An Arithmetic for Matrix Pencils

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Abstract

We define an algebra on matrix pencils that is a natural extension of sums, products and quotients of real numbers. The classical algebra of linear transformations may be regarded as a special case of the algebra of pencils. The sum and product defined here preserve right deflating subspaces. We show below that the matrix sign function and the inverse-free algorithms can be derived from an algebra of linear relations. The linear algebra of relations suggests generalizations and variations of these algorithms.

1 Linear Relations

A linear relation is a set of ordered pairs of the form

$$\boldsymbol{R_{E,A}} = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n \mid Ey = Ax\} \qquad (1)$$

for some matrices $E \in \mathbf{R}^{m\times n}$ and $A \in \mathbf{R}^{m\times n}$ and $m \leq 2n$. The representation of the linear relation (1) in terms of E and A is not unique. If $M \in \mathbf{R}^{m\times m}$ is nonsingular, then $R_{E,A} = R_{ME,MA}$. If E has full column rank n, then the relation is a linear transformation represented by $y = (E^TE)^{-1}E^TAx$. From another point of view, a linear relation is just a vector subspace of $\mathbf{R}^{2n}$. In this short paper, for brevity we will consider only the case $m = n$.

Sum and product relations defined below provide a tool to rederive the matrix sign function and the inverse-free spectral divide and conquer algorithms. Linear relations suggests variations and generalizations.

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2 Sums and Products

If R_{E_1,A_1} and R_{E_2,A_2} are linear relations, then the composite or product relation $R_{E_2,A_2}R_{E_1,A_1}$ may be defined as

$$R_{E_2,A_2}R_{E_1,A_1} = \left\{ (x,z) \in \mathbf{R}^n \times \mathbf{R}^n \;\middle|\; \begin{array}{l} \text{For some } y \in \mathbf{R}^n, \\ (y,z) \in R_{E_2,A_2} \text{ and} \\ (x,y) \in R_{E_1,A_1} \end{array} \right\}.$$

The inverse relation $R_{E,A}^{-1}$ is defined by $(x,y) \in R_{E,A}^{-1}$ if and only if $(y,x) \in R_{E,A}$. If both E and A are nonsingular, then $R_{E,A}R_{E,A}^{-1}$ is the identity relation. However, this does not hold for general E and A.

Although we are interested in the case that E and/or A is singular, there are some advantages to representing $R_{E,A}$ in terms of two matrices even when both E and A are nonsingular. If E is ill-conditioned with respect to inversion, then for numerical stability, it may be best to avoid forming $E^{-1}A$ explicitly. Also, with the two matrix representation, the inverse relation $R_{E,A}^{-1}$ is trivially represented by $R_{A,E}$. Matrix inversion is a substantial calculation, so $R_{M,I}$ is an inexpensive representation of $R_{I,M^{-1}}$. It is this observation that makes variations of the AB algorithm [20, 21, 22] and inverse-free algorithms [4, 5] free of inverses.

The following theorem shows how to identify matrices that represent a product relation. It is a generalization of the formula for the product of scalar fractions $a_1/e_1 \times a_2/e_2 = (a_1a_2)/(e_1e_2)$.

Theorem 1 *Let R_{E_1,A_1} and R_{E_2,A_2} be linear relations. If $\tilde{A}_2,\ \tilde{E}_1 \in \mathbf{R}^{n\times n}$ satisfy*

$$\mathbf{null}[\tilde{A}_2,\ \tilde{E}_1] = \mathbf{range}\begin{bmatrix} -E_1 \\ A_2 \end{bmatrix}, \qquad (2)$$

then

$$R_{E_2,A_2}R_{E_1,A_1} = R_{\tilde{E}_1E_2,\tilde{A}_2A_1} = \left\{ (x,z) \in \mathbf{R}^n \times \mathbf{R}^n \;\middle|\; \tilde{E}_1E_2z = \tilde{A}_2A_1x \right\} \qquad (3)$$

Proof If $(x, z) \in (R_{E_2,A_2} R_{E_1,A_1})$, then by definition, there exists $y \in \mathbf{R}^{n\times n}$ such that $(x, y) \in R_{E_1,A_1}$ and $(y, z) \in R_{E_2,A_2}$. Hence, $E_1 y = A_1 x$ and $E_2 z = A_2 y$. Hypothesis (2) implies $\tilde{E}_1 A_2 = \tilde{A}_2 E_1$, so

$$\tilde{E}_1 E_2 z = \tilde{E}_1 A_2 y = \tilde{A}_2 E_1 y = \tilde{A}_2 A_1 y.$$

Therefore the left-hand-side of (3) is contained in the right-hand-side.

