MPC FÜR NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN PARABOLISCHEN TYPS

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Overview

Control Problems for Nonlinear PDEs

$$\min_{u \in L^2(0,T_f;\mathcal{U})} \frac{1}{2} \int_0^{T_f} \langle y, Q\, y \rangle_{\mathcal{Y}} + \langle u, R\, u \rangle_{\mathcal{U}}\, dt + G(x(T_f)),$$

with $T_f \in (0, \infty]$ and $G \equiv 0$ if $T_f = \infty$, subject to the semilinear (abstract) state equation

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + B\, u(t) + F\, v(t) \quad \text{for } t > 0, \quad x(0) = x_0 + \eta_0, \\ y(t) &= C\, x(t) + w(t). \end{aligned}$$

Here,

- $x(t) \in \mathcal{X}$ are the states ($\mathcal{X} = H_1(\Omega)$ or $\mathcal{X} = \mathbb{R}^n$ after semi-discretization);
- $u(t) \in \mathcal{U}$ are the inputs and $v(t)$ is the input noise (here assume $\mathcal{U} = \mathbb{R}^m$);
- $y(t) \in \mathcal{Y}$ are the outputs and $w(t)$ is the output noise ($\mathcal{Y}$ Hilbert space or $\mathcal{Y} = \mathbb{R}^p$ after semi-discretization);
- η_0 is the noise in the initial condition.

Control Problems for Nonlinear PDEs

$$\min_{u \in L^2(0,T_f;\mathcal{U})} \frac{1}{2} \int_0^{T_f} \langle y, Q\, y \rangle_{\mathcal{Y}} + \langle u, R\, u \rangle_{\mathcal{U}}\, dt + G(x(T_f)),$$

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Examples of semilinear PDEs:

- reaction-diffusion equation: $f(x) = \alpha\Delta x + p(x)$, p – polynomial;
- Allen-Cahn equation: $f(x) = \alpha\Delta x + x(1 - x^2)$;
- Kuramoto-Sivashinsky equation: $f(x) = -\alpha\Delta x - \beta\Delta^2 x - x\nabla x$;
- Burgers equation: $f(x) = \alpha\Delta x - x\nabla x$.

Control Problems for Nonlinear PDEs

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General form of semilinear PDE control problems considered here:

$$\frac{\partial}{\partial t}\mathbf{x}(t,\xi) = \mathbf{A}(\xi)\mathbf{x}(t,\xi) + \mathbf{N}(\xi, \mathbf{x}, \frac{\partial}{\partial x}\mathbf{x}) + \mathbf{B}(\xi)u(t),$$

$\xi \in \Omega \subset \mathbb{R}^d$; $t \in [0, T_f]$; $\mathbf{A} : \mathcal{X} \to \mathcal{X}$ linear, elliptic operator;
$\mathbf{N}$ sufficiently smooth; together with suitable boundary conditions

Motivation
Abrasive Waterjet Cutting

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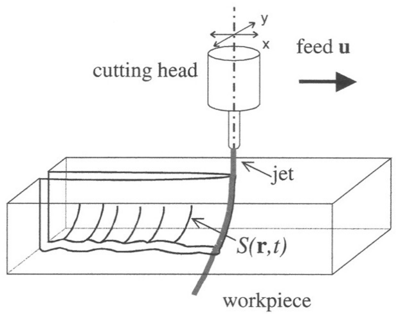
- Used to cut titanium, ceramics, and other compound materials.
- Material is cut using a focused beam (diam $\approx$ 1mm) in which abrasive particles (sand, garnet) are accelerated by water or air to velocity $\approx$ 900m/sec;
- cut has depth of several centimeters.
- Problem: ripple formation at sidewalls, degradation of cut quality.
- Similar problems arise in laser/ion/electron beam cutting.
- Modeled by generalized KS equation.



$$\begin{aligned} \mathbf{r} &= (x, y) \\ S(\mathbf{r}, t) &= \text{erosion front} \\ S(\mathbf{r}, 0) &= 0 \\ u &= 0 \quad \rightsquigarrow \text{drilling.} \end{aligned}$$

Source: Friedrich/Radons/Ditzinger/Henning, *Phys. Rev. Lett.*, 85(23), 2000.

Goal: control speed $u(t)$ to achieve smooth cut, avoid ripples.

