γ-Iteration for Descriptor Systems Using Structured Matrix Pencils

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1 Introduction

The optimal infinite-horizon output (or measurement) feedback $\mathcal{H}_\infty$ control problem is one of the central tasks in robust control, see, e.g., [12, 13, 22, 28, 30]. For standard state space systems, where the dynamics of the system are modelled by a linear constant coefficient ordinary differential equation, the analysis of this problem is well studied and numerical methods have been developed and integrated in control software packages such as [2, 8, 14]. These methods work well for a wide range of problems in computing close to optimal (suboptimal) controllers, but the exact computation of the optimal value γ in $\mathcal{H}_\infty$ control is considered a challenge [9]. In order to avoid some of the numerical difficulties that arise when approaching the optimum, in [6, 7] several improvements of the previously known methods were presented. These are based on the solution of structured eigenvalue problems with structured methods.

In this paper we study the more general case that the dynamics is constrained, i.e. described by a *differential-algebraic equation (DAE) or descriptor system.* Descriptor systems arise in the control of constrained mechanical systems, see e.g. [10, 26], in electrical circuit simulation, see e.g. [15], and in particular in heterogeneous systems, where different models are coupled [21].

Robust control for descriptor systems has been studied in [23–25] using linear matrix inequalities (LMIs) and employing methods of semidefinite programming to find γ_{mo}. This is attractive, because easy-to-use methods for semidefinite programming are available, see, e.g., [20]. In such an approach, LMIs in $\mathcal{O}(n^2)$ variables need to be solved which in general results in a complexity of $\mathcal{O}(n^6)$. Despite recent progress in reducing this complexity based on exploiting duality in the related semidefinite programs [1, 29], the best complexity achievable is still larger than $\mathcal{O}(n^4)$ as compared to the $\mathcal{O}(n^3)$ cost of the procedure discussed here. Robust control for descriptor systems has also been studied in [27] via generalized Riccati equations and J-spectral factorization. Unfortunately, there are several numerical difficulties associated with Riccati methods. Primary among these is the fact that often as γ approaches γ_{mo}, one of the ARE solutions X_H or X_J either diverges to ∞ or becomes highly ill-conditioned, i.e., tiny errors in the Hamiltonian matrices $H(\gamma)$ or $J(\gamma)$ may lead to large errors in X_H or X_J. In contrast to these approaches, we extend the analysis and the robust numerical methods that were derived via deflating subspaces in [6, 7]. We

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discuss descriptor systems of the form

$$\mathbf{P}:\left\{\begin{aligned} E\dot{x}(t) &= \ \ Ax(t) + B_1 w(t) \ \ + B_2 u(t), \qquad x(t_0) = x^0, \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t), \end{aligned}\right. \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n,m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$ for $i,j = 1,2$. (Here, by $\mathbb{R}^{k,l}$ we denote the set of real $k \times l$ matrices.)

In this system, $x(t) \in \mathbb{R}^n$ is the descriptor vector, $u(t) \in \mathbb{R}^{m_2}$ is the control input vector, and $w(t) \in \mathbb{R}^{m_1}$ is an exogenous input that may include noise, linearization errors and un-modelled dynamics. The vector $y(t) \in \mathbb{R}^{p_2}$ contains measured outputs, while $z(t) \in \mathbb{R}^{p_1}$ is a regulated output or an estimation error. Our approach can also be extended to rectangular systems and systems in behavior formulation, using a remodelling as it was suggested in [17], see also [16], but here we only study the formulation in (1).

The optimal $\mathcal{H}_\infty$ control problem is typically formulated in frequency domain. For this we need the following notation. The space $\mathcal{H}_\infty^{p,m}$ consists of all $\mathbb{C}^{p,m}$-valued functions that are analytic and essentially bounded in the complex half plane $\mathbb{C}^+ = \{s \in \mathbb{C} \,:\, \mathrm{Re}(s) > 0\}$. For $F \in \mathcal{H}_\infty^{p,m}$ the $\mathcal{H}_\infty$-norm is given by $\|F\|_\infty = \sup_{s\in\mathbb{C}^+} \sigma_{\max}(F(s))$, where $\sigma_{\max}(F(s))$ denotes the maximal singular value of the matrix $F(s)$.

In robust control, $\|F\|_\infty$ is used as a measure of the worst case influence of the disturbances w on the output z, where in this case F is the transfer function mapping noise or disturbance inputs to error signals [30].

