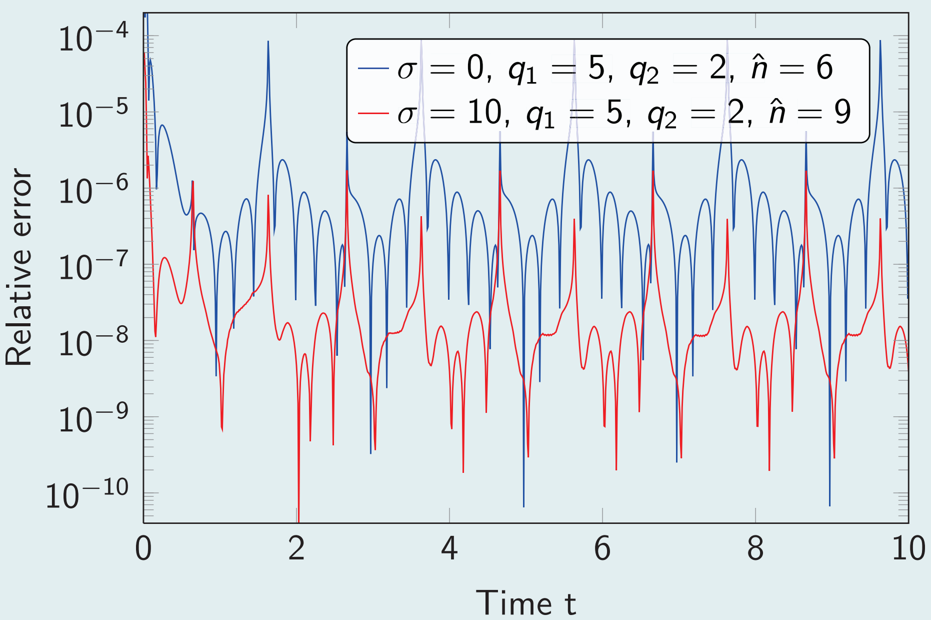
Quadratic-Bilinear DAEs
Viscous Burgers' Equation

Transient responses for $k = 10000$ and $u(t) = \cos(\pi t)$



Quadratic-Bilinear DAEs
Viscous Burgers' Equation

Relative errors for $k = 10000$ and $u(t) = \cos(\pi t)$



Quadratic-Bilinear DAEs
Nonlinear Advection-Diffusion-Reaction System

Next, let us focus on a nonlinear PDE arising in jet diffusion flame models

$$\frac{\partial w}{\partial t} + U \cdot \nabla w - \nabla(\kappa \nabla w) + f(w) = 0, \quad (x,t) \in (0,1) \times (0,T),$$

with Arrhenius type term $f(w) = Aw(c-w)e^{-\frac{E}{d-w}}$ and constant parameters U, A, E, c, d, κ, see [Galbally'09]. Again define initial and boundary conditions

$$w(x,0) = 0, \quad x \in [0,1], \qquad w(0,t) = u(t), \quad w(1,t) = 0, \quad t \geq 0.$$

$$\text{Output: } w_{center} = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}.$$