Solution Strategy

Problems are

1. Nonlinearity → Formulation as an optimization problem leads to a nonlinear program → large computational effort.
2. Nominal states may not be available because of
   - noise in input, output (measurements) and initial condition (noise may also represent modeling errors);
   - states are not fully accessible (by measurements or simulations).

Strategy (based on Ito/Kunisch, *Receding Horizon Control with Incomplete Observations*, SIAM J. Control Optim., Vol. 45, No. 1, March 2006)

1. Linearize the nonlinear state equation on sub-intervals (Model Predictive Control (MPC) or Receding Horizon Control (RHC)).
2. Find estimates of the states (Linear Quadratic Gaussian Design (LQG)) on the sub-intervals.

MPC/LQG Design

Linearization

(1) Spatial semi-discretization by FEM or FDM.

(2) Linearization on $[T_i, T_i + T]$

Use a given reference trajectory $(x^*(t), u^*(t))$ and determine an operating point $\bar{x}$, for example

$$\bar{x} = \frac{1}{T} \int_{T_i}^{T_i+T} x^*(t)\, dt.$$

Linearize around this operating point to obtain a linear equation on $[T_i, T_i + T]$:

$$\frac{d}{dt}(x(t) - x^*(t)) = A(x(t) - x^*(t)) + B(u(t) - u^*(t)) + Fv(t)$$

where $A = f'(\bar{x})$.

(3) LQG Design on $[T_i, T_i + T]$

Assumptions:

- The linearized system is controllable and observable.
- w and v are zero-mean stochastic processes, that is $E[w] = 0$, $E[v] = 0$, $E[vw^T] = 0$, $E[ww^T] = W$, $E[vv^T] = V$.
  V and W are symmetric positive definite covariance matrices.
- $E[\eta] = 0$, $E[\eta\, v^T] = 0$, $E[\eta\, w^T] = 0$

Best estimate $\hat{x}(t)$ of $x(t)$: Kalman filter

$$\dot{\hat{x}}(t) = A(\hat{x}(t) - x^*(t)) + f(x^*(t)) + Bu(t) + G_f(y(t) - C\hat{x}(t)),$$

where $\hat{x}(0) = x_0 + \eta_0$, $G_f = \Sigma_* C^T W^{-1}$ and Σ_* is the solution of the Filter Algebraic Riccati Equation (FARE)

$$0 = A\Sigma + \Sigma A^T - \Sigma C^T W^{-1} C \Sigma + FVF^T.$$

MPC/LQG Design

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(3) LQG Design on $[T_i, T_i + T]$

The control on $[T_i, T_i + T]$ is given by the feedback law

$$u_i(t) = -G\,\hat{x}(t)$$

where $G = R^{-1}B^T X_*$ and X_* is the solution of the Algebraic Riccati Equation (ARE)

$$0 = X A + A^T X - X BR^{-1}B^T X + C^T QC.$$

(4) Update

$$u(t) = u_i(t), t \in [T_i, T_i + \delta), \ \delta \le T.$$

Repeat the whole procedure on the next interval $[T_{i+1}, T_{i+1} + T]$.

MPC/LQG Design

Effort on every sub-interval $[T_i, T_i + T]$:

1. Reference trajectory $(x^*(t), u^*(t))$ (known/desired trajectory or solve an undisturbed problem)
2. Linearization around an operating point based on the reference trajectory
3. Solve the ARE and FARE to obtain the two gain matrices G and G_f.
4. Solve two ODEs:
   - ODE for measurements/simulation of measurement (MODE) $\longrightarrow$ uses computed control from EODE
   - ODE for estimated states (EODE) $\longrightarrow$ uses measurements from MODE

Theoretical Aspects

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Performance of the compensator (Ito/Kunisch)

$$E(t) = \left[\langle x - x^*, X(x - x^*)\rangle + \langle x - \hat{x}, \Sigma^{-1}(x - \hat{x})\rangle\right]^{\frac{1}{2}}$$

Assumptions:

$E(0) \leq \frac{\delta}{2}$, and $|x^*(t) - \bar{x}| \leq \delta$ on $[0, T]$ for some $\delta > 0$ and ...

Conclusion:

It can be shown, that $E(t) < \frac{\delta}{2}(1 + \alpha)$ $\forall t \in [0, T]$, $\alpha \in (0, 1)$.