The optimal $\mathcal{H}_\infty$ control problem is the task of designing a dynamic controller that minimizes (or at least approximately minimizes) this measure, leading to a closed-loop system as in the following figure.

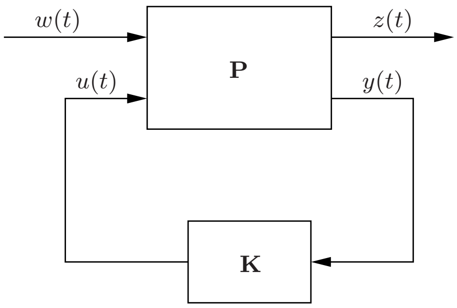

Figure 1. *Plant interconnected with Controller*

Put more rigorously, the optimal $\mathcal{H}_\infty$ control problem is the following.

Definition 1 (The Optimal $\mathcal{H}_\infty$ control problem). *For the descriptor system (1), determine a controller (dynamic compensator)*

$$\mathbf{K}:\left\{\begin{aligned} \hat{E}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}y(t), \\ u(t) &= \hat{C}\hat{x}(t) + \hat{D}y(t) \end{aligned}\right. \tag{2}$$

with $\hat{E}, \hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$, *and a transfer function* $K(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$ *such that the closed-loop system resulting from the combination of (1) and (2) as in Figure 1, has the following properties.*

1.) Internal stability, *i.e., the solution* $\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$ *of the system with* $w \equiv 0$ *is* asymptotically stable.

2.) *The closed-loop transfer function $T_{zw}(s)$ from w to z satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ and is minimized in the $\mathcal{H}_\infty$-norm.*

In principle, there is no restriction on the dimension N of the auxiliary state $\hat{x}$ in (2), although, smaller dimensions N are preferred for practical implementation and computation.

As in the case of the optimal $\mathcal{H}_\infty$ control problems for ordinary state space systems it is also necessary to study two closely related optimization problems, the *modified optimal $\mathcal{H}_\infty$ control problem* and the *suboptimal $\mathcal{H}_\infty$ control problem.*

Definition 2 (The modified optimal $\mathcal{H}_\infty$ control problem). *For the descriptor system (1) let Γ be the set of positive real numbers γ for which there exists an internally stabilizing dynamic controller of the form (2) so that the transfer function $T_{zw}(s)$ of the closed loop system satisfies $\|T_{zw}\|_\infty < \gamma$. Determine $\gamma_{mo} = \inf \Gamma$. (If no internally stabilizing dynamic controller exists, we set $\Gamma = \emptyset$ and $\gamma_{mo} = \infty$.)*

Note that it is possible that there is no internally stabilizing dynamic controller with the property $\|T_{zw}\|_\infty = \gamma_{mo}$. Moreover, the minimizing controller may be fragile and in practice, the minimization condition is often relaxed and rather than the optimal $\mathcal{H}_\infty$ control problem, the following suboptimal $\mathcal{H}_\infty$ control problem is solved.

Definition 3 (The suboptimal $\mathcal{H}_\infty$ control problem.). *For the descriptor system (1) and $\gamma \in \Gamma$ with $\gamma > \gamma_{mo}$, determine an internally stabilizing dynamic controller of the form (2) such that the closed loop transfer function satisfies $\|T_{zw}\|_\infty < \gamma$. We call such a controller* γ-suboptimal controller *or simply* suboptimal controller.

To state our results we will also need some notions of controllability and observability for descriptor systems

Definition 4. *Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n,m}$ and $C \in \mathbb{R}^{p,n}$. Further, let T_∞, S_∞ be matrices with $\operatorname{Im} T_\infty = \ker E^T$ and $\operatorname{Im} S_\infty = \ker E$.*

i) The triple (E, A, B) is called finite dynamics stabilizable *if* $\operatorname{rank}[\lambda E - A, B] = n$ *for all $\lambda \in \mathbb{C}^+$;*

ii) (E, A, B) is impulse controllable *if* $\operatorname{rank}[E, AS_\infty, B] = n$*;*

iii) (E, A, B) is strongly stabilizable *if it is both finite dynamics stabilizable and impulse controllable;*

iv) The triple (E, A, C) is finite dynamics detectable *if* $\operatorname{rank}[\lambda E^T - A^T, C^T] = n$ *for all $\lambda \in \mathbb{C}^+$;*

v) (E, A, C) is impulse observable *if* $\operatorname{rank}[E^T, A^T T_\infty, C^T] = n$*;*

vi) $(\lambda E - A, C)$ is strongly detectable *if it is both finite dynamics detectable and impulse observable.*