Conversely, if $\tilde{E}_1 E_2 z = \tilde{A}_2 A_1 x$, then $\begin{bmatrix} -A_1 x \\ E_2 z \end{bmatrix} \in$ $\mathbf{null}[\tilde{A}_2,\ \tilde{E}_1]$. By (2), there exists $y \in \mathbf{R}^n$ such that

$$\begin{bmatrix} -A_1 x \\ E_2 z \end{bmatrix} = \begin{bmatrix} -E_1 \\ A_2 \end{bmatrix} y.$$

Hence, $(x, y) \in R_{E_1,A_1}$, $(y, z) \in R_{E_2,A_2}$ and the right-hand-side of (3) is contained in the left-hand-side. □

If $E_1 = I$ and $E_2 = I$, then $\tilde{E}_1 = I$ and $\tilde{A}_2 = A_2$ is a possibility in (2). This choice results in conventional matrix multiplication $R_{I,A_2} R_{I,A_1} = R_{I,A_2 A_1}$. If $E_2 = I$ and $A_2 = cI$ for some scalar $c \in \mathbf{R}$, then $\tilde{E}_1 = E_1 = I$ and $\tilde{A}_2 = A_2 = cI$ is a possibility in (2). This choice results in conventional scalar multiplication $R_{I,cI} R_{E_1,A_1} = R_{E_1,cA_1}$.

The sum relation $R_{E_1,A_1} + R_{E_2,A_2}$ is defined by

$$R_{E_1,A_1} + R_{E_2,A_2} = \left\{ (x, z) \in \mathbf{R}^n \;\middle|\; \begin{array}{l} \text{There exists } y_1,\ y_2 \in \mathbf{R}^{n\times n} \\ \text{such that } (x, y_1) \in R_{E_1,A_1}, \\ (x, y_2) \in R_{E_2,A_2} \text{ and } z = \\ y_1 + y_2 \end{array} \right\}.$$

The following theorem shows how to identify matrices that represent a sum linear relation. It is a generalization of the formula for the sum of scalar fractions $a_1/e_1 + a_2/e_2 = (e_2 a_1 + e_1 a_2)/(e_1 e_2)$.

Theorem 2 *Let R_{E_1,A_1} and R_{E_2,A_2} be linear relations. If $\tilde{E}_2,\ \tilde{E}_1 \in \mathbf{R}^{n\times n}$ satisfy*

$$\mathbf{null}[\tilde{E}_2,\ \tilde{E}_1] = \mathbf{range} \begin{bmatrix} -E_1 \\ E_2 \end{bmatrix}, \tag{4}$$

then

$$R_{E_2,A_2} + R_{E_1,A_1} = R_{\tilde{E}_1 E_2, \tilde{E}_2 A_1 + \tilde{E}_1 A_2} = \left\{ (x, z) \in \mathbf{R}^n \times \mathbf{R}^n \;\middle|\; \tilde{E}_1 E_2 z = (\tilde{E}_2 A_1 + \tilde{E}_1 A_2) x \right\} \tag{5}$$

Proof Similar to the proof of Theorem 1. □

Observe that (4) implies that $\tilde{E}_1 E_2 = \tilde{E}_2 E_1$, so the representation (5) is symmetric and reflects commutativity. If $E_1 = E_2 = I$, then $\tilde{E}_1 = \tilde{E}_2 = I$ is a possibility in (4). This choice gives conventional matrix addition $R_{I,A_1} + R_{I,A_2} = R_{I,A_1+A_2}$.

3 Inverse Free Spectral Divide and Conquer

Let $M \in \mathbf{R}^{n\times n}$ have no eigenvalue on the unit circle and let $\mathcal{V}_0$ be the invariant subspace of M corresponding to eigenvalues inside the open unit disk. It is easy to show that

$$\mathcal{V}_0 = \mathbf{range}\left(\lim_{j\to\infty} (I + M^j)^{-1} \right) \tag{6}$$

This is essentially the power method. Although the power method forms the mathematical foundation for several practical numerical methods [17], it is impractical to use it directly, because M^j can not be represented accurately enough in the commonly used finite precision arithmetic. (Usually, the invariant subspace information in M^j is lost to rounding errors before overflow or underflow stops the computation.)

