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**Adaptive Discontinuous Galerkin
Approximation of Optimal Control
Problems Governed by Transient
Convection-Diffusion Equations**



Abstract

In this paper, we investigate an a posteriori error estimate of a control constrained optimal control problem governed by a time-dependent convection diffusion equation. Control constraints are handled by using the primal-dual active set algorithm as a semi-smooth Newton method or by adding a Moreau-Yosida-type penalty function to the cost functional. An adaptive mesh refinement indicated by a posteriori error estimates is applied for both approaches. A symmetric interior penalty Galerkin method in space and a backward Euler in time are applied in order to discretize the optimization problem. Numerical results are presented, which illustrate the performance of the proposed error estimator.

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1 Introduction

Optimal control problems (OCPs) governed by convection diffusion partial differential equations (PDEs) arise in environmental modeling, petroleum reservoir simulation and in many other engineering applications [9, 10, 27]. Efficient numerical methods are essential to successful applications of such optimal control problems.

Several well-established techniques have been proposed to enhance stability and accuracy of the optimal control problems governed by the steady convection diffusion equations, i.e., the streamline upwind/Petrov Galerkin (SUPG) finite element method [8], the local projection stabilization [4], the edge stabilization [18, 35], discontinuous Galerkin methods [22, 36, 37, 38, 39]. Also, only few papers are published so far for unsteady optimal control problems governed by convection diffusion equations, i.e., the characteristic finite element method [11, 12], the streamline upwind/Petrov Galerkin (SUPG) finite element method [31], the local discontinuous Galerkin (LDG) method [41], the nonsymmetric interior penalty Galerkin (NIPG) [33], and the symmetric interior penalty Galerkin (SIPG) method [1].

Adaptive finite element approximations are particularly attractive for the solution of optimal control problems governed by elliptic convection dominated partial differential equations (PDEs), since the solution of the governing state PDE or the solution of the associated adjoint PDE may exhibit boundary and/or interior layers, localized regions where the derivative of the PDE solution is large. It allows local mesh refinement around the layers as needed, thereby achieving a desired residual error bound with as few degrees of freedom as possible. The a posteriori error analysis of the optimal control problems governed by parabolic equations is discussed in [25, 26, 34]. For the optimal control problems governed by time-dependent convection diffusion equations, the a posteriori error analyses are investigated by using a characteristic finite element discretization in [13] and by using the edge stabilization in [40].

We here will derive an a posteriori error analysis of the optimal control problems governed by the transient convection diffusion equations using the discontinuous Galerkin method in space and the backward Euler method in time. We apply discontinuous Galerkin (DG) discretization for convection dominated optimal control problems due to their better convergence behavior, local mass conservation, flexibility in approximating rough solutions on complicated meshes and mesh adaptation. We would like to refer to [3, 17, 30] for the discontinuous Galerkin methods in details. To solve the optimization problem, we use both the primal-dual active set strategy and the Moreau-Yosida regularization. Suitable error estimators are introduced for both cases. However, the a posteriori error analysis of the Moreau-Yosida regularized optimization problem depends on the regularization parameter δ . Therefore, we formally assume that the Moreau-Yosida regularization parameter to be fixed in advance as done in [15, 36].

The rest of the paper is organized as follows: in the next section, we introduce control constrained optimal control problems governed by transient convection diffusion equation. We apply the symmetric interior penalty Galerkin (SIPG) method for the diffusion and the upwind discretization for the convection in order to discretize the optimization problem in space. The primal-dual active set strategy as a semi-smooth Newton method is also introduced to solve the optimality system. The error estimator of the primal-dual active set approach and the reliability of the error estimator are derived in Section 3. The other approach to solve the control constrained optimal control problem, the Moreau-Yosida regularization, is given in

Section 4. Section 5 contains the numerical experiments to illustrate the performance of the proposed error estimators.

2 Approximation schemes for the optimal control problem

In this section, we introduce the discontinuous Galerkin finite element discretization in space and the backward Euler discretization in time for the approximations of the distributed linear-quadratic optimal control problems governed by unsteady convection diffusion PDEs.

We adopt the standard notations for Sobolev spaces on computational domains and their norms. Ω and Ω_U are bounded open sets in \mathbb{R}^2 with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$, respectively. Although adaptive finite element methods provide a real benefit on non-convex domains, for example such with reentrant corners in practical applications, we assume that Ω and Ω_U are convex polygons for simplicity. The inner products in $L^2(\Omega_U)$ and $L^2(\Omega)$ are denoted by $(\cdot, \cdot)_U$ and (\cdot, \cdot) , respectively. Here, $a \lesssim b$ means that $a \leq Cb$ for some positive constant C . Further, we consider the Bochner spaces of functions mapping the time interval $(0, T)$ to a Bänsch space V in which the norm $\|\cdot\|_V$ is defined. For $r \geq 1$, we define

$$L^r(0, T; V) = \{z : [0, T] \rightarrow V \text{ measurable} : \int_0^T \|z(\cdot)\|_V^r dt < \infty\}$$

with

$$\|z(\cdot)\|_{L^r(0, T; V)} = \begin{cases} \left(\int_0^T \|z(\cdot)\|_V^r dt \right)^{1/r}, & \text{if } 1 \leq r < \infty, \\ \text{ess sup}_{t \in (0, T]} \|z(\cdot)\|_V, & \text{if } r = \infty. \end{cases}$$

In this paper, we shall take the state space $W = L^2(0, T; V)$ with $V = H_0^1(\Omega)$, and the control space $X = L^2(0, T; U)$ with $U = L^2(\Omega_U)$. We are interested in the following distributed optimal control problem governed by a transient convection diffusion equation:

$$\min_{u \in U_{ad} \subseteq X} J(y, u) := \int_0^T \left(\frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|u - u_d\|_{L^2(\Omega_U)}^2 \right) dt, \quad (1)$$

subject to

$$\partial_t y - \varepsilon \Delta y + \beta \cdot \nabla y = f + Bu, \quad x \in \Omega, \quad t \in (0, T], \quad (2a)$$

$$y(x, t) = 0, \quad x \in \partial\Omega, \quad t \in (0, T], \quad (2b)$$

$$y(x, 0) = y_0(x), \quad x \in \Omega, \quad (2c)$$

where the closed convex admissible set of control constraints is given by

$$U_{ad} = \{u \in X : u_a \leq u \leq u_b, \text{ a.e. in } \Omega_U \times (0, T]\} \quad (3)$$

with the constant bounds, $u_a \leq u_b$. The function u_d , called desired control, is a guideline for the control, see, e.g., [7]. Note that this formulation also allows for the special (and most common case) $u_d = 0$, i.e. there is no a priori information on the optimal control. B is a

linear continuous operator from X to $L^2(0, T; V')$ realizing the transition between Ω_U and Ω . Generally, Ω_U can be a subset of Ω . In the special case $\Omega_U = \Omega$, $B = I$ is the identity operator on $L^2(\Omega)$.

We make the following assumptions for the functions and parameters in the optimal control problem (1)-(3):

(i) The source function f , the desired state y_d , and the desired control u_d satisfy the following regularity:

$$f, y_d \in L^2(0, T; L^2(\Omega)) \quad \text{and} \quad u_d \in L^2(0, T; U).$$

(ii) The initial condition is defined as $y_0(x) \in V = H_0^1(\Omega)$.

(iii) β denotes a velocity field. It belongs to $(W^{1, \infty}(\Omega))^2$ and satisfies the incompressibility condition, i.e. $\nabla \cdot \beta = 0$. The diffusion parameter ε is also taken as $0 < \varepsilon \ll 1$.