The solution and many properties of the *free descriptor system (with $u, w = 0$)* can be characterized in terms of the *Weierstraß canonical form (WCF).*

Theorem 5. *[11] For a regular matrix pencil $\lambda E - A$, there exist matrices $W_f, V_f \in \mathbb{R}^{n,n_f}$, $W_\infty, V_\infty \in \mathbb{R}^{n,n_\infty}$ with the property that $W = \begin{bmatrix} W_f & W_\infty \end{bmatrix}$, $V = \begin{bmatrix} V_f & V_\infty \end{bmatrix}$ are square and* *invertible, with*

$$W^T EV = \begin{bmatrix} W_f^T \\ W_\infty^T \end{bmatrix} E \begin{bmatrix} V_f & V_\infty \end{bmatrix} = \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix}, \tag{3a}$$

$$W^T AV = \begin{bmatrix} W_f^T \\ W_\infty^T \end{bmatrix} A \begin{bmatrix} V_f & V_\infty \end{bmatrix} = \begin{bmatrix} A_f & 0 \\ 0 & I_{n_\infty} \end{bmatrix}, \tag{3b}$$

$A_f \in \mathbb{R}^{n_f, n_f}$ *is in real Jordan canonical form and* N *is a nilpotent matrix, also in Jordan canonical form. We call* n_f, n_∞ *the number of finite or infinite eigenvalues, respectively.*

The index of nilpotency of the nilpotent matrix N in (3a) is called the *index* of the system and if E is nonsingular, then the pencil is said to have *index zero.*

To simplify notation, the term *eigenvalue* is used both for eigenvalues of matrices and for pairs $(\alpha, \beta) \neq (0, 0)$ for which $\det(\alpha E - \beta A) = 0$. These pairs are not unique. If $\beta \neq 0$ then we identify (α, β) with $(\alpha/\beta, 1)$ and $\lambda = \alpha/\beta$. Pairs $(\alpha, 0)$ with $\alpha \neq 0$ are called *infinite eigenvalues.*

By $\Lambda(E, A)$, we denote the set of eigenvalues of $\alpha E - \beta A$ including finite and infinite eigenvalues both counted according to multiplicity.

We will denote by $\Lambda_-(E, A)$, $\Lambda_0(E, A)$ and $\Lambda_+(E, A)$ the set of finite eigenvalues of $\alpha A - \beta E$ with negative, zero, and positive real parts, respectively. The set of infinite eigenvalues is denoted by $\Lambda_\infty(E, A)$. Multiple eigenvalues are repeated in $\Lambda_-(E, A)$, $\Lambda_0(E, A)$, $\Lambda_+(E, A)$ and $\Lambda_\infty(E, A)$ according to algebraic multiplicity. The set of all eigenvalues counted according to multiplicity is $\Lambda(E, A) := \Lambda_-(E, A) \cup \Lambda_0(E, A) \cup \Lambda_+(E, A) \cup \Lambda_\infty(E, A)$. Similarly, we denote by $\mathrm{Def}_-(E, A)$, $\mathrm{Def}_0(E, A)$, $\mathrm{Def}_+(E, A)$ and $\mathrm{Def}_\infty(E, A)$ the right deflating subspaces corresponding to $\Lambda_-(E, A)$, $\Lambda_0(E, A)$, $\Lambda_+(E, A)$ and $\Lambda_\infty(E, A)$, respectively.

2 Main result

In this section we approach the problem of determining γ_{mo} for a given system (1). As in the case of standard state space systems, see [12,13], we need several assumptions on the system matrices. In the following we set $r = \operatorname{rank} E$.

Assumptions:

A1) The triple (E, A, B_2) is strongly stabilizable and the triple (E, A, C_2) is strongly detectable.

A2) $\operatorname{rank} \begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all $\omega \in \mathbb{R}$.

A3) $\operatorname{rank} \begin{bmatrix} A - i\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all $\omega \in \mathbb{R}$.