After spatial discretization of order k, define new state variables

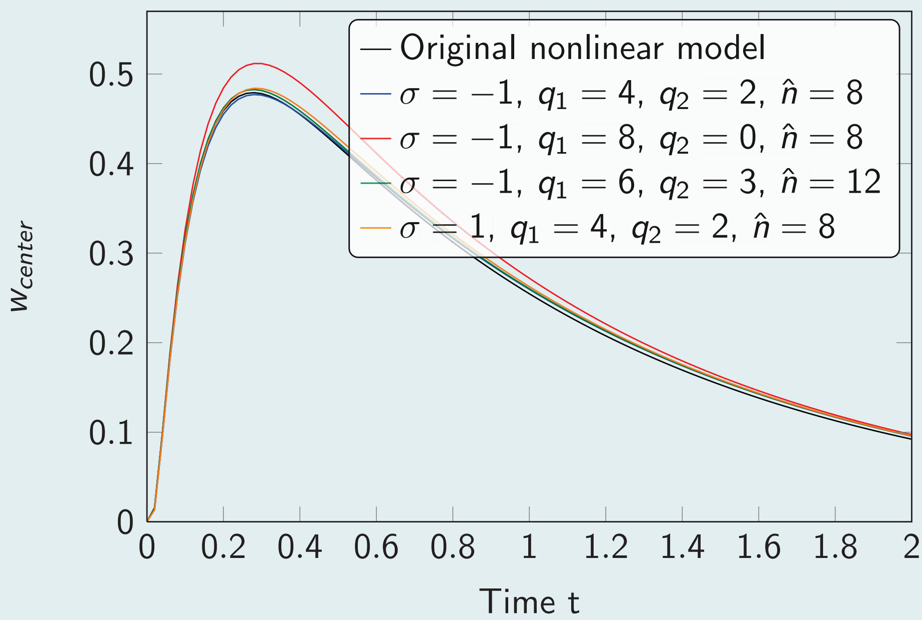
$$z_i := -\frac{\beta}{\delta - w_i}, \quad q_i := e^{z_i},$$

and iteratively construct a system of QBDAEs

$\rightsquigarrow$ state dimension increases to $n = 8 \cdot k$.

Quadratic-Bilinear DAEs
Nonlinear Advection-Diffusion-Reaction System

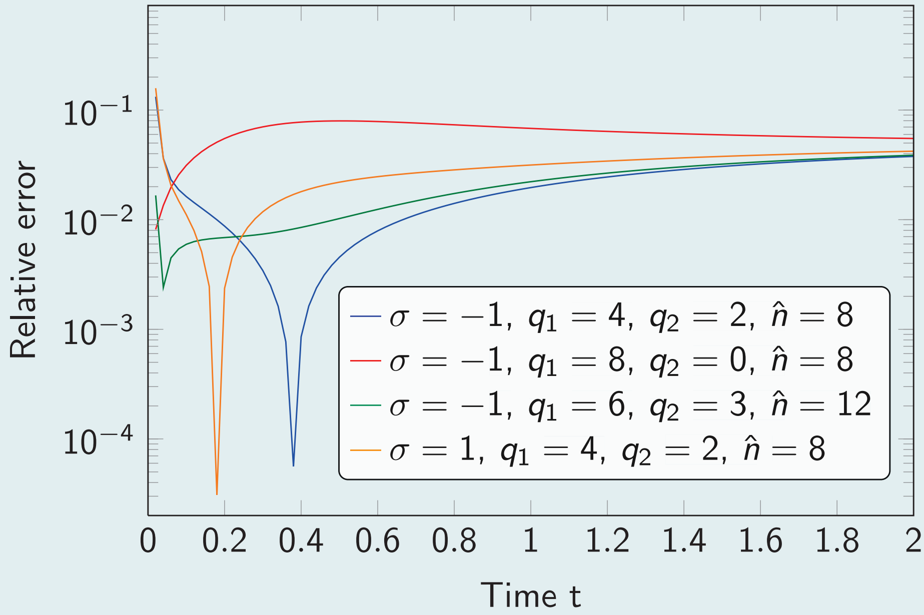
Transient responses for $k = 1500$ and $u(t) = e^{-t}$



Quadratic-Bilinear DAEs

Nonlinear Advection-Diffusion-Reaction System

Relative errors for $k = 1500$ and $u(t) = e^{-t}$



Improvements and Further Investigations

Tensor Approximations and Two-Sided Methods?

Note that $V \in \mathbb{R}^{n \times q}$ in general is dense
$\rightsquigarrow$ computation of $\hat{A}_2 = V^T A_2 V \otimes V$ might cause memory problems,
$\rightsquigarrow$ find approximations:

$$A_2 \approx g_1 \otimes G_1 + \ldots g_r \otimes G_r,$$

with $g_i^T \in \mathbb{R}^n,\ G_i \in \mathbb{R}^{n \times n}$ possibly sparse and low-rank and $r \ll n$.