Using the assumptions defined above, the following result on regularity of the state solution can be stated.

Proposition 2.1 ([24]) *Under the assumptions defined above and for a given control $u \in L^2(0, T; L^2(\Omega_U))$, the state y satisfies the following regularity condition*

$$y \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

and the weak formulation

$$\begin{aligned} (\partial_t y(u), v) + a(y, v) &= (f + Bu, v) \quad \forall v \in V, \\ y(x, 0) &= y_0, \end{aligned} \quad (4)$$

where the (bi)-linear forms are defined by

$$a(y, v) = \int_{\Omega} (\varepsilon \nabla y \cdot \nabla v + \beta \cdot \nabla y v) dx, \quad (f, v) = \int_{\Omega} f v dx.$$

Then, the variational formulation corresponding to (1)-(3) can be written as

$$\min_{u \in U_{ad}} J(y, u) := \int_0^T \left(\frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|u - u_d\|_{L^2(\Omega_U)}^2 \right) dt \quad (5a)$$

$$\text{subject to } (\partial_t y, v) + a(y, v) = (f + Bu, v) \quad \forall v \in V, \quad t \in (0, T], \quad (5b)$$

$$y(x, 0) = y_0,$$

$$(y, u) \in H^1(0, T; L^2(\Omega)) \cap W \times U_{ad}.$$

It can be derived by the standard techniques (see, e.g., [14] and [23]) that the control problem (5) has a unique solution (y, u) , and that (y, u) is the solution of (5) if and only if there exists an adjoint $p \in H^1(0, T; L^2(\Omega)) \cap W$ such that (y, u, p) satisfies the following optimality system for $t \in (0, T]$

$$(\partial_t y, v) + a(y, v) = (f + Bu, v) \quad \forall v \in V, \quad y(x, 0) = y_0, \quad (6a)$$

$$-(\partial_t y, \psi) + a(\psi, p) = (y - y_d, \psi) \quad \forall \psi \in V, \quad p(x, T) = 0, \quad (6b)$$

$$\int_0^T (\alpha(u - u_d) + B^* p, w - u)_{U} dt \geq 0 \quad \forall w \in U_{ad}, \quad (6c)$$

where B^* denotes the adjoint of B . From the second equation (6b), we deduce that the adjoint p satisfies the following transient convection diffusion equation:

$$-\partial_t p - \varepsilon \Delta p - \boldsymbol{\beta} \cdot \nabla y = y - y_d \quad x \in \Omega, \quad t \in (0, T], \quad (7a)$$

$$p(x, t) = 0 \quad x \in \partial\Omega, \quad t \in (0, T], \quad (7b)$$

$$p(x, T) = 0 \quad x \in \Omega. \quad (7c)$$

2.1 Discontinuous Galerkin (DG) scheme

In the following, we construct the discontinuous Galerkin finite element scheme for the state equation (2).

Let $\{\mathcal{T}_h\}_h$ be a family of shape-regular simplicial triangulations of Ω . Each mesh \mathcal{T}_h consists of closed triangles such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} \bar{K}$ holds. We assume that the mesh is regular in the following sense: for different triangles $K_i, K_j \in \mathcal{T}_h$, $i \neq j$, the intersection $K_i \cap K_j$ is either empty or a vertex or an edge, i.e., hanging nodes are not allowed. The diameter of an element K and the length of an edge E are denoted by h_K and h_E , respectively. Further, the maximum value of the element diameter is denoted by $h = \max_{K \in \mathcal{T}_h} h_K$.

We split the set of all edges \mathcal{E}_h into the set \mathcal{E}_h^0 of interior edges and the set \mathcal{E}_h^∂ of boundary edges so that $\mathcal{E}_h = \mathcal{E}_h^\partial \cup \mathcal{E}_h^0$. Let \mathbf{n} denote the unit outward normal to $\partial\Omega$. The inflow and outflow parts of $\partial\Omega$ are denoted by Γ^- and Γ^+ , respectively,

$$\Gamma^- = \{x \in \partial\Omega : \boldsymbol{\beta} \cdot \mathbf{n}(x) < 0\}, \quad \Gamma^+ = \{x \in \partial\Omega : \boldsymbol{\beta} \cdot \mathbf{n}(x) \geq 0\}.$$

Similarly, the inflow and outflow boundaries of an element K are defined by

$$\partial K^- = \{x \in \partial K : \boldsymbol{\beta} \cdot \mathbf{n}_K(x) < 0\}, \quad \partial K^+ = \{x \in \partial K : \boldsymbol{\beta} \cdot \mathbf{n}_K(x) \geq 0\},$$

where \mathbf{n}_K is the unit normal vector on the boundary ∂K of an element K .

Let the edge E be a common edge for two elements K and K^e . For a piecewise continuous scalar function y , there are two traces of y along E , denoted by $y|_E$ from inside K and $y^e|_E$ from inside K^e . The jump and average of y across the edge E are defined by:

$$[[y]] = y|_E \mathbf{n}_K + y^e|_E \mathbf{n}_{K^e}, \quad \{\{y\}\} = \frac{1}{2}(y|_E + y^e|_E). \quad (8)$$

Similarly, for a piecewise continuous vector field ∇y , the jump and average across an edge E are given by

$$[[\nabla y]] = \nabla y|_E \cdot \mathbf{n}_K + \nabla y^e|_E \cdot \mathbf{n}_{K^e}, \quad \{\{\nabla y\}\} = \frac{1}{2}(\nabla y|_E + \nabla y^e|_E). \quad (9)$$

For a boundary edge $E \in K \cap \Gamma$, we set $\{\{\nabla y\}\} = \nabla y$ and $[[y]] = y\mathbf{n}$, where \mathbf{n} is the outward normal unit vector on Γ .

We only consider discontinuous piecewise linear finite element spaces to define the discrete spaces of the state and test functions

$$V_h = W_h = \{y \in L^2(\Omega) : y|_K \in \mathbb{P}^1(K) \quad \forall K \in \mathcal{T}_h\}. \quad (10)$$

Remark 2.2 *When the state equation (2) contains nonhomogeneous Dirichlet boundary conditions, the space of discrete states W_h and the space of test functions V_h can still be taken to be the same due to the weak treatment of boundary conditions in DG methods.*

We now consider the discretization of the control variable. Let $\{\mathcal{T}_h^U\}_h$ be also a family of shape-regular simplicial triangulations of Ω_U such that $\overline{\Omega_U} = \bigcup_{K_U \in \mathcal{T}_h^U} \overline{K_U}$ holds. For $K_U^i, K_U^j \in \mathcal{T}_h^U, i \neq j$, the intersection $K_U^i \cap K_U^j$ is either empty or a vertex or an edge. The maximum diameter is defined by $h_U = \max_{K_U \in \mathcal{T}_h^U} h_{K_U}$, where h_{K_U} denotes the diameter of an element K_U . The discrete space of the control variable associated with $\{\mathcal{T}_h^U\}_h$ is also a discontinuous piecewise linear finite element space

$$X_h = \{u \in L^2(\Omega_U) : u|_{K_U} \in \mathbb{P}^1(K_U) \quad \forall K_U \in \mathcal{T}_h^U\}. \quad (11)$$