Nevertheless, the inverse-free spectral divide and conquer (IFSDC) algorithm uses (6) to compute a basis for $\mathcal{V}_0$. The IFSDC algorithm is described and analyzed in [4] and derived in [8, 16, 24, 25]. It is related to the AB algorithms [20, 21, 22]. The algorithm works because it represents the relation R_{I,M^j} in the form R_{E_j,A_j}. It uses (3) to calculate E_{j+1} and A_{j+1} from E_j and A_j. At the end, it uses the representation $(I + M^j)^{-1} = (E_j + A_j)^{-1} E_j$ to obtain a basis of $\mathcal{V}_0$. (There are several computational details necessary to the success of IFSDC. See [4] for details.)

4 Matrix Sign Function

If $M \in \mathbf{R}^{n\times n}$ has no non-positive real eigenvalue, then there is a unique matrix square root for which the eigenvalues have positive real part. Denoting this particular matrix square root by $M^{1/2}$, we may define $\mathbf{sign}(A)$ by $\mathbf{sign}(A) = A(A^2)^{-1/2}$ [18]. If A has no eigenvalue with zero real part, then $\mathbf{sign}(A)$ is well-defined. If $\mathcal{V}_-$ is the invariant subspace of $A \in \mathbf{R}^{n\times n}$ corresponding to eigenvalues with negative real part, then $\mathcal{V}_- = \mathbf{range}(\mathbf{sign}(A) - I)$. The matrix sign function may be used to find invariant subspaces and to solve related problems like finding solutions of the algebraic Riccati equation and stable Lyapunov equation [2, 6, 10, 15, 19, 23, 26].

The matrix sign function is attracting much attention. The survey [19] lists over 100 references. The rounding error analysis and perturbation theory are becoming understood [2, 9, 10, 11, 13, 27]. Because it is rich in matrix-matrix operations, the matrix sign function is well suited to computers with advanced architectures [1, 2, 3, 15, 23]. It has been demonstrated on matrices of order greater than 10,000 [1]. A sign of its maturity and acceptance is its presence in the ScaLAPACK library [7].

If A has no eigenvalue with zero real part, then

$\mathbf{sign}(A)$ is defined and the Newton iteration

$$\begin{aligned} A_0 &= A \\ A_{j+1} &= \frac{1}{2}(A_j + A_j^{-1}) \end{aligned} \tag{7}$$

converges globally and quadratically to $\mathbf{sign}(A)$ [12, 19, 26]. Rewriting (7) in terms of relations we get

$$\mathrm{R}_{E_{j+1},A_{j+1}} = R_{I,\frac{1}{2}I}\left(R_{E_j,A_j} + R^{-1}_{E_j,A_j}\right). \tag{8}$$

Representing $R^{-1}_{E_j,A_j}$ by the trivial inverse R_{A_j,E_j}, (4) becomes

$$\mathbf{null}[\tilde{A}_j,\ \tilde{E}_j] = \mathbf{range}\begin{bmatrix} -E_j \\ A_j \end{bmatrix}.$$

If we choose $\tilde{A}_j = I$ and $\tilde{E}_j = E_j A_j^{-1}$, then $E_{j+1} = E_j = I$ and $A_{j+1} = \frac{1}{2}(A_j + A_j^{-1})$, so (8) reduces to (7).

A generalization of the matrix sign function to matrix pencils was introduced in [14] to solve the generalized algebraic Riccati equation

$$Q + A^T X E + E^T X A - E^T X G X E = 0 \tag{9}$$

where $Q,\ A,\ E,\ G \in \mathrm{R}^{n\times n}$ are known coefficient matrices, $G = G^T$ and $Q = Q^T$ and E nonsingular. Frequently a solution X is desired that is stabilizing in the sense that the eigenvalues of the pencil $(A - GXE) - \lambda E$ have negative real part. It was shown in [14] that the desired stabilizing solution is the solution of the overdetermined but consistent system of equations $(\mathbf{lim}_{j\to\infty} H_j + K)\begin{bmatrix} I \\ -XE \end{bmatrix} = 0$ where $H_0 = H$, and

$$H_{j+1} = \frac{1}{2}\left(H_j + K H_j^{-1} K\right). \tag{10}$$

The matrix pencil $\lambda K - \mathbf{lim}_{j\to\infty} H_j$ is a generalization of the matrix sign function to pencils. In terms of relations, (10) is an instance of the iteration of relations (8) when identifying H_j by A_j and K_j by E_j. Representing $R^{-1}_{K_j,H_j}$ by the trivial inverse R_{H_j,K_j}, (4) becomes

$$\mathbf{null}[\tilde{H}_j,\ \tilde{K}_j] = \mathbf{range}\begin{bmatrix} -K_j \\ H_j \end{bmatrix}.$$