For a two-sided method, the output Krylov space will be of dimension n^2
$\rightsquigarrow$ need for approximations of the corresponding vectors by tensor products.

Choice of interpolation points for optimal $\mathcal{H}_2$-norm reduction?
$\rightsquigarrow$ need for a reasonable generalization of the linear $\mathcal{H}_2$-norm.

Bilinear Control Systems
State-Space and Output Representation

For the special case $E = I, A_2 = 0$, we arrive at continuous-time bilinear systems

$$\Sigma : \quad \begin{cases} \dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

where $A, N \in \mathbb{R}^{n\times n},\ B,\ C^T \in \mathbb{R}^n$.

Output Characterization: Volterra series

$$y(t) = \sum_{j=1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{j-1}} h(t_1, \dots, t_j) u(t - t_1 - \dots - t_j) \cdots u(t - t_j) dt_j \cdots dt_1,$$

with kernels $h(t_1, \dots, t_j) = Ce^{At_j} N \cdots e^{At_2} N e^{At_1} B$.
Multivariable Laplace-transform:

$$H(s_1, \dots, s_j) = C(s_j I - A)^{-1} N \cdots (s_2 I - A)^{-1} N (s_1 I - A)^{-1} B.$$

Bilinear Control Systems
Approximation via Carleman Bilinearization

Approximate nonlinear state evolution function f by Taylor polynomial, e.g.

$$\dot{x} = f(x) + Bu \approx A_1 x + A_2 (x \otimes x) + Bu.$$

Construct enlarged bilinear system as

$$\frac{d}{dt}\begin{bmatrix} x \\ x\otimes x \end{bmatrix} \approx \underbrace{\begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \otimes I + I \otimes A_1 \end{bmatrix}}_{A^{\otimes}} \begin{bmatrix} x \\ x\otimes x \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ B\otimes I + I \otimes B & 0 \end{bmatrix}}_{N^{\otimes}} \begin{bmatrix} x \\ x\otimes x \end{bmatrix} u + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B^{\otimes}} u,$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C^{\otimes}} \begin{bmatrix} x \\ x\otimes x \end{bmatrix}.$$

Interpolation-Based MOR

Error System and Optimal Interpolation Points

We can show that $\mathcal{H}_2$-norm of the error system is given as

$$||\Sigma - \hat{\Sigma}||^2_{\mathcal{H}_2} = \sum_{j=1}^{\infty} \sum_{\ell_1,\ldots,\ell_j}^{n} \Phi_{\lambda_{\ell_1},\ldots,\lambda_{\ell_j}} \left(H_j(-\lambda_{\ell_1},\ldots,-\lambda_{\ell_j}) - \hat{H}_j(-\lambda_{\ell_1},\ldots,-\lambda_{\ell_j}) \right)$$
$$+ \sum_{j=1}^{\infty} \sum_{\hat{\ell}_1,\ldots,\hat{\ell}_j}^{\hat{n}} \hat{\Phi}_{\hat{\lambda}_{\hat{\ell}_1},\ldots,\hat{\lambda}_{\hat{\ell}_j}} \left(\hat{H}_j(-\hat{\lambda}_{\hat{\ell}_1},\ldots,-\hat{\lambda}_{\hat{\ell}_j}) - H_j(-\hat{\lambda}_{\hat{\ell}_1},\ldots,-\hat{\lambda}_{\hat{\ell}_j}) \right),$$

with generalized residues

$$\Phi_{\lambda_{\ell_1},\ldots,\lambda_{\ell_j}} = \lim_{s_k \to \lambda_{\ell_k}} H_j(s_1,\ldots,s_j)(s_1-\lambda_{\ell_1})\cdots(s_j-\lambda_{\ell_j}).$$