We can now give the DG discretizations of the state equation (2) in space for a fixed control u . The DG method proposed here is based on the upwind discretization of the convection term and on the SIPG discretization of the diffusion term. Recall that in discontinuous Galerkin methods we do not explicitly impose continuity constraints on the trial and test functions across the element interfaces. As a consequence, weak formulations include jump terms across interfaces, and penalty terms are typically added to control the jump terms. We refer to [3, 30] for a rigorous derivation of the following (bi-)linear forms applied to $y_h \in H^1(0, T; W_h)$ for a fixed control u_h and $\forall t \in (0, T]$:

$$(\partial_t y_h, v_h) + a_h(y_h, v_h) = (f + Bu_h, v_h) \quad \forall v_h \in V_h, \quad (12)$$

where

$$\begin{aligned} a_h(y, v) = & \sum_{K \in \mathcal{T}_h} \int_K \varepsilon \nabla y \cdot \nabla v \, dx - \sum_{E \in \mathcal{E}_h} \int \left(\{\{\varepsilon \nabla y\}\} \cdot \llbracket v \rrbracket + \{\{\varepsilon \nabla v\}\} \cdot \llbracket y \rrbracket \right) ds + \sum_{E \in \mathcal{E}_h} \frac{\sigma \varepsilon}{h_E} \int \llbracket y \rrbracket \cdot \llbracket v \rrbracket \, ds \\ & + \sum_{K \in \mathcal{T}_h} \int_K \beta \cdot \nabla y v \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K^- \setminus \Gamma} \beta \cdot \mathbf{n} (y^e - y) v \, ds - \sum_{K \in \mathcal{T}_h} \int_{\partial K^- \cap \Gamma^-} \beta \cdot \mathbf{n} y v \, ds \end{aligned} \quad (13)$$

with the nonnegative real parameter σ being called the penalty parameter. We choose σ to be sufficiently large, independent of the mesh size h and the diffusion coefficient ε to ensure the stability of the DG discretization as described in [30, Sec. 2.7.1].

2.2 Primal-dual active set (PDAS) strategy

We here explain our first approach to solve the control constrained optimal control problem (1)-(3), called the primal-dual active set (PDAS) strategy introduced in [5]. We first define the semi-discrete approximation of the optimal control problem (5) as follows:

$$\min_{u_h \in U_h^{ad}} \int_0^T \left(\frac{1}{2} \sum_{K \in \mathcal{T}_h} \|y_h - y_d\|_{L^2(K)}^2 + \frac{\alpha}{2} \sum_{K_U \in \mathcal{T}_h^U} \|u_h - u_d\|_{L^2(K_U)}^2 \right) dt, \quad (14a)$$

$$\begin{aligned} \text{subject to } & (\partial_t y_h, v_h) + a_h(y_h, v_h) = (f + Bu_h, v_h) \quad \forall v_h \in V_h, \quad t \in (0, T], \\ & y_h(x, 0) = y_h^0(x), \\ & (y_h, u_h) \in H^1(0, T; W_h) \times U_h^{ad}, \end{aligned} \quad (14b)$$

where

$$U_h^{ad} = \{u_h \in L^2(0, T; X_h) : u_a \leq u_h \leq u_b \text{ a.e. in } \Omega_U \times (0, T]\} \quad (14c)$$

is a closed convex set in $L^2(0, T; X_h)$. For ease of exposition, we also assume $U_h^{ad} \subset U_{ad} \cap L^2(0, T; X_h)$.

Let $J(\cdot)$ be a continuous functional in $L^2(\Omega)$. Then, there exists a least one solution for the optimization problem (14) since the discrete state $y(u_h)$ can be bounded in the given norm as shown in [1, 33]. Then, it follows that the control problem (14) has a unique solution $(y_h, u_h) \in H^1(0, T; W_h) \times U_h^{ad}$ (see, e.g., [23]) and that a pair (y_h, u_h) is the solution of (14) if and only if there is an adjoint $p_h \in H^1(0, T; W_h)$ such that the triple (y_h, u_h, p_h) satisfies the following optimality system:

$$(\partial_t y_h, v_h) + a_h(y_h, v_h) = (f + Bu_h, v_h) \quad \forall v_h \in V_h, \quad (15a)$$

$$y_h(x, 0) = y_h^0,$$

$$-(\partial_t p_h, \Psi_h) + a_h(\Psi_h, p_h) = (y_h - y_d, \Psi_h) \quad \forall \Psi_h \in V_h, \quad (15b)$$

$$p_h(x, T) = 0,$$

$$\int_0^T (\alpha(u_h + u_d) + B^* p_h, w_h - u_h)_U dt \geq 0 \quad \forall w_h \in U_h^{ad}. \quad (15c)$$

We now consider the fully-discrete approximation for the optimal control problem (1)-(3) using the standard backward Euler scheme in time and the discontinuous Galerkin discretization in space.

Let N_T be a positive integer. The discrete time interval $\bar{I} = [0, T]$ is defined as

$$0 = t_0 < t_1 < \dots < t_{N_T-1} < t_{N_T} = T$$

with size $k_n = t_n - t_{n-1}$ for $n = 1, \dots, N_T$ and $k = \max_{n=1, \dots, N_T} k_n$.

Then, the fully-discrete approximation scheme of the semi-discrete problem (14) is

$$\min_{u_{h,n} \in U_{h,n}^{ad}} \sum_{n=1}^{N_T} k_n \left(\frac{1}{2} \sum_{K \in \mathcal{T}_h} \|y_{h,n} - y_n^d\|_{L^2(K)}^2 + \frac{\alpha}{2} \sum_{K_U \in \mathcal{T}_h^U} \|u_{h,n} - u_n^d\|_{L^2(K_U)}^2 \right), \quad (16a)$$

subject to

$$\left(\frac{y_{h,n} - y_{h,n-1}}{k_n}, v \right) + a_h(y_{h,n}, v) = (f_n + Bu_{h,n}, v) \quad \forall v \in V_h, \quad (16b)$$

$$y_{h,0}(x, 0) = y_h^0(x),$$

where

$$U_{h,n}^{ad} = \{u_{h,n} \in X_h : u_a \leq u_{h,n} \leq u_b \text{ a.e. in } \Omega_U\} \quad \text{for } n = 1, 2, \dots, N_T. \quad (16c)$$

The fully discretized minimization problem (16) has at least one solution due to the boundedness of the solution as shown in [1, Lemma 6]. Then, the fully discretized control problem (16) has a unique solution $(Y_{h,n}, U_{h,n}) \in W_h \times U_h^{ad}$, $n = 1, 2, \dots, N_T$, and $(Y_{h,n}, U_{h,n})$, $n = 1, 2, \dots, N_T$ is the solution of (16) if and only if there is an adjoint $P_{h,n-1} \in V_h$, $i = 1, 2, \dots, N_T$, and such that $(Y_{h,n}, U_{h,n}, P_{h,n-1}) \in W_h \times U_h^{ad} \times V_h$ satisfies the following optimality system:

$$\left(\frac{Y_{h,n} - Y_{h,n-1}}{k_n}, v\right) + a_h(Y_{h,n}, v) = (f_n + BU_{h,n}, v) \quad \forall v \in V_h, \quad (17a)$$

$$Y_{h,0} = y_h^0 \quad n = 1, 2, \dots, N_T,$$

$$\left(\frac{P_{h,n-1} - P_{h,n}}{k_n}, q\right) + a_h(q, P_{h,n-1}) = (Y_{h,n} - y_n^d, q) \quad \forall q \in V_h, \quad (17b)$$

$$P_{h,T} = 0 \quad n = N_T, \dots, 2, 1,$$

$$\left(\alpha(U_{h,n} - u_n^d) + B^*P_{h,n-1}, w - U_{h,n}\right)_U \geq 0 \quad \forall w \in U_{h,n}^{ad}, \quad n = 1, 2, \dots, N_T. \quad (17c)$$