A4) For matrices $T_\infty, S_\infty \in \mathbb{R}^{n, n-r}$ with $\operatorname{Im} S_\infty = \ker E$ and $\operatorname{Im} T_\infty = \ker E^T$ the rank conditions $\operatorname{rank} \begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} = n + m_2 - r$, $\operatorname{rank} \begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} = n + p_1 - r$ hold.

It is well known for standard state space systems that Assumption **A1)** is essential for the existence of a controller that internally stabilizes the system. We will see that a similar result holds for the descriptor case. Assumptions **A2)** and **A3)** correspond to the typical claim that the system does not have transmission zeros on the imaginary axis. This is assumed in many works about $\mathcal{H}_\infty$-control of standard state space systems, since eigenvalues on the imaginary axis of the Hamiltonian matrices that are used in the computation of an optimal controller usually lead to problems in the computation of a semi-stable subspace, see [19].

Further typical assumptions used when treating the $\mathcal{H}_\infty$-control problem for standard state space systems are that D_{12}, D_{21}^T have full column rank, see [13,30]. The conditions in **A4)** reduce

to these rank conditions if E is invertible.
In the following we will make use of even matrix pencils as well as of skew-Hamiltonian/Hamiltonian matrix pencils: a matrix pencil $\lambda N - M$ is called even if N is skew-symmetric and M is symmetric. A matrix pencil $\lambda S - H$ with $S, H \in \mathbb{R}^{2n,2n}$ is called skew-Hamiltonian/Hamiltonian if $S\mathcal{J}$ is skew symmetric and $H\mathcal{J}$ is symmetric, where $\mathcal{J} = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$. For a brief overview of the properties of skew-Hamiltonian/Hamiltonian pencils, see, for example [5].

For the solution of the modified optimal $\mathcal{H}_\infty$ control problem we will make use of the following two even matrix pencils, which generalize the pencils constructed in [7]. The contruction of these pencils is based on the observation that the solution of the well known Riccati equations for $\mathcal{H}_\infty$ control can be expressed in terms of certain Lagrangian invariant subspaces for Hamiltonian matrices [30], which in turn are replaced by skew-Hamiltonain/Hamiltonian or by even matrix pencils to avoid explicit inverses and improve the condition of the problem. We will call a subspace $\mathcal{L} \subset \mathbb{R}^{2n}$ isotropic if $x^T \mathcal{J} y = 0$ for all $x, y \in \mathcal{L}$, where $\mathcal{J} = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$. An isotropic subspace with $\dim \mathcal{L} = n$ is called Lagrangian. Let

$$\lambda N_H + M_H(\gamma) = \lambda \left[\begin{array}{cc|ccc} 0 & -E^T & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & -A^T & 0 & 0 & -C_1^T \\ -A & 0 & -B_1 & -B_2 & 0 \\ \hline 0 & -B_1^T & -\gamma^2 I_{m_1} & 0 & -D_{11}^T \\ 0 & -B_2^T & 0 & 0 & -D_{12}^T \\ -C_1 & 0 & -D_{11} & -D_{12} & -I_{p_1} \end{array}\right] \quad (4)$$

and

$$\lambda N_J + M_J(\gamma) = \lambda \left[\begin{array}{cc|ccc} 0 & -E & 0 & 0 & 0 \\ E^T & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & -A & 0 & 0 & -B_1 \\ -A^T & 0 & -C_1^T & -C_2^T & 0 \\ \hline 0 & -C_1 & -\gamma^2 I_{p_1} & 0 & -D_{11} \\ 0 & -C_2 & 0 & 0 & -D_{21} \\ -B_1^T & 0 & -D_{11}^T & -D_{21}^T & -I_{m_1} \end{array}\right]. \quad (5)$$