If $E(0) \leq \frac{\delta}{2}$, and $|x^*(t) - \bar{x}| \leq \delta$ on $[T_i, T_{i+1}]$ $\forall i = 0, \ldots$ and ...
$\longrightarrow$ this can be expanded to all $t > 0$.

Optimal Control Problem for the Burgers Equation

Burgers Equation: 1D convection-diffusion equation

$$x_t(t,\xi) + x(t,\xi)\, x_\xi(t,\xi) = \nu x_{\xi\xi}(t,\xi)$$

- ν – viscosity parameter

Optimal control problem for the Burgers equation

$$\min_{u\in L^2(0,T_f;\mathcal{U})} \frac{1}{2}\int_0^\infty \int_{\Omega_y} y(t,\xi)^T\, y(t,\xi)\, d\xi + u(t)^T R\, u(t)\, dt$$

subject to the Burgers equation

$$\begin{aligned} x_t(t,\xi) &= \nu\, x_{\xi\xi}(t,\xi) - x(t,\xi)\, x_\xi(t,\xi) + B(\xi)u(t) + F(\xi)v(t),\\ x(t,0) &= x(t,1) = 0, \quad t > 0,\\ x(0,\xi) &= x_0(\xi) + \eta(\xi), \quad \xi \in (0,1),\\ y(t,\xi) &= C\, x(t,\xi) + w(t,\xi). \end{aligned}$$

Optimal Control Problem for the Burgers Equation

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Burgers Equation: 1D convection-diffusion equation

$$x_t(t,\xi) + x(t,\xi)\, x_\xi(t,\xi) = \nu x_{\xi\xi}(t,\xi)$$

- ν – viscosity parameter

Optimal control problem for the Burgers equation

FEM discretization in space: N sub-intervals $[\xi_i, \xi_{i+1}]$, $i = 0, .., N-1$

$$\rightsquigarrow \quad \text{ODE:} \quad M\dot{x}^N(t) = \nu\, K\, x^N(t) + S(x^N(t)) + B^N u(t) + F^N v(t),$$

where M is the mass matrix, K the stiffness matrix and $S(x^N(t))$ the nonlinear part.

MPC/LQG

Consider a sub-interval $[T_i, T_i + T]$

1. Reference trajectory: from two-point boundary value problem for the undisturbed control problem
2. Linearization: $A = f'(\bar{x}^N(t)) = -M^{-1}(\nu K + S_x(\bar{x}^N))$
   ($S_x(\bar{x}^N)$ - Jacobian of S at point $\bar{x}^N$)
3. Solve the two matrix Riccati Equations

$$\begin{aligned} 0 &= XA + A^TX - XB^NR^{-1}(B^N)^TX + C^TQC \to X_* \\ 0 &= A\Sigma + \Sigma A^T - \Sigma C^TW^{-1}C\Sigma + F^NV(F^N)^T \to \Sigma_* \end{aligned}$$

   Gain matrices: $G = R^{-1}(B^N)^T X_*, \quad G_f = \Sigma_* C^T W^{-1}$.
4. ODEs for measured and estimated states in every time step:

$$\begin{aligned} \textbf{MODE:} \quad & M\dot{x}^N(t) = -\nu K x^N(t) - S(x^N(t)) + F^N v(t) + B^N u(t) \\ & y^N(t) = C x^N(t) + M^{-1} w^N(t) \\ \textbf{EODE:} \quad & \dot{\hat{x}}(t) = A(\hat{x}(t) - x^*(t)) + f(x^*(t)) + B^N u(t) \\ & \qquad + G_f(y^N(t) - C\hat{x}(t)) \\ & u(t) = u^*(t) - G(\hat{x}(t) - x^*(t)) \end{aligned}$$

First Results (without noise)