By following the strategy introduced in [26], we define for $n = 1, 2, \dots, N_T$

$$Y_h|_{(t_{n-1}, t_n]} = ((t_n - t)Y_{h,n-1} + (t - t_{n-1})Y_{h,n})/k_i, \quad (18a)$$

$$P_h|_{(t_{n-1}, t_n]} = ((t_n - t)P_{h,n-1} + (t - t_{n-1})P_{h,n})/k_i, \quad (18b)$$

$$U_h|_{(t_{n-1}, t_n]} = U_{h,n}. \quad (18c)$$

Let $\widehat{w}(x, t)|_{t \in (t_{n-1}, t_n]} = w(x, t_n)$ and $\widetilde{w}(x, t)|_{t \in (t_{n-1}, t_n]} = w(x, t_{n-1})$ for any function $w \in C(0, T; L^2(\Omega))$. Then, the optimality system (17) can be restated as

$$\left(\frac{\partial Y_h}{\partial t}, v\right) + a_h(\widehat{Y}_h, v) = (\widehat{f} + BU_h, v) \quad \forall v \in V_h, \quad (19a)$$

$$Y_h(x, 0) = y_h^0(x) \quad n = 1, 2, \dots, N_T,$$

$$-\left(\frac{\partial P_h}{\partial t}, q\right) + a_h(q, \widetilde{P}_h) = (\widehat{Y}_h - \widehat{y}_d, q) \quad \forall q \in V_h, \quad (19b)$$

$$P_h(x, T) = 0 \quad n = N_T, \dots, 2, 1,$$

$$\left(\alpha(U_h - \widehat{u}_d) + B^*\widetilde{P}_h, w - U_h\right)_U \geq 0 \quad \forall w \in U_{h,n}^{ad}, \quad n = 1, 2, \dots, N_T. \quad (19c)$$

We solve the optimality system (17) by using the primal-dual active set (PDAS) algorithm as a semi-smooth Newton method [5]. To use this approach, we first need to define the active sets

$$\mathcal{A}_-^n = \{x \in \Omega : -P_{h,n-1} - \alpha(u_a - u_n^d) < 0\},$$

$$\mathcal{A}_+^n = \{x \in \Omega : -P_{h,n-1} - \alpha(u_b - u_n^d) > 0\},$$

and the inactive set $I^n = \Omega \setminus (\mathcal{A}_-^n \cup \mathcal{A}_+^n)$ for each time step t_n . For $n = 1, 2, \dots, N_T$, the discretized optimality system (17) is equivalent to

$$(\mathcal{M} + k_n \mathcal{K})Y_n - \mathcal{M}Y_{n-1} = \ell(f_n) + \mathcal{M}U_n, \quad (20a)$$

$$(k_n \mathcal{K}^T + \mathcal{M})P_{n-1} - \mathcal{M}P_n = \mathcal{M}Y_n - \ell(y_n^d), \quad (20b)$$

$$\alpha \mathcal{M}U_n - \alpha \chi_{I^n} \ell(u_n^d) + \mathcal{M} \chi_{I^{n-1}} P_{n-1} = \alpha \mathcal{M} \left(\chi_{\mathcal{A}_-^n} u_a + \chi_{\mathcal{A}_+^n} u_b \right), \quad (20c)$$

where \mathcal{K} is the stiffness matrix corresponding to $a_h(\cdot, \cdot)$ and \mathcal{M} is the mass matrix. $\chi_{\mathcal{A}_-^n}$, $\chi_{\mathcal{A}_+^n}$, and χ_{I^n} denote the characteristic functions of \mathcal{A}_-^n , \mathcal{A}_+^n , and I^n , respectively. Further, $\ell(z) = \int_{\Omega} z v \, dx$ with $v \in V_h$. By considering all time steps, we then apply the active set algorithm described in Algorithm 1 for the iteration number k .

Algorithm 1 Active set algorithm

Choose initial values for $y^{(0)}, u^{(0)}$, and $p^{(0)}$.
Set the active sets $\mathcal{A}_-^{(0)}, \mathcal{A}_+^{(0)}$ and inactive set $I^{(0)}$.
for $k = 1, 2, \dots$ **do**
 Solve (20) and update active sets $\mathcal{A}_-^{(k)}, \mathcal{A}_+^{(k)}$ and inactive set $I^{(k)}$ for all time steps.
 if $\mathcal{A}_-^{(k)} = \mathcal{A}_-^{(k+1)}, \mathcal{A}_+^{(k)} = \mathcal{A}_+^{(k+1)}$, and $I^{(k)} = I^{(k+1)}$ **then**
 STOP
 end if
end for

3 A posteriori error estimates

We here analyse the a posteriori error estimates of the optimal control problem governed by transient convection diffusion equations discretized by the discontinuous Galerkin scheme in space and the backward Euler scheme in time. In general, a posteriori error analysis of the unsteady optimal control problems is more complicated than the ones of the steady optimal control problems due to the fact that the properties of the time variable and its discretization are quite different from those of the space variables. Thus, different approaches are needed to handle the two groups of variables, and their interactions.

To derive a sharp estimator for the control, we divide Ω_U for each time $n = 1, 2, \dots, N_T$ as follows:

$$\begin{aligned}\Omega_U^{n,a} &= \{x \in \Omega_U : (B^*P_h)(x, t_{n-1}) > \alpha(u_d(x, t_n) - u_a), U_{h,n} = u_a\}, \\ \Omega_U^{n,a+} &= \{x \in \Omega_U : (B^*P_h)(x, t_{n-1}) > \alpha(u_d(x, t_n) - u_a), U_{h,n} > u_a\}, \\ \Omega_U^{n,b} &= \{x \in \Omega_U : (B^*P_h)(x, t_{n-1}) < \alpha(u_d(x, t_n) - u_b), U_{h,n} = u_b\}, \\ \Omega_U^{n,b-} &= \{x \in \Omega_U : (B^*P_h)(x, t_{n-1}) < \alpha(u_d(x, t_n) - u_b), U_{h,n} < u_b\}, \\ \Omega_U^{n,0} &= \{x \in \Omega_U : \alpha(u_d(x, t_n) - u_b) \leq (B^*P_h)(x, t_{n-1}) \leq \alpha(u_d(x, t_n) - u_a)\}.\end{aligned}\tag{21}$$

It is assumed that the intersection of the above sets is empty, i.e., $\Omega_U^{n,i} \cap \Omega_U^{n,j} = \emptyset$ for $i \neq j$, $i, j \in \{0, a, a+, b, b-\}$ and

$$\Omega_U(t_n) = \Omega_U^{n,0}(t_n) \cup \Omega_U^{n,a}(t_n) \cup \Omega_U^{n,a+}(t_n) \cup \Omega_U^{n,b}(t_n) \cup \Omega_U^{n,b-}(t_n).$$

To ease the notation, we define

$$\Omega_U^{n,*} = \Omega_U^{n,0} \cup \Omega_U^{n,a+} \cup \Omega_U^{n,b-}.$$

In the following lemma, we derive an estimate of the control variable in the optimization problem (1)-(3) by making a connection with the adjoint variable.