Our approach is based on considering deflating subspaces of the matrix pencils (4) and (5), where the subspaces are spanned by the columns of the matrices X_H and X_J that are partitioned conformably with the pencils, i.e.,

$$\begin{aligned} X_H^T(\gamma) &= \left[\begin{array}{ccccc} X_{H,1}^T(\gamma) & X_{H,2}^T(\gamma) & X_{H,3}^T(\gamma) & X_{H,4}^T(\gamma) & X_{H,5}^T(\gamma) \end{array}\right], \\ X_J^T(\gamma) &= \left[\begin{array}{ccccc} X_{J,1}^T(\gamma) & X_{J,2}^T(\gamma) & X_{J,3}^T(\gamma) & X_{J,4}^T(\gamma) & X_{J,5}^T(\gamma) \end{array}\right], \end{aligned} \quad (6)$$

with $X_{H,1}(\gamma)$, $X_{H,2}(\gamma), X_{J,1}(\gamma)$, $X_{J,2}(\gamma) \in \mathbb{R}^{n,r}$, $X_{H,4}(\gamma) \in \mathbb{R}^{m_2,r}$, $X_{J,4}(\gamma) \in \mathbb{R}^{p_2,r}$, $X_{H,3}(\gamma)$, $X_{J,5}(\gamma) \in \mathbb{R}^{m_1,r}$, $X_{H,5}(\gamma), X_{J,3}(\gamma) \in \mathbb{R}^{p_1,r}$.

We extend the results in [7] to general descriptor systems and use deflating subspaces of the even pencils (4) and (5) to characterize the elements of the set Γ in Definition 2. For this we introduce the following conditions which will be shown to be necessary for the existence of a controller with the desired properties associated with a parameter $\gamma \in \Gamma$.

C1) The index of both pencils (4) and (5) is at most one.

C2) There exists a matrix $X_H(\gamma)$ as in (6) such that

- **C2.a)** $\operatorname{Im} X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H(\gamma)$ and $\operatorname{Im} \left[\begin{smallmatrix} EX_{H,1} \\ X_{H,2} \end{smallmatrix}\right]$ is an r-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
- **C2.b)** $\operatorname{rank} EX_{H,1}(\gamma) = r$.

C3) There exists a matrix $X_J(\gamma)$ as in (6) such that

- **C3.a)** $\operatorname{Im} X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J + M_J(\gamma)$ and $\operatorname{Im} \left[\begin{smallmatrix} E^T X_{J,1} \\ X_{J,2} \end{smallmatrix}\right]$ is an r-dimensional isotropic subspace of $\mathbb{R}^{2n}$;

C3.b) $\operatorname{rank} E^T X_{J,1}(\gamma) = r$.

C4) The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} -\gamma X_{H,2}^T(\gamma) E X_{H,1}(\gamma) & X_{H,2}^T(\gamma) E X_{J,2}(\gamma) \\ X_{J,2}^T(\gamma) E^T X_{H,2}(\gamma) & -\gamma X_{J,2}(\gamma)^T E^T X_{J,1}(\gamma) \end{bmatrix} \tag{7}$$

is symmetric, positive semi-definite and satisfies $\operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$, where $\hat{k}_H = \operatorname{rank} E^T X_{H,2}(\gamma)$ and $\hat{k}_J = \operatorname{rank} E X_{J,2}(\gamma)$.

Using these conditions we can state the following result which is proved in [18].

Theorem 6. *Consider system (1) and the even pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ as in (4) and (5), respectively. Suppose that assumptions* **A1)** – **A4)** *hold.*

Then there exists an internally stabilizing controller such that the transfer function from w to z satisfies $\|T_{zw}\|_\infty < \gamma$ if and only if γ is such that the conditions **C1)** – **C4)** *hold.*

Furthermore, the set of γ satisfying the conditions **C1)** – **C4)** *is nonempty.*

Sketch of proof:
First, one can show that there exists an index reducing a priori feedback such that the Assumptions **A1)** – **A4)** and the Conditions **C1)** – **C4)** remain unchanged. Then the resulting system of index one can be transformed to Weierstraß canonical form [11] and the resulting special structure can be used to rewrite the system as a standard system ($E = I$). It is then possible to show that the resulting standard system fulfills the well known assumptions and conditions for the existence of an internally stabilizing controller, given for example in [30], such that the transfer function from w to z satisfies $\|T_{zw}\|_\infty < \gamma$, if and only if the original system satisfies **A1)** – **A4)** and **C1)** – **C4)** respectively. Finally showing that the combination of optimal controller and index reducing feedback results in the same closed loop system as by applying the optimal controller to the system written as a standard system concludes the proof. □

3 Computation of γ_{mo}

In this section we give a numerical method for the computation of γ_{mo} that is similar to the procedure proposed in [7] and uses a bisection method.

Procedure 1: (Classification of γ)
Input: Data of system (1), value $\gamma \geq 0$.
Output: Decision whether $\gamma < \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$.

