DISK FUNCTIONS AND THEIR RELATIONSHIP TO THE MATRIX SIGN FUNCTION

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Abstract

This short paper investigates a generalization of the matrix sign function to matrix pencils.

1 Introduction

The problem of extracting an invariant subspace of a matrix or a deflating subspace of a matrix pencil arises in many control computations including solving Lyapunov, Sylvester, and Riccati equations [16, 18, 19, 32, 38] and computing H_∞ norms [7, 6]. Numerical methods related to the matrix sign function are particularly attractive for machines with advanced architectures [2, 16, 27].

The matrix sign function [37, 38] has many equivalent definitions [21, 26]. One of the more convenient (but less common) definitions is the following. The sign of a matrix $A \in \mathbf{R}^{n\times n}$ is the anti-stabilizing solution $S = \mathrm{sign}(A)$ to the (non-symmetric) algebraic Riccati equation

$$A - SAS = 0, \tag{1}$$

i.e., the solution for which the eigenvalues of AS lie in the open right half plane. (Equation (1) is related to work in [23, 27].) The "quadratic formula" form of the solution is

$$\mathrm{sign}(A) = A^{-1}(A^2)^{1/2} = A(A^2)^{-1/2}. \tag{2}$$

Here, $M^{1/2}$ denotes the unique matrix square root with with eigenvalues in the open right half complex plane. (This remarkably concise formula for $\mathrm{sign}(A)$ is derived by other means in [21].) The Riccati equation (1) is equivalent to

$$\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ S & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ S & -I \end{bmatrix} \begin{bmatrix} AS & -A \\ 0 & -SA \end{bmatrix}. \tag{3}$$

So, $S = \mathrm{sign}(A)$ if and only if the columns of $\begin{bmatrix} I \\ S \end{bmatrix}$ span the n-dimensional invariant subspace of $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ corresponding to eigenvalues with positive real part. Computing invariant subspaces, and Riccati, Sylvester, and Lyapunov solutions using $\mathrm{sign}(A)$ is typically more efficient than by conventional methods like the QR iteration. This is particularly so on computers with advanced architectures. (See [22, 24, 26, 36] or [27] for a survey.)

We will use the following notation. The open left half plane is $\mathbf{C}_-$ and the open right half plane is C_+. The set of eigenvalues of a matrix M is $\sigma(M)$. The invariant subspace of A corresponding to eigenvalues with negative real part, the stable invariant subspace, is denoted by $\mathcal{V}_- = \mathcal{V}_-(A)$. The invariant subspace of A corresponding to eigenvalues with positive real part, the unstable invariant subspace, is denoted by $\mathcal{V}_+ = \mathcal{V}_+(A)$. The eigenprojection $P_+ = P_+(A)$ is the skew projection onto $\mathcal{V}_+$ parallel to $\mathcal{V}_-$ and the eigenprojection $P_- = P_-(A)$ is the skew projection onto $\mathcal{V}_-$ parallel to $\mathcal{V}_+$. The norm $\|\cdot\|$ represents the spectral norm and $\|\cdot\|_F$ represents the Frobenius or Euclidean norm $\|M\| = \sqrt{\mathrm{trace}(M^TM)}$.

The familiar properties of the matrix sign function follow easily from (1) and (2) including, in particular, $SA = AS$ and $S^2 = I$. The fundamental property upon which most (but not all) applications of the matrix sign function rest is

$$\operatorname{sign}(A) = P_+(A) - P_-(A). \tag{4}$$

In particular, $\mathcal{V}_-(A) = \operatorname{range}(I - \operatorname{sign}(A))$. This property is used to compute invariant subspaces [2, 3, 5, 13, 14] and to solve Riccati and Sylvester equations [10, 16, 17, 27].

Recently, a natural generalization of the matrix sign function to matrix pencils appeared [3, 8, 20, 33, 34, 35] and been applied to the computation of invariant and deflating subspaces and solution of Riccati equations [4]. The same generalization is suggested by (3) and (4). This short paper investigates some of its mathematical and computational properties.