Lemma 3.1 *Let (y, u, p) and (Y_h, U_h, P_h) be the solutions of (6) and (17), respectively. Then, we have the following estimate*

$$\|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2 \leq C \left(\mathfrak{n}_u^2 + \|\tilde{P}_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 \right),\tag{22}$$

where

$$\eta_u = \sum_{n=1}^{N_T} \int_{t_{n-1}}^{t_n} \int_{\Omega_U^{n,*}} (\alpha(U_h - \hat{u}_d) + B^* \tilde{P}_h) dx dt + \|\alpha(u_d - \hat{u}_d)\|_{L^2(0,T;L^2(\Omega_U))}^2, \quad (23)$$

and the auxiliary solutions, i.e., $y(U_h), p(U_h) \in H^1(0, T; L^2(\Omega)) \cap W$, are defined as follows:

$$\left(\frac{\partial}{\partial t} y(U_h), w \right) + a(y(U_h), w) = (f + BU_h, w), \quad \forall w \in V, \quad (24a)$$

$$y(U_h)(x, t)|_{\partial\Omega} = 0, \quad y(U_h)(x, 0) = y_0(x), \quad x \in \Omega,$$

$$- \left(\frac{\partial}{\partial t} p(U_h), q \right) + a(q, p(U_h)) = (y(U_h) - y_d, q), \quad \forall q \in V, \quad (24b)$$

$$p(U_h)(x, t)|_{\partial\Omega} = 0, \quad p(U_h)(x, T) = 0, \quad x \in \Omega.$$

Proof. The inequality (6c) gives us

$$\begin{aligned} \alpha \|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2 &= \int_0^T (\alpha u, u - U_h)_U dt - \int_0^T (\alpha U_h, u - U_h)_U dt \\ &\leq \int_0^T (\alpha u_d - B^* p, u - U_h)_U dt - \int_0^T (\alpha U_h, u - U_h)_U dt \\ &= \underbrace{\int_0^T (\alpha(U_h - \hat{u}_d) + B^* \tilde{P}_h, U_h - u)_U dt}_{M_1} + \underbrace{\int_0^T (\alpha(u_d - \hat{u}_d), u - U_h)_U dt}_{M_2} \\ &\quad + \underbrace{\int_0^T (B^*(\tilde{P}_h - p(U_h)), u - U_h)_U dt}_{M_3} + \underbrace{\int_0^T (B^*(p(U_h) - p), u - U_h)_U dt}_{M_4}. \end{aligned} \quad (25)$$

We first derive an estimate of M_1 for any $t \in (t_{i-1}, t_i]$,

$$\begin{aligned} (\alpha(U_h - \hat{u}_d) + B^* \tilde{P}_h, U_h - u)_U &= \int_{\Omega_U^{n,*}} (\alpha(U_h - \hat{u}_d) + B^* \tilde{P}_h)(U_h - u) dx \\ &\quad + \int_{\Omega_U^{n,a} \cup \Omega_U^{n,b}} (\alpha(U_h - \hat{u}_d) + B^* \tilde{P}_h)(U_h - u) dx. \end{aligned} \quad (26)$$

By the definitions of $\Omega_U^{n,a}$ and $\Omega_U^{n,b}$ in (21), we have

$$\begin{aligned} &\int_{\Omega_U^{n,a} \cup \Omega_U^{n,b}} (\alpha(U_h - \hat{u}_d) + B^* \tilde{P}_h)(U_h - u) dx \\ &= \int_{\Omega_U^{n,a}} \underbrace{(\alpha(u_a - \hat{u}_d) + B^* \tilde{P}_h)}_{>0} \underbrace{(u_a - u)}_{\leq 0} dx + \int_{\Omega_U^{n,b}} \underbrace{(\alpha(u_b - \hat{u}_d) + B^* \tilde{P}_h)}_{<0} \underbrace{(u_b - u)}_{\geq 0} dx \\ &\leq 0. \end{aligned}$$

Then, with the help of Young's inequality, we

$$\begin{aligned}
M_1 &= \int_0^T (\alpha(U_h - \hat{u}_d) + B^* \tilde{P}_h, U_h - u)_{\Omega_U^{n,*}} dt \\
&\lesssim \sum_{n=1}^{N_T} \int_{t_{n-1}}^{t_n} \|(\alpha(U_h - \hat{u}_d) + B^* \tilde{P}_h)\|_{L^2(\Omega_U^{n,*})}^2 dt + \sum_{n=1}^{N_T} \int_{t_{n-1}}^{t_n} \|u - U_h\|_{L^2(\Omega_U^{n,*})}^2 dt \\
&\lesssim \eta_u^2 + \|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2.
\end{aligned} \tag{27}$$

Next, we estimate M_2 and M_3 by invoking again Young's inequality

$$\begin{aligned}
M_2 &= \int_0^T (\alpha(u_d - \hat{u}_d), u - U_h)_U dt \\
&\lesssim \|\alpha(u_d - \hat{u}_d)\|_{L^2(0,T;L^2(\Omega_U))}^2 + \|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2,
\end{aligned} \tag{28}$$

$$\begin{aligned}
M_3 &= \int_0^T (B^*(\tilde{P}_h - p(U_h)), u - U_h)_U dt \\
&\lesssim \int_0^T \|B^*(\tilde{P}_h - p(U_h))\|_{L^2(\Omega_U)}^2 dt + \int_0^T \|u - U_h\|_{L^2(\Omega_U)}^2 dt \\
&\lesssim \|\tilde{P}_h - p(U_h)\|_{L^2(0,T;L^2(\Omega_U))}^2 + \|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2.
\end{aligned} \tag{29}$$

Finally, the auxiliary equations in (24) yield

$$\begin{aligned}
M_4 &= \int_0^T (p(U_h) - p, B(u - U_h))_U dt \\
&= \int_0^T (\partial_t(y - y(U_h)), p(U_h) - p) dt + \int_0^T (a(y - y(U_h)), p(U_h) - p) dt \\
&= \int_0^T (\partial_t(y - y(U_h)), p(U_h) - p) dt + \int_0^T (\partial_t(p(U_h) - p), y - y(U_h)) dt \\
&\quad + \int_0^T (y(U_h) - y, y - y(U_h)) dt.
\end{aligned}$$

Application of integration by parts on time derivatives by using the fact $(y - y(U_h))|_{t=0} = 0$ and $(p(U_h) - p)|_{t=T} = 0$ yields

$$M_4 = \int_0^T (y(U_h) - y, y - y(U_h)) dt \leq 0. \tag{30}$$

By inserting the estimates (27-30) of $M_1 - M_4$ into (25), we obtain the desired result. \square

Before deriving error estimates for the state and adjoint equations, we need the following result for the Lagrange interpolation operator Π_h , and the trace inequality.

Lemma 3.2 ([6]) *Let Π_h be the standard Lagrange interpolation operator. For $m = 0, 1$, $q > 1$ and $v \in W^{2,q}(\Omega)$, there exists a positive constant C such that*

$$|v - \Pi_h v|_{W^{m,q}(\Omega)} \leq Ch^{2-m} |v|_{W^{2,q}(\Omega)}.$$

Lemma 3.3 ([21]) For all $v \in W^{1,q}(\Omega)$, $1 \leq q < \infty$,

$$\|v\|_{W^{0,q}(\partial K)} \leq C \left(h_K^{-1/q} \|v\|_{W^{0,q}(K)} + h_K^{1-1/q} |v|_{W^{1,q}(K)} \right).$$

We have the following inequalities, derived in [2],

$$\|v\|_{L^2(E)}^2 \leq Ch_E^{-1} \|v\|_{L^2(K)}^2, \quad \|\mathbf{n}_E \cdot \nabla v\|_{L^2(E)}^2 \leq Ch_E^{-1} \|\nabla v\|_{L^2(K)}^2, \quad (31)$$

where the constant C depends on the shape regularity of the mesh. Then, the above inequalities yield the following estimation

$$\sum_{E \in \mathcal{E}_h} |h_E| \|\{\{\nabla v\}\}\|_{L^2(E)}^2 \leq C \sum_{K \in \mathcal{T}_h} \|\nabla v\|_{L^2(K)}^2, \quad \forall v \in V_h. \quad (32)$$

We finally define the following stability results derived in [19] for convection diffusion equations.