If we choose $\tilde{H}_j = I$ and $\tilde{K}_j = K_j H_j^{-1}$, then $K_{j+1} = K_j = K$ and $H_{j+1} = \frac{1}{2}\left(H_j + K H_j^{-1} K\right)$. Hence, the relation (8) written in terms of H_j, K_j reduces to (10).

If the Riccati equations studied in [14] are well posed, then both K and the H_j's are nonsingular for all j. Nevertheless, some of the H_j may be ill-conditioned, so the presence of H_j^{-1} is a possible source of numerical instability. Theorem 2 shows that the explicit inverse can be avoided by replacing $[I,\ K_j H_j^{-1}]$ by a different representation of the left null space of $\begin{bmatrix} -K_j \\ H_j \end{bmatrix}$. For example, such a basis of the left null space can be taken from the QR factorization of $\begin{bmatrix} -K_j \\ H_j \end{bmatrix}$ [4]. Thus, (10) can be replaced by the following. Initially set $H_0 = H$ and $K_0 = K$. For $j = 0, 1, 2, \ldots$, choose $[\tilde{H}_j,\ \tilde{K}_j]$ satisfying

$$\mathbf{null}[\tilde{H}_j,\ \tilde{K}_j] = \mathbf{range}\begin{bmatrix} -K_j \\ H_j \end{bmatrix} = 0. \tag{11}$$

Set

$$K_{j+1} = \tilde{K}_j H_j = \tilde{H}_j K_j \tag{12}$$

and

$$H_{j+1} = \frac{1}{2}\left(\tilde{H}_j H_j + \tilde{K}_j K_j\right). \tag{13}$$

The limits $\mathbf{lim}_{j\to\infty} H_j$ and $\mathbf{lim}_{j\to\infty} K_j$ may or may not exist depending on the choices of $\tilde{H}_j$ and $\tilde{K}_j$. However, The right deflating subspaces of the pencils $H_j - \lambda K_j$ are invariant, and $K_j^{-1}(H_j + K_j)$ converges to $\mathbf{sign}(K^{-1}H) - I$. If X satisfies the overdetermined but consistent system of equations $\mathbf{lim}_{j\to\infty} K_j^{-1}(H_j + K_j)\begin{bmatrix} I \\ -XE \end{bmatrix} = 0$, then X is the desired stabilizing solution to (9).

The iteration (11), (12), (13) gives what is essentially an inverse-avoiding matrix sign function algorithm to solve the algebraic Riccati equation.

5 Conclusions

The sum and product of relations defined above generalize matrix multiplication and matrix addition to linear relations. This arithmetic of linear relations gives a new way to evaluate a generalized matrix sign function without using matrix inverses.

References

[1] Z. Bai and J. Demmel, *Design of a parallel nonsymmetric eigenroutine toolbox, Part I*, in Proceedings of the sixth SIAM Conference on Parallel Processing for Scientific Computing, SIAM Publications, Philadelphia, PA, 1993.

[2] ———, *Design of a parallel nonsymmetric eigenroutine toolbox, Part II*, Tech. Rep. 95-11, Dept. of Mathematics, University of California, Berkeley, 1995.

[3] Z. Bai, J. Demmel, J. Dongarra, A. Petitet, H. Robinson, and K. Stanley, *The spectral decomposition of nonsymmetric matrices on distributed memory parallel computers*, SIAM J. Sci. Comput., 18 (1997), pp. 1446–1461.

[4] Z. Bai, J. Demmel, and M. Gu, *An inverse free parallel spectral divide and conquer algorithm for nonsymmetric eigenproblems*, Numer. Math., 76 (1997), pp. 279–308.

[5] P. Benner and R. Byers, *Disk functions and their relationship to the matrix sign function*, in Proceedings of the 4th European Control Conference, Brussels, July 1–4, 1997 (CD-ROM), BELWARE Information Technology, Waterloo (B), 1997. Paper 936.