1. Form the pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$.
2. Compute the deflating subspace matrices X_H and X_J associated with the eigenvalues in the closed left half plane.
3. IF the dimension of one/both of these subspaces is less than r, then $\gamma < \gamma_{mo}$,

   ELSE

      IF the rank of $EX_{H,1}$ and/or $E^T X_{J,1}$ is less than r, THEN $\gamma < \gamma_{mo}$,

      ELSE

         Form the matrix $\mathcal{Y}$.
         IF $\mathcal{Y}$ is not positive semi-definite and/or $\operatorname{rank} \mathcal{Y} < \hat{k}_H + \hat{k}_J$, THEN $\gamma < \gamma_{mo}$,
         ELSE $\gamma \geq \gamma_{mo}$.
         END

      END

END

With this procedure we can determine γ_{mo} using a bisection method. The computation of the deflating subspace matrices X_H and Y_H in Step 2. of Procedure 1 should respect the structure of the matrix pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. This is achieved by using the procedure described in the following section.

4 Structured Computation of Deflating Subspaces

In this section we will consider skew-Hamiltonian/Hamiltonian matrix pencils rather than even ones, since an even matrix pencil of even size* can be made skew-Hamiltonian/Hamiltonian by simply multiplying with $\mathcal{J}$ of appropriate size from the left leaving the deflating subspaces unchanged.
If $\alpha\mathcal{N} - \beta\mathcal{H}$ is a real skew-Hamiltonian/Hamiltonian pencil, then for any real nonsingular matrix $\mathcal{X}$, $(\mathcal{J}\mathcal{X}^T\mathcal{J})(\alpha\mathcal{N} - \beta\mathcal{H})\mathcal{X}$ is still a real skew-Hamiltonian/Hamiltonian pencil. For a real skew-Hamiltonian/Hamiltonian pencil $\alpha\mathcal{N} - \beta\mathcal{H}$, we call the condensed form

$$(\mathcal{J}\mathcal{Q}^T\mathcal{J}^T)(\alpha\mathcal{N} - \beta\mathcal{H})\mathcal{Q} = \alpha \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix} - \beta \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^T \end{bmatrix}$$

a *structured Schur form*, where $\mathcal{Q}$ is real orthogonal, N_{11} is upper triangular and H_{11} is quasi upper triangular. Note that not every skew-Hamiltonian/Hamiltonian pencil has such a structured Schur form [5]. But using the following *generalized symplectic URV decomposition* and embedding the pencil in one of double size we can still efficiently compute the deflating subspaces in a structure preserving way for all regular skew-Hamiltonian/Hamiltonian pencils.

Theorem 7. *Let $\alpha\mathcal{S} - \beta\mathcal{H}$ be a regular real skew-Hamiltonian/Hamiltonian pencil. Then there exist orthogonal matrices $\mathcal{Q}_1, \mathcal{Q}_2$ such that*

$$\mathcal{Q}_1^T\mathcal{S}\mathcal{J}\mathcal{Q}_1\mathcal{J}^T = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix}, \ \mathcal{J}\mathcal{Q}_2^T\mathcal{J}^T\mathcal{S}\mathcal{Q}_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{11}^T \end{bmatrix}, \ \mathcal{Q}_1^T\mathcal{H}\mathcal{Q}_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

where S_{11}, T_{11}, H_{11} are upper triangular and H_{22}^T is quasi upper triangular.

A proof can be found in [3]. Based on this theorem, we can now compute a structured Schur form yielding the desired deflating subspaces if we embed the original pencil in a pencil of double size, $\mathcal{B}_{\mathcal{S}} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{bmatrix}$, $\mathcal{B}_{\mathcal{H}} = \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\mathcal{H} \end{bmatrix}$. Introduce the orthogonal matrices

$$\mathcal{Y}_r = \frac{\sqrt{2}}{2} \begin{bmatrix} I_{2n} & I_{2n} \\ -I_{2n} & I_{2n} \end{bmatrix}, \ \mathcal{P} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}, \tag{8}$$