Lemma 3.4 ([19]) Assume that Ω is a convex domain. Let ϕ and ψ be the solutions of the dual problems (33) and (34), respectively. Then, for given $F \in L^2(0, T; L^2(\Omega))$

$$\begin{aligned} \|v\|_{L^\infty(0, T; L^2(\Omega))} &\lesssim \|F\|_{L^2(0, T; L^2(\Omega))}, \\ \|\nabla v\|_{L^2(0, T; L^2(\Omega))} &\lesssim \|F\|_{L^2(0, T; L^2(\Omega))}, \\ \|\Delta v\|_{L^2(0, T; L^2(\Omega))} &\lesssim \|F\|_{L^2(0, T; L^2(\Omega))}, \\ \|v_t\|_{L^2(0, T; L^2(\Omega))} &\lesssim \|F\|_{L^2(0, T; L^2(\Omega))}, \end{aligned}$$

where $v \in \{\phi, \psi\}$ satisfies

$$\begin{aligned} \phi_t - \varepsilon \Delta \phi + \beta \cdot \nabla \phi &= F, & (x, t) \in \Omega \times (0, T], \\ \phi(x, t) &= 0, & (x, t) \in \partial\Omega \times [0, T], \\ \phi(x, 0) &= 0, & x \in \Omega. \end{aligned} \quad (33)$$

or

$$\begin{aligned} -\psi_t - \varepsilon \Delta \psi - \beta \cdot \nabla \psi &= F, & (x, t) \in \Omega \times (0, T], \\ \psi(x, t) &= 0, & (x, t) \in \partial\Omega \times [0, T], \\ \psi(x, 0) &= 0, & x \in \Omega. \end{aligned} \quad (34)$$

Now, we turn to estimate the error $\|P_h - p(U_h)\|_{L^2(0, T; L^2(\Omega))}^2$.

Lemma 3.5 Let (y, u, p) and (Y_h, U_h, P_h) be the solutions of (6) and (17), respectively. The auxiliary solutions $y(U_h)$ and $p(U_h)$ are defined by the system (24). Assume that Ω is a convex domain, then,

$$\|P_h - p(U_h)\|_{L^2(0, T; L^2(\Omega))}^2 \lesssim \|Y_h - y(U_h)\|_{L^2(0, T; L^2(\Omega))}^2 + \sum_{i=1}^7 \eta_i^2,$$

where

$$\begin{aligned}
\eta_1^2 &= \int_0^T \sum_{K \in \mathcal{T}_h} h_K^4 \int_K \left(\widehat{Y}_h - \widehat{y}_d + \frac{\partial P_h}{\partial t} + \varepsilon \Delta \widetilde{P}_h + \boldsymbol{\beta}_h \cdot \nabla \widetilde{P}_h \right)^2 dx dt, \\
\eta_2^2 &= \int_0^T \sum_{E \in \mathcal{E}_h} h_E^3 \int_E \left[\left[\varepsilon \nabla \widetilde{P}_h \right] \right]^2 ds dt, \\
\eta_3^2 &= \|Y_h - \widehat{Y}_h\|_{L^2(0,T;L^2(\Omega))}^2 + \|\widehat{y}_d - y_d\|_{L^2(0,T;L^2(\Omega))}^2, \\
\eta_4^2 &= \int_0^T \sum_{E \in \mathcal{E}_h} h_E \int_E \left[\left[\widetilde{P}_h \right] \right]^2 ds dt, \\
\eta_5^2 &= \int_0^T \int_{\Omega} (|\varepsilon \nabla (P_h - \widetilde{P}_h)|^2 + |\boldsymbol{\beta} \cdot \nabla (P_h - \widetilde{P}_h)|^2) dx dt, \\
\eta_6^2 &= \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K^+ \setminus \Gamma} h_E^3 \left(\boldsymbol{\beta} \cdot \mathbf{n}_E \left[\left[\widetilde{P}_h \right] \right] \right)^2 ds dt + \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K^+ \cap \Gamma^+} h_E^3 (\boldsymbol{\beta} \cdot \mathbf{n}_E \widetilde{P}_h)^2 ds dt, \\
\eta_7^2 &= \int_0^T \sum_{E \in \mathcal{E}_h} h_E^3 \int_E \left[\left[\boldsymbol{\beta} (P_h - \widetilde{P}_h) \right] \right]^2 ds dt.
\end{aligned}$$

Proof. Let ϕ be solution of (33) with $F = P_h - p(U_h)$. Let $\phi_I = \Pi_h \phi$ be the Lagrange interpolation of ϕ defined as in Lemma 3.2. Then, by using the adjoint equation (19b), the auxiliary equation (24), and the dual problem (33), we obtain

$$\begin{aligned}
\|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 &= \int_0^T (P_h - p(U_h), F) dt \\
&= \int_0^T (P_h - p(U_h), \phi_t - \varepsilon \Delta \phi + \boldsymbol{\beta} \cdot \nabla \phi) dt \\
&= \int_0^T \left(-\left(\frac{\partial}{\partial t} (P_h - p(U_h)), \phi \right) + a(\phi, P_h - p(U_h)) \right) dt \\
&= \int_0^T \left(-\left(\frac{\partial}{\partial t} (P_h - p(U_h)), \phi - \phi_I \right) + a(\phi - \phi_I, \widetilde{P}_h - p(U_h)) \right) dt \\
&\quad + \int_0^T \left(-\left(\frac{\partial}{\partial t} (P_h - p(U_h)), \phi_I \right) + a(\phi_I, \widetilde{P}_h - p(U_h)) + a(\phi, P_h - \widetilde{P}_h) \right) dt \\
&= \int_0^T \left(\left(-\frac{\partial P_h}{\partial t} - y(U_h) + y_d, \phi - \phi_I \right) + a(\phi, \widetilde{P}_h) \right) dt \\
&\quad + \int_0^T \left(-a_h(\phi_I, \widetilde{P}_h) + (\widehat{Y}_h - \widehat{y}_d, \phi_I) - (y(U_h) - y_d, \phi_I) + a(\phi, P_h - \widetilde{P}_h) \right) dt.
\end{aligned}$$