[6] P. Benner and E. S. Quintana-Ortí, *Solving stable generalized Lyapunov equations with the matrix sign function*, Tech. Rep. SFB393/97-23, Sonderforschungsbereich 393 – Numerische Simulation auf massiv parallelen Rechnern, TU Chemnitz , 1997.

[7] S. Blackford, J. Choi, A. Cleary, E. D'Azevedo, J. Demmel, I. Dhillon, J. Dongarra, S. Hammarling, G. Henry, A. Petitet, K. Stanley, D. Walker, and R. C. Whaley, *ScaLAPACK Users' Guide*, SIAM, Philadelphia, PA, 1997.

[8] A. Y. Bulgakov and S. K. Godunov, *Circular dichotomy of a matrix spectrum*, Sibirsk. Mat. Zh., 29 (1988), pp. 59–70, 237.

[9] R. Byers, *Numerical stability and instability in matrix sign function based algorithms*, in Computational and Combinatorial Methods in Systems Theory, C. I. Byrnes and A. Lindquist, eds., North-Holland, New York, 1986, pp. 185–200.

[10] ——, *Solving the algebraic Riccati equation with the matrix sign function*, Linear Algebra Appl., 85 (1987), pp. 267–279.

[11] R. Byers, C. He, and V. Mehrmann, *The matrix sign function method and the computation of invariant subspaces*, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 615–632.

[12] E. Denman and A. Beavers, *The matrix sign function and computations in systems*, Appl. Math. and Comput., 2 (1976), pp. 63–94.

[13] J. Gardiner, *A stabilized matrix sign function algorithm for solving algebraic Riccati equations*, in Proceedings of the Fifth SIAM Conference on Applied Linear Algebra, J. G. Lewis, ed., 1994, pp. 75–79.

[14] J. Gardiner and A. Laub, *A generalization of matrix-sign-function solution for algebraic Riccati equations*, Internat. J. Control, 44 (1986), pp. 823–832.

[15] ——, *Parallel algorithms for algebraic Riccati equations*, Internat. J. Control, 54 (1991), pp. 1317–1333.

[16] S. K. Godunov, *The problem of the dichotomy of the spectrum of a matrix*, Sibirsk. Mat. Zh., 27 (1986), pp. 24–37, 204. English translation: Siberian Math. J. 27 (1987), 649–660.

[17] G. Golub and C. van Loan, *Matrix Computations*, The Johns Hopkins Press, Baltimore, 1989.

[18] N. J. Higham, *The matrix sign decomposition and its relation to the polar decomposition*, Linear Algebra Appl., 212/213 (1994), pp. 3–20.

[19] C. Kenney and A. Laub, *The matrix sign function*, IEEE Trans. Automat. Control, 40 (1995), pp. 1330–1348.

[20] V. Khazanov and V. Kublanovskaya, *Sectral problems for matrix pencils. Methods and algorithms. I*, Soviet J. Numer. Anal. Math. Modelling, 3 (1988), pp. 337–371.

[21] V. Kublanovskaya, *The **AB**-algorithm and its properties*, LOMI, 102 (1980), pp. 42–60. Leningrad Department, Steklov Mathematical Institute, USSR Academy of Science, Leningrad, (in Russian).

[22] ——, ***AB**–algorithm and its modifications for the spectral problem of linear pencils of matrices*, Numer. Math., 43 (1984), pp. 329–342.

[23] A. Laub and J. Gardiner, *Hypercube implementation of some parallel algorithms in control*, in Advanced computing concepts and techniques in control engineering (Il Ciocco, 1987), vol. 47 of NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., Springer, Berlin, 1988, pp. 361–390.

[24] A. Malyshev, *Computing invariant subspaces of a regular linear pencil of matrices*, Siberian Math. J., 30 (1989), pp. 559–567.

[25] ——, *Parallel algorithms for solving some spectral problems of linear algebra*, Linear Algebra Appl., 188/189 (1993), pp. 489–520.

[26] J. Roberts, *Linear model reduction and solution of the algebraic Riccati equation by use of the sign function*, Internat. J. Control, 32 (1980), pp. 677–687. Reprint of Tech. Rep. No. TR-13, CUED/B-Control, Cambridge University, Engineering Dept., 1971.

[27] J.-g. Sun, *Perturbation analysis of the matrix sign function*, Linear Algebra Appl., 250 (1997), pp. 177–206.