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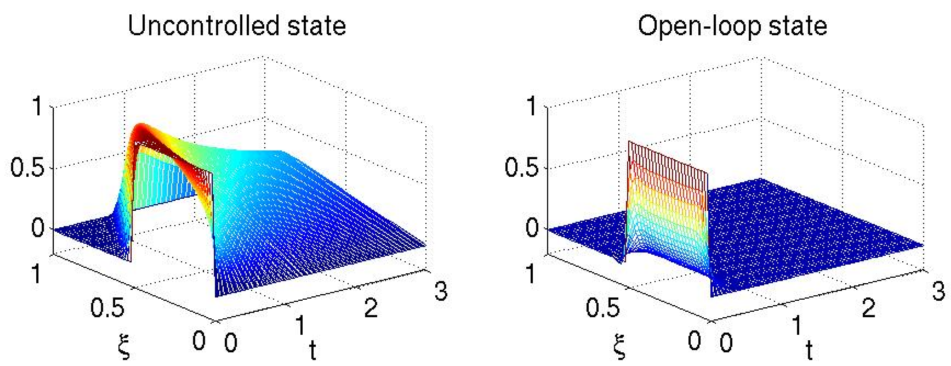
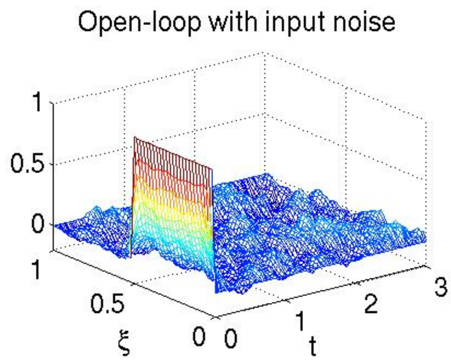
Parameters

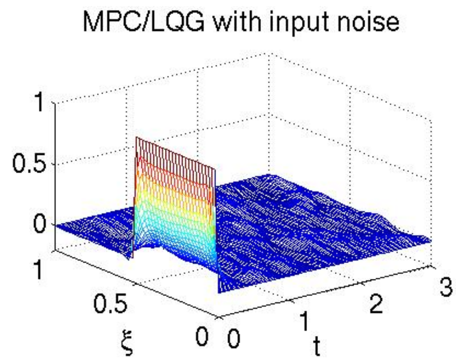
$$T_f = 3,\ T = 0.5,\ h = \tau = \tfrac{1}{50},\ \nu = 0.01,\ C = \mathrm{I},\ B = F = 1_{\Omega_u}(\xi),$$
$$Q = 0.1\mathrm{I},\ R = 0.001\mathrm{I},\ V = 4\mathrm{I},\ W = 0.01\mathrm{I}$$



First Results (with noise)



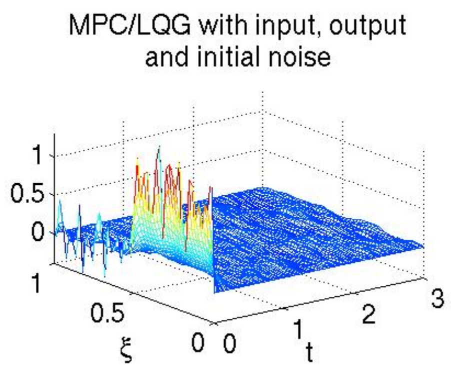
STC: 0.02218



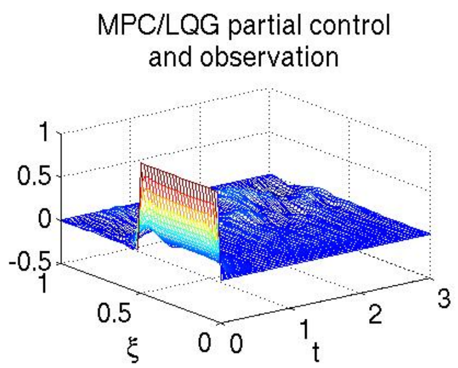
STC: 0.00199

$$\left(\text{State Tracking Cost (STC):} \int_0^3 (x - x^*)^T C^T Q C (x - x^*)\, dt \right)$$

First Results



STC: 0.00301



STC: 0.00706

Summary

Summary

Case study for MPC/LQG approach based on Ito/Kunisch with taking into account disturbance in

- input
- output
- initial value.

$\rightsquigarrow$ robust control scheme in the presence of uncertainties

Outlook

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- More efficient implementation:
  - large-scale Riccati solvers, especially for FARE;
  - adapted meshes for time frames;
  - Riccati differential equation solvers for time-varying linearizations.
- Application to more challenging nonlinear PDEs in 2D and 3D, in particular to abrasive water cutting problem.
- Generalize finite-dimensional convergence theory to more abstract setting.

References

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