Integrating by parts, we obtain

$$\|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 = \underbrace{\int_0^T \left(-\frac{\partial P_h}{\partial t} - \varepsilon \Delta \widetilde{P}_h - \boldsymbol{\beta} \cdot \nabla \widetilde{P}_h - \widehat{Y}_h + \widehat{y}_d, \phi - \phi_I \right) dt}_{I_1}$$

$$\begin{aligned}
& + \underbrace{\int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\varepsilon \nabla \tilde{P}_h \cdot \mathbf{n})(\phi - \phi_I) ds dt}_{I_2} + \underbrace{\int_0^T (\hat{Y}_h - y(U_h) + y_d - \hat{y}_d, \phi) dt}_{I_3} \\
& + \underbrace{\int_0^T \sum_{E \in \mathcal{E}_h} \int_E \left(\left\{ \left\{ \varepsilon \nabla \tilde{P}_h \right\} \right\} \cdot [\![\phi_I]\!] + \left\{ \left\{ \varepsilon \nabla \phi_I \right\} \right\} \cdot [\![\tilde{P}_h]\!] \right) ds dt}_{I_4} \\
& - \underbrace{\int_0^T \sum_{E \in \mathcal{E}_h} \frac{\varepsilon \sigma}{h_E} \int_E [\![\tilde{P}_h]\!] [\![\phi_I]\!] ds dt}_{I_5} + \underbrace{\int_0^T \int_{\Omega} (\varepsilon \nabla (P_h - \tilde{P}_h) \nabla \phi - \beta \cdot \nabla (P_h - \tilde{P}_h) \phi) dx dt}_{I_6} \\
& + \underbrace{\int_0^T \left(\sum_{K \in \mathcal{T}_h} \int_{\partial K^+ \setminus \Gamma} \beta \cdot \mathbf{n} (\tilde{P}_h - \tilde{P}_h^e) \phi_I ds + \sum_{K \in \mathcal{T}_h} \int_{\partial K^+ \cap \Gamma^+} \beta \cdot \mathbf{n} \tilde{P}_h \phi_I ds \right) dt}_{I_7} \\
& + \underbrace{\int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\beta \cdot \mathbf{n})(P_h - \tilde{P}_h) \phi ds dt}_{I_8}. \tag{35}
\end{aligned}$$

We now estimate the terms on the right-hand side of (35) term by term. To estimate the first term in the right-hand side of (35), we use Lemma 3.2 and Lemma 3.4 such that

$$\begin{aligned}
I_1 & \leq \int_0^T \sum_{K \in \mathcal{T}_h} h_K^4 \int_K \left(\hat{Y}_h - \hat{y}_d + \frac{\partial P_h}{\partial t} + \varepsilon \Delta \tilde{P}_h + \beta \cdot \nabla \tilde{P}_h \right)^2 dx dt + \int_0^T |\phi|_{H^2(\Omega)}^2 dt \\
& \lesssim \eta_1^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2. \tag{36}
\end{aligned}$$

Next, if we rewrite the term I_2 in terms of the jump of $\nabla \tilde{P}_h$ and use the Lemmas 3.2-3.4, we obtain

$$\begin{aligned}
I_2 & = \int_0^T \sum_{E \in \mathcal{E}_h} \int_E [\![\varepsilon \nabla \tilde{P}_h]\!] (\Phi - \Phi_I) ds dt \\
& \lesssim \int_0^T \sum_{E \in \mathcal{E}_h} h_E^3 \int_E [\![\varepsilon \nabla \tilde{P}_h]\!]^2 ds dt + \int_0^T |\phi|_{H^2(\Omega)}^2 dt \\
& \lesssim \eta_2^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2. \tag{37}
\end{aligned}$$

Then, Lemma 3.4, Young's and the triangle inequalities give us

$$\begin{aligned}
I_3 & \lesssim \|\hat{Y}_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 + \|\hat{y}_d - y_d\|_{L^2(0,T;L^2(\Omega))}^2 + \|\phi\|_{L^2(0,T;L^2(\Omega))}^2 \\
& \lesssim \eta_3^2 + \|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2. \tag{38}
\end{aligned}$$

Similarly, using Young's inequality, Lemma 3.3, Lemma 3.4 and the inequality in (32), we obtain

$$I_4 \lesssim \eta_2^2 + \eta_4^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2, \quad (39)$$

$$I_5 \lesssim \eta_4^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2, \quad (40)$$

$$\begin{aligned} I_6 &\lesssim \int_0^T \int_{\Omega} (|\varepsilon \nabla(P_h - \tilde{P}_h)|^2 + |\beta \cdot \nabla(P_h - \tilde{P}_h)|^2) dx dt + \int_0^T (\|\nabla \phi\|_{L^2(\Omega)}^2 + \|\phi\|_{L^2(\Omega)}^2) dt \\ &\lesssim \eta_5^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2, \end{aligned} \quad (41)$$

$$I_7 \lesssim \eta_6^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2. \quad (42)$$

Finally, rewriting the term I_8 in terms of the jump operator and using Lemma 3.3 and Lemma 3.4, we have

$$I_8 \lesssim \eta_7^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2. \quad (43)$$

Inserting (36-43) into (35), the desired result is obtained. \square

Now, we need to estimate $\|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}$.

Lemma 3.6 *Let (y, u, p) and (Y_h, U_h, P_h) be the solutions of (6) and (17), respectively. The auxiliary solutions $y(U_h)$ and $p(U_h)$ are defined by the system (24). Assume that Ω is a convex domain, then,*

$$\|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 \lesssim \sum_{i=8}^{14} \eta_i^2,$$

where

$$\eta_8^2 = \int_0^T \sum_{K \in \mathcal{T}_h} h_K^4 \int_K \left(\hat{f}_h - BU_h - \frac{\partial Y_h}{\partial t} + \varepsilon \Delta \hat{Y}_h - \beta_h \cdot \nabla \hat{Y}_h \right)^2 dx dt,$$

$$\eta_9^2 = \int_0^T \sum_{E \in \mathcal{E}_h} h_E^3 \int_E \left[\left[\varepsilon \nabla \hat{Y}_h \right] \right]^2 ds dt,$$

$$\eta_{10}^2 = \int_0^T \sum_{E \in \mathcal{E}_h} h_E \int_E \left[\left[\hat{Y}_h \right] \right]^2 ds dt,$$

$$\eta_{11}^2 = \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K \setminus \Gamma} h_E^3 \left(\beta \cdot \mathbf{n}_E \left[\left[\hat{Y}_h \right] \right] \right)^2 ds dt + \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K \cap \Gamma^-} h_E^3 \left(\beta \cdot \mathbf{n}_E \hat{Y}_h \right)^2 ds dt,$$

$$\eta_{12}^2 = \|\hat{f} - f\|_{L^2(0,T;L^2(\Omega))}^2 + \|\tilde{P}_h - P_h\|_{L^2(0,T;L^2(\Omega))}^2,$$

$$\eta_{13}^2 = \int_0^T \int_{\Omega} (|\varepsilon \nabla(Y_h - \hat{Y}_h)|^2 + |\beta \cdot \nabla(Y_h - \hat{Y}_h)|^2) dx dt,$$

$$\eta_{14}^2 = \|Y_h(x, 0) - y_0(x)\|_{L^2(\Omega)}^2.$$

Proof. Similar as before, let ψ be the solution of (34) with $F = Y_h - y(U_h)$. Let $\psi_I = \Pi_h \psi$ be the Lagrange interpolation of ψ defined as in Lemma 3.2. Then, we conclude from (19a), (24), and (34), that

$$\begin{aligned} \|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 &= \int_0^T (Y_h - y(U_h), F) dt \\ &= \int_0^T (Y_h - y(U_h), -\psi_t - \varepsilon \Delta \psi - \beta \cdot \nabla \psi) dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^T \left(\frac{\partial Y_h}{\partial t} - \varepsilon \Delta \widehat{Y}_h + \beta \cdot \nabla \widehat{Y}_h - \widehat{f}_h - BU_h, \Psi - \Psi_I \right) dt + \int_0^T \sum_{K \in \mathcal{T}_h} \int_K (\varepsilon \nabla \widehat{Y}_h \cdot \mathbf{n})(\Psi - \Psi_I) ds dt \\
&\quad - \int_0^T \sum_{E \in \mathcal{E}_h} \frac{\varepsilon \sigma}{h_E} \int_E \left[[\widehat{Y}_h] \right] \llbracket \Psi_I \rrbracket ds dt + \int_0^T \sum_{E \in \mathcal{E}_h} \int_E \left(\{ \{ \varepsilon \nabla \Psi_I \} \} \cdot [\widehat{Y}_h] + \{ \{ \varepsilon \nabla \widehat{Y}_h \} \} \cdot \llbracket \Psi_I \rrbracket \right) ds dt \\
&\quad - \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K^- \setminus \Gamma} (\beta \cdot \mathbf{n})(\widehat{Y}_h^e - \widehat{Y}_h) \Psi_I ds dt + \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K^- \cap \Gamma^-} (\beta \cdot \mathbf{n}) \widehat{Y}_h \Psi_I ds dt \\
&\quad + \int_0^T (\widehat{f}_h - f, \Psi) dt + \int_0^T a(Y_h - \widehat{Y}_h, \Psi) dt + ((Y_h - y(U_h))(x, 0), \Psi(x, 0)). \tag{44}
\end{aligned}$$

Applying the same arguments as in (36-43), the desired result is obtained. \square

From Lemma 3.1, 3.5 and 3.6, we have the following a posteriori error estimate.