2 Disk Functions

Suppose that the pencil $A - \lambda E$, A, $E \in \mathbf{R}^{n\times n}$, is regular and has no eigenvalue on the unit disk. The right disk function is the pencil $\operatorname{disk}_R(A - \lambda E) = U_R - \lambda V_R$ where U_R, $V_R \in \mathbf{R}^{n\times n}$ satisfy

$$\begin{aligned} V_R + U_R &= I \\ V_R - U_R &= \operatorname{sign}((A-E)^{-1}(A+E)). \end{aligned} \tag{5}$$

Similarly, the left disk function is $\operatorname{disk}_L(A - \lambda E) = U_L - \lambda V_L$ where U_L, $V_L \in \mathbf{R}^{n\times n}$ satisfy

$$\begin{aligned} V_L + U_L &= I \\ V_L - U_L &= \operatorname{sign}((A+E)(A-E)^{-1}). \end{aligned} \tag{6}$$

If A and E commute (in particular, if ($E = I$), then the two disk functions coincide. They agree with the disk function of $Z - \lambda I$, $\operatorname{disc}(0, 1, Z)$, mentioned in [38] and with the projectors $P_{R,|z|>1}$, $P_{R,|z|<1}$, $P_{L,|z|>1}$, and $P_{L,|z|<1}$ studied in [3].

Equations (5) and (6) determine $U_R - \lambda V_R$ and $U_L - \lambda V_L$ uniquely, so it is easy to verify that

$$\begin{aligned} U_R &= P_-((A-E)^{-1}(A+E)), \\ V_R &= P_+((A-E)^{-1}(A+E)), \\ U_L &= P_-((A+E)(A-E)^{-1}), \\ V_L &= P_+((A+E)(A-E)^{-1}). \end{aligned}$$

A trivial consequence of the Weierstraß Canonical form [15, 29, 39, 40] is that, if $A - \lambda E$ is regular, then there exist nonsingular matrices X, $Y \in \mathbf{R}^{n\times n}$ such that

$$XAY = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & I \end{bmatrix} \tag{7}$$

$$XEY = \begin{bmatrix} I & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$

where $\Lambda_1 \in \mathbf{R}^{k\times k}$, $\Lambda_2 \in \mathbf{R}^{n-k\times n-k}$ and both $\sigma(\Lambda_1)$ and $\sigma(\Lambda_2)$ lie in the unit disk. In this notation, (5) and (6) reduce to

$$Y^{-1}U_R Y = XU_L X^{-1} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

$$Y^{-1}V_R Y = XV_L X^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix} \tag{9}$$

where I_j denotes the j-by-j identity matrix [3]. In particular, $U_R - \lambda V_R$ and $U_L - \lambda V_L$ are regular with semi-simple eigenvalues at $\lambda = 0$ and $\lambda = \infty$. The matrix U_R is the eigenprojection onto the right deflating subspace of $A - \lambda E$ corresponding to eigenvalues inside the open unit disk. Similarly, V_R is the eigenprojection onto the right deflating subspace of $A - \lambda E$ corresponding to eigenvalues outside the unit disk. The corresponding eigenprojections onto the left deflating subspaces are U_L and V_L. Moreover, $\operatorname{trace}(U_R) = \operatorname{rank}(U_R) = k$ and $\operatorname{trace}(V_R) = \operatorname{rank}(V_R) = n - k$. (Similar identities hold for $U_L - \lambda V_L$.)

It also follows from (8) and (9) that the disk_L and disk_R operators are idempotent and a little more, that is

$$\begin{aligned} \operatorname{disk}_R(U_R - \lambda V_R) &= U_R - \lambda V_R \\ \operatorname{disk}_L(U_R - \lambda V_R) &= U_R - \lambda V_R \end{aligned}$$

with similar identities for $U_L - \lambda V_L$.

The analog to (3) is the $2n$-by-$2n$ deflating subspace relationship

$$\begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} U_R & V_R \\ V_R & U_R \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_R \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} U_R & V_R \\ V_R & U_R \end{bmatrix} \begin{bmatrix} W_R & 0 \\ 0 & I \end{bmatrix} \tag{10}$$

where $\sigma(W_R)$ lies in the open unit disk. With the additional requirement that $U_R + V_R = I$, (10) defines the right disk function uniquely and

$$\begin{aligned} W_R &= (AV_R + EU_R)^{-1}(AU_R + EV_R) \\ &= Y \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} Y^{-1}. \end{aligned}$$

The columns of $\begin{bmatrix} U_R \\ V_R \end{bmatrix}$ span the the right deflating subspace of $\begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & A \end{bmatrix}$ corresponding to eigenvalues inside the unit disk and the columns of $\begin{bmatrix} U_R \\ V_R \end{bmatrix}$ span the right deflating subspace corresponding to eigenvalues outside the unit disk. Of course, the left disk pencil has an analogous property.