and let $\mathcal{X}_r = \mathcal{Y}_r\mathcal{P}$. Now setting $\tilde{\mathcal{Q}} = \mathcal{P}^T \operatorname{diag}(\mathcal{J}\mathcal{Q}_1\mathcal{J}^T, \mathcal{Q}_2)\mathcal{P}$, we can transform the skew-Hamiltonian matrix $\mathcal{B}_{\mathcal{S}}^r := \mathcal{X}_r^T\mathcal{B}_{\mathcal{S}}\mathcal{X}$ and the Hamiltonian matrix $\mathcal{B}_{\mathcal{H}}^r := \mathcal{X}_r^T\mathcal{B}_{\mathcal{H}}\mathcal{X}_r$ to

$$\begin{aligned} \mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{B}_{\mathcal{S}}^r\tilde{\mathcal{Q}} &= \left[\begin{array}{cc|cc} S_{11} & 0 & S_{12} & 0 \\ 0 & T_{11} & 0 & T_{12} \\ \hline 0 & 0 & S_{11}^T & 0 \\ 0 & 0 & 0 & T_{11}^T \end{array}\right] =: \begin{bmatrix} \tilde{\mathcal{S}}_{11} & \tilde{\mathcal{S}}_{12} \\ 0 & \tilde{\mathcal{S}}_{11}^T \end{bmatrix}, \\ \mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{B}_{\mathcal{H}}^r\tilde{\mathcal{Q}} &= \left[\begin{array}{cc|cc} 0 & H_{11} & 0 & H_{12} \\ -H_{22}^T & 0 & H_{12}^T & 0 \\ \hline 0 & 0 & 0 & H_{22} \\ 0 & 0 & -H_{11}^T & 0 \end{array}\right] =: \begin{bmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} \\ 0 & -\tilde{\mathcal{H}}_{11}^T \end{bmatrix}. \end{aligned} \tag{9}$$

*Odd size pencils can be treated analogously [5].

Note that $\mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{B}_{\mathcal{S}}^r\tilde{\mathcal{Q}}$ and $\mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{B}_{\mathcal{H}}^r\tilde{\mathcal{Q}}$ are Hamiltonian and skew-Hamiltonian respectively, as $\tilde{Q}$ is nonsingular (see above). We now determine orthogonal matrices $\mathcal{Q}_3$ and $\mathcal{Q}_4$ such that $\mathcal{H}_{11} = \mathcal{Q}_4^T\tilde{\mathcal{H}}_{11}\mathcal{Q}_3$, $\mathcal{S}_{11} = \mathcal{Q}_4^T\tilde{\mathcal{S}}_{11}\mathcal{Q}_3$ are quasi upper triangular and upper triangular, respectively. Setting $\mathcal{Q} = \tilde{\mathcal{Q}}\,\mathrm{diag}(\mathcal{Q}_3, \mathcal{Q}_4)$ with $\mathcal{S}_{12} := \mathcal{Q}_4^T\tilde{\mathcal{S}}_{12}\mathcal{Q}_4$ and $\mathcal{H}_{12} := \mathcal{Q}_4^T\tilde{\mathcal{H}}_{12}\mathcal{Q}_4$, we get the structured Schur form

$$\tilde{\mathcal{B}}_{\mathcal{S}}^r := \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{S}}^r\mathcal{Q} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{11}^T \end{bmatrix}, \quad \tilde{\mathcal{B}}_{\mathcal{H}}^r := \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{H}}^r\mathcal{Q} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^T \end{bmatrix}.$$

By a proper reordering of the eigenvalues we can now compute the desired deflating subspaces of $\alpha\mathcal{S} - \beta\mathcal{H}$ due to the following theorem wich is proved in [4].

Theorem 8. *Let $\alpha\mathcal{S} - \beta\mathcal{H}$ be a skew-Hamiltonian/Hamiltonian pencil and consider the extended matrices $\mathcal{B}_{\mathcal{S}} = \mathrm{diag}(\mathcal{S}, \mathcal{S})$ and $\mathcal{B}_{\mathcal{H}} = \mathrm{diag}(\mathcal{H}, -\mathcal{H})$.*