$\Rightarrow$ Similar to the linear case, we aim at:

$$H_j(-\hat{\lambda}_{i_1},\ldots,-\hat{\lambda}_{i_j}) = \hat{H}_j(-\hat{\lambda}_{i_1},\ldots,-\hat{\lambda}_{i_j}),$$
$$\frac{\partial}{\partial s_k} H_j(-\hat{\lambda}_{i_1},\ldots,-\hat{\lambda}_{i_j}) = \frac{\partial}{\partial s_k} \hat{H}_j(-\hat{\lambda}_{i_1},\ldots,-\hat{\lambda}_{i_j}),$$

for $i_1,\ldots,i_j \leq n,\ k = 1,\ldots,j$ and $j = 1,2,3,\ldots$

Interpolation-Based MOR

Generalized Rational Interpolation

Theorem

Let Σ be a bilinear SISO system. Assume that V and W are given as bases of the unions of the column spaces

$$V_1 = \left[(\sigma_1 I - A)^{-1}B,\ldots,(\sigma_q I - A)^{-1}B\right],$$
$$V_j = \left[(\sigma_1 I - A)^{-1}NV_{j-1},\ldots,(\sigma_q I - A)^{-1}NV_{j-1}\right], \quad j \leq r,$$
$$W_1 = \left[(\sigma_1 I - A^T)^{-1}C,\ldots,(\sigma_q I - A^T)^{-1}C\right],$$
$$W_j = \left[(\sigma_1 I - A^T)^{-1}N^T W_{j-1},\ldots,(\sigma_q I - A^T)^{-1}N^T W_{j-1}\right], \quad j \leq r.$$

If $\hat{\Sigma}$ is constructed by projection $\hat{A} = W^TAV$, $\hat{N} = W^TNV$, $\hat{B} = W^TB$, $\hat{C} = CV$, it holds:

$$H_j(s_1,\ldots,s_j) = \hat{H}_j(s_1,\ldots,s_j), \qquad j \leq 2r,$$
$$\frac{\partial}{\partial s_k} H_j(s_1,\ldots,s_j) = \frac{\partial}{\partial s_k}\hat{H}_j(s_1,\ldots,s_j), \qquad j = 1,\ldots,r, \quad k = 1,\ldots,j.$$

Algorithm 1 Bilinear Iterative Rational Krylov Algorithm (Bilinear-IRKA)

Input: A,N,B,C,r,q
Output: $\hat{A},\hat{N},\hat{B},\hat{C}$
1: Make an initial selection $\{\sigma_1,\dots,\sigma_q\}$.
2: **while** (change in $\sigma_i > \epsilon$) **do**
3: Compute $V=[V_1,\dots,V_r]$ and $W=[W_1,\dots,W_r]\in\mathbb{R}^{n\times(q+\dots+q^r)}$.
4: Compute truncated SVD V_q and W_q of V and W.
5: $\hat{A}=(W_q^TV_q)^{-1}W_q^TAV_q$
6: $\sigma_i\leftarrow-\lambda_i(\hat{A})$
7: **end while**
8: $\hat{N}=(W_q^TV_q)^{-1}W_q^TNV_q,\ \hat{B}=(W_q^TV_q)^{-1}W_q^TB,\ \hat{C}=CV_q$

Remark: Exact interpolation properties are lost due to SVD.