3 Numerical Methods

One way to compute the disk function is to use (5) to reduce the problem to computing $\operatorname{sign}((A-E)^{-1}(A+E))$.

Much work has gone to developing algorithms to evaluate the matrix sign function. Most are quite efficient relative to conventional invariant subspace algorithms like the QR-algorithm. A conventional implementation of the Newton iteration with determinant scaling is theoretically and empirically twice as fast as the QR iteration on a conventional, serial computer [10]. (It is this relative efficiency that generates interest in using the matrix sign function.) Recently developed numerical methods successfully take advantage of computers with advanced architectures. See, for example, [2, 12, 24, 26, 36] or [27] for a survey.

The oldest and most common numerical method for evaluating the sign(A) is the scaled Newton iteration:

$$\begin{aligned} A_0 &= A \\ A_{k+1} &= \frac{1}{2}\left((\beta_k A_k) + (\beta_k A_k)^{-1}\right). \end{aligned}$$

Under reasonable assumptions on the scale factors β_k, $\mathrm{sign}(A) = \lim_{k\to\infty} A_k$. The scale factor β_k is chosen to accelerate convergence and promote numerical stability. Typical choices are $\beta_k = |\det(A_k)|^{-1/n}$ and $\beta_k \approx \sqrt{\|A_k^{-1}\|/\|A_k\|}$ [10, 11, 25].

Computing $\mathrm{sign}((A-E)^{-1}(A+E))$ requires an inverse of $A-E$ followed by several other matrix inverses. The assumption that $A-\lambda E$ has no eigenvalue on the unit circle implies that $A-E$ is nonsingular. Nevertheless, $A-E$ may be ill-conditioned for inversion. Moreover, in rare cases, even the well-studied scaled Newton iteration exhibits numerical instabilities [11, 9, 27].

An alternative way to compute disk functions is the inverse-free, implicit squaring algorithm [3, 8, 20, 33, 34, 35] which is related to the AB-algorithm [28, 30, 31]:

$$U_R - \lambda V_R = \lim_{k\to\infty} (U_k + V_k)^{-1}(U_k - \lambda V_k)$$

where

$$\begin{aligned} U_0 &= A \\ V_0 &= E \\ U_{k+1} &= Q_{k,12}^H U_k \\ V_{k+1} &= Q_{k,22}^H V_k \end{aligned}$$

and

$$\begin{bmatrix} V_k \\ -U_k \end{bmatrix} = \begin{bmatrix} Q_{k,11} & Q_{k,12} \\ Q_{k,21} & Q_{k,22} \end{bmatrix} \begin{bmatrix} R_j \\ 0 \end{bmatrix}$$

is a QR factorization partitioned into n-by-n blocks.

This approach has the disadvantage of needing two to six times as much the arithmetic as the Newton iteration for evaluating the matrix sign function. However, it does have a favorable rounding error analysis [3] and is well suited to computation on machines with advanced architectures [1, 3].

4 Remarks

A different generalization of the matrix sign function to matrix pencils without infinite eigenvalues appears in [16]. This generalization calculates $\mathrm{sign}(AE^{-1})$ in the factored form $A_\infty E^{-1}$. The factor A_∞ is the limit of the sequence $A_{k+1} = (A_k + EA_k^{-1}E)/2$ with initial condition $A_0 = A$.

Another disk function is the central disk function. In the notation of (7), the central disk function is $\mathrm{disk}_C(A-\lambda E) = U_C - \lambda V_C$ where

$$\begin{aligned} XU_CY &= \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \\ XV_CY &= \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix}. \end{aligned}$$

The central disk function preserves the information in both the left and right disk functions. This disk function is less easily computed than the left and right hand disk function.

5 Conclusions

We have surveyed some of the properties of a generalization of the matrix sign function to matrix pencils and examined its relationship to the matrix sign function.

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