Let $\mathcal{V}, \mathcal{W}$ be orthogonal matrices such that

$$\mathcal{W}^H\mathcal{B}_{\mathcal{S}}\mathcal{V} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{22} \end{bmatrix}, \quad \mathcal{W}^H\mathcal{B}_{\mathcal{H}}\mathcal{V} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & \mathcal{H}_{22} \end{bmatrix}, \tag{10}$$

where $\mathcal{S}_{11}, \mathcal{H}_{11} \in \mathbb{R}^{m,m}$, $\Lambda_-(\mathcal{B}_{\mathcal{S}}, \mathcal{B}_{\mathcal{H}}) \subset \Lambda(\mathcal{S}_{11}, \mathcal{H}_{11})$ and $\Lambda(\mathcal{S}_{11}, \mathcal{H}_{11}) \cap \Lambda_+(\mathcal{B}_{\mathcal{S}}, \mathcal{B}_{\mathcal{H}}) = \emptyset$. Let $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{R}^{2n,m}$ be the first m columns of $\mathcal{V}$, then

$$\begin{aligned} \mathrm{range}\, V_1 &= \mathrm{Def}_-(\mathcal{S}, \mathcal{H}) + \mathbb{L}_1, \quad \mathbb{L}_1 \subseteq \mathrm{Def}_0(\mathcal{S}, \mathcal{H}) + \mathrm{Def}_\infty(\mathcal{S}, \mathcal{H}), \\ \mathrm{range}\, V_2 &= \mathrm{Def}_+(\mathcal{S}, \mathcal{H}) + \mathbb{L}_2, \quad \mathbb{L}_2 \subseteq \mathrm{Def}_0(\mathcal{S}, \mathcal{H}) + \mathrm{Def}_\infty(\mathcal{S}, \mathcal{H}). \end{aligned} \tag{11}$$

If $\Lambda(\mathcal{S}_{11}, \mathcal{H}_{11}) = \Lambda_-(\mathcal{B}_{\mathcal{S}}, \mathcal{B}_{\mathcal{H}})$, and $\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ are the first m columns of $\mathcal{W}$, then there exist orthogonal matrices Q_V, Q_W such that

$$\begin{aligned} V_1 &= [P_V^-, 0]Q_V, W_1 = [P_W^-, 0]Q_W, \\ V_2 &= [0, P_V^+]Q_V, W_2 = [0, P_W^+]Q_W, \end{aligned}$$

and the columns of P_V^- and P_V^+ form orthogonal bases of $\mathrm{Def}_-(\mathcal{S}, \mathcal{H})$ and $\mathrm{Def}_+(\mathcal{S}, \mathcal{H})$, respectively. Moreover, the matrices P_W^-, P_W^+ have orthonormal columns and the following relations are satisfied

$$\begin{aligned} \mathcal{S}P_V^- &= P_W^-\tilde{S}_{11}, \quad \mathcal{H}P_V^- = P_W^-\tilde{H}_{11}, \\ \mathcal{S}P_V^+ &= P_W^+\tilde{S}_{22}, \quad \mathcal{H}P_V^+ = -P_W^+\tilde{H}_{22}. \end{aligned} \tag{12}$$

Here $\tilde{S}_{i,i}$ and $\tilde{H}_{i,i}$ for $i = 1, 2$ are matrices such that $\Lambda(\tilde{S}_{11}, \tilde{H}_{11}) = \Lambda(\tilde{S}_{22}, \tilde{H}_{22}) = \Lambda_-(\mathcal{S}, \mathcal{H})$.

Computing the deflating subspaces in this manner not only has the advantage of preserving the Hamiltonian spectral symmetry, but also evades numerical difficulities of classical methods, which suffer from bad conditioning especially close to the optimal value of γ.

5 Numerical Example

To illustrate the functionality of our approach, consider the following example from [27] which is also discussed in [23–25]. The descriptor system is given by (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = 1, \quad D_{11} = D_{22} = 0.$$

The pencil $\lambda E - A$ is of index 2 and the associated pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ have index 1 for $\gamma \neq 0$. The goal is to find the minimum value γ that satisfies the conditions **C1) – C4)**. Using our experimental code for the structured computation of the deflating subspaces associated with eigenvalues in the closed left half plane and using the Procedure 2 to determine the optimal value for γ, we computed γ_{opt} given by $\gamma^{\rho} = 0.7678$, which is smaller than the sub-optimal values obtained in [23–25, 27].

6 Conclusions

We have developed conditions for optimal and suboptimal $\mathcal{H}_\infty$-control for descriptor systems of arbitrary index and expressed criteria for the existence of an internally stabilizing controller in terms of the deflating subspaces of even pencils. Furthermore we have presented a method to compute these subspaces in a structure preserving way. Combined, this yields a numerically robust γ-iteration for descriptor systems.

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