Consider again

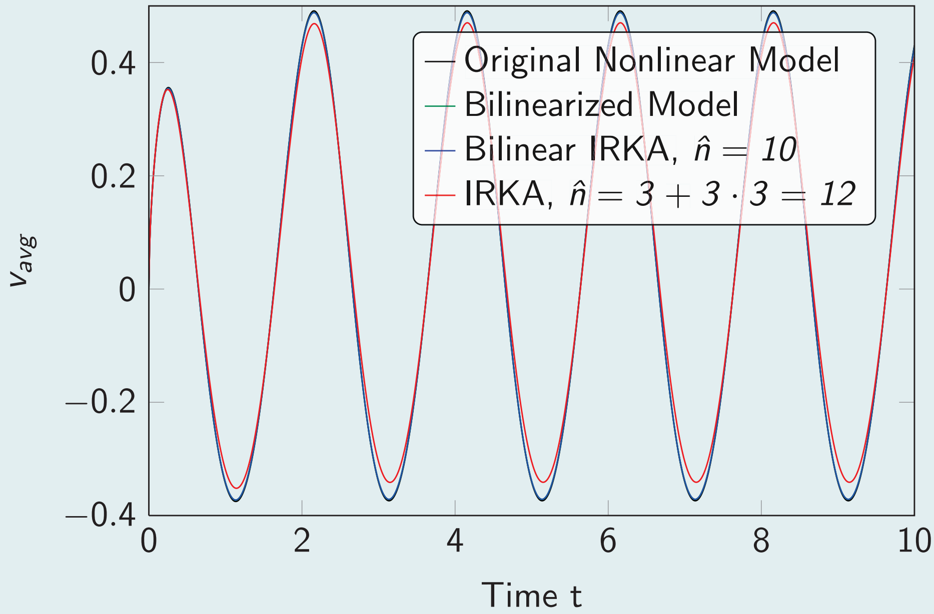
$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\nu\frac{\partial^2 v}{\partial x^2},\quad (x,t)\in(0,1)\times(0,T),$$

subject to initial and boundary conditions

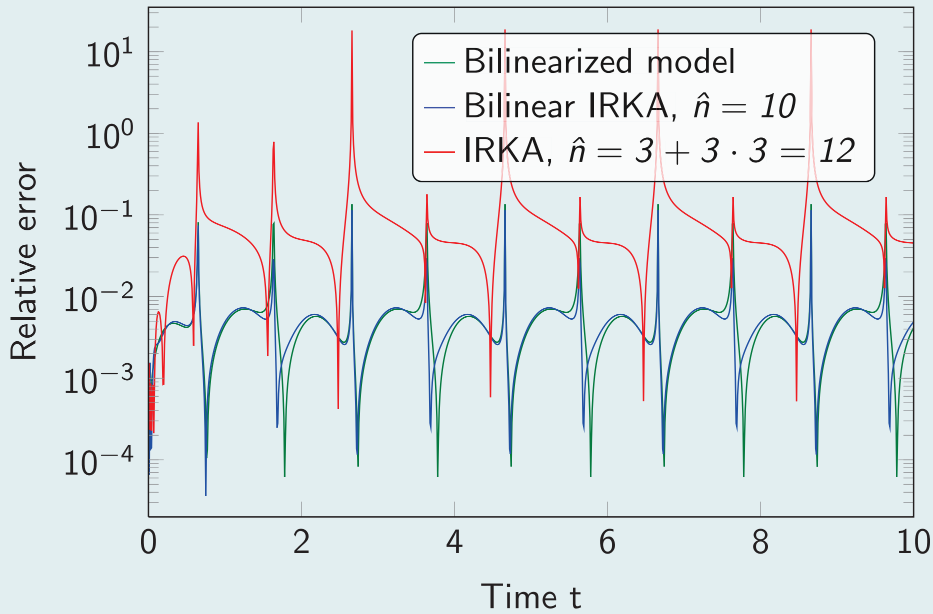
$$v(x,0)=0,\quad x\in[0,1],\qquad v(0,t)=u(t),\quad v(1,t)=0,\quad t\geq 0.$$

Spatially discretize the PDE and approximate the occurring quadratic nonlinearity by a Carleman bilinearized system of dimension $n=k+k^2$.

Transient responses for $k = 100,\ r = 2$ and $u(t) = \cos(\pi t)$



Relative errors for $k = 100,\ r = 2$ and $u(t) = \cos(\pi t)$



Outlook

To do:

- investigate possible two-sided reduction methods for quadratic-bilinear systems.
- Extend Krylov-based techniques to partial realization problem.
- Approximations for Kronecker products of the form $V \otimes V$.
- Lyapunov-based reduction possible?
- Improve $\mathcal{H}_2$-model reduction approach for bilinear systems.

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