Theorem 3.7 *Let (y, u, p) and (Y_h, U_h, P_h) be the solutions of (6) and (17), respectively. The auxiliary solutions $y(U_h)$ and $p(U_h)$ are defined by the system (24). Assume that Ω is a convex domain, then,*

$$\|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2 + \|y - Y_h\|_{L^2(0,T;L^2(\Omega))}^2 + \|p - P_h\|_{L^2(0,T;L^2(\Omega))}^2 \lesssim \eta_u^2 + \sum_{i=1}^{14} \eta_i^2.$$

Proof. It follows from (6) and (24) that

$$\|y(U_h) - y\|_{L^2(0,T;L^2(\Omega))}^2 \lesssim \|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2, \tag{45a}$$

$$\|p(U_h) - p\|_{L^2(0,T;L^2(\Omega))}^2 \lesssim \|y(U_h) - y\|_{L^2(0,T;L^2(\Omega))}^2. \tag{45b}$$

Lemma 3.1, 3.5 and 3.6 yield

$$\begin{aligned}
\|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2 &\lesssim \eta_u^2 + \|\widetilde{P}_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 \\
&\lesssim \eta_u^2 + \|\widetilde{P}_h - P_h\|_{L^2(0,T;L^2(\Omega))}^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 \\
&\lesssim \eta_u^2 + \sum_{i=1}^{14} \eta_i^2. \tag{46}
\end{aligned}$$

Then, the desired result is obtained by applying the triangle inequality and using the inequalities (45)-(46) with Lemmas 3.1, 3.5 and 3.6. \square

4 Moreau-Yosida regularization

The Moreau-Yosida regularization is a popular technique for the optimal control problems with state constraints. Some recent progress in this area has been summarised in [15, 16, 20, 36], and the references cited therein. However, it also provides challenges for the control constrained case, see, e.g., [28, 32].

We penalize the control constraint, i.e. $u_a \leq u \leq u_b$, with a Moreau-Yosida-based regularization by modifying the objective functional $J(y, u)$ in (1). Now, we wish to minimize

$$J(y, u) + \frac{1}{2\delta} \int_0^T (\|\max\{0, u - u_b\}\|_{L^2(\Omega)}^2 + \|\min\{0, u - u_a\}\|_{L^2(\Omega)}^2) dt \quad (47)$$

subject to the state system (2). Here, δ is the Moreau-Yosida regularization parameter. The min- and max-expressions in the regularized objective functional arises from regularizing the indicator function corresponding to the set of admissible controls.

The unconstrained optimal control problem (47) has a unique solution $(y, u) \in W \times X$ if and only if there is an adjoint $p \in W$ such that (y, u, p) satisfies the following system for $t \in (0, T]$

$$(\partial_t y, v) + a(y, v) = (f + Bu, v) \quad \forall v \in V, \quad (48a)$$

$$y(x, 0) = y_0,$$

$$-(\partial_t y, \psi) + a(\psi, p) = (y - y_d, \psi) \quad \forall \psi \in V, \quad (48b)$$

$$p(x, T) = 0,$$

$$\int_0^T (\alpha(u - u_d) + B^* p + \sigma, w - u)_U dt = 0 \quad \forall w \in U, \quad (48c)$$

where the multiplier corresponding to the control constraint is

$$\sigma = \frac{1}{\delta} (\max\{0, u - u_b\} + \min\{0, u - u_a\}).$$

Then, the fully discretized optimality system of the regularized optimal control problem (47) is written as

$$\left(\frac{Y_{h,n} - Y_{h,n-1}}{k_n}, v \right) + a_h(Y_{h,n}, v) = (f_n + BU_{h,n}, v) \quad \forall v \in V_h, \quad (49a)$$

$$Y_{h,0} = y_h^0 \quad n = 1, 2, \dots, N_T,$$

$$\left(\frac{P_{h,n-1} - P_{h,n}}{k_n}, q \right) + a_h(q, P_{h,n-1}) = (Y_{h,n} - y_n^d, q) \quad \forall q \in V_h, \quad (49b)$$

$$P_{h,T} = 0 \quad n = N_T, \dots, 2, 1,$$

$$\left(\alpha(U_{h,n} - u_n^d) + B^* P_{h,n-1} + \sigma_{h,n}, w - U_{h,n} \right)_U = 0 \quad \forall w \in X_h, \quad n = 1, 2, \dots, N_T. \quad (49c)$$

where

$$\sigma_{h,n} = \frac{1}{\delta} (\max\{0, U_{h,n} - u_b\} + \min\{0, U_{h,n} - u_a\}).$$

As in the previous section, we restate the optimality system (49) as follows:

$$\left(\frac{\partial Y_h}{\partial t}, v \right) + a_h(\widehat{Y}_h, v) = (\widehat{f}_h + BU_h, v) \quad \forall v \in V_h, \quad (50a)$$

$$Y_h(x, 0) = y_h^0(x) \quad n = 1, 2, \dots, N_T,$$

$$\left(\frac{\partial P_h}{\partial t}, q \right) + a_h(q, \widetilde{P}_h) = (\widehat{Y}_h - \widehat{y}_d, q) \quad \forall q \in V_h, \quad (50b)$$

$$P_h(x, T) = 0 \quad n = N_T, \dots, 2, 1,$$

$$\left(\alpha(U_h - \widehat{u}_d) + B^* \widetilde{P}_h + \widehat{\sigma}_h, w - U_h \right)_U = 0 \quad \forall w \in X_h, \quad n = 1, 2, \dots, N_T, \quad (50c)$$

where $\widehat{\mathfrak{G}}_h = \frac{1}{\delta} \left(\max\{0, U_h - u_b\} + \min\{0, U_h - u_a\} \right)$.

The optimality system (49) of the Moreau-Yosida approach leads to the following linear system for $n = 1, \dots, N_T$

$$\begin{aligned} (\mathcal{M} + k_n \mathcal{K}) Y_n - \mathcal{M} Y_{n-1} &= \ell(f_n) + \mathcal{M} U_n, \\ (k_n \mathcal{K}^T + \mathcal{M}) P_{n-1} - \mathcal{M} P_n &= \mathcal{M} Y_n - \ell(y_n^d), \\ \left(\alpha \mathcal{M} + \frac{1}{\delta} \chi_{\mathcal{A}_n} \mathcal{M} \chi_{\mathcal{A}_n} \right) U_n - \alpha \ell(u_n^d) + \mathcal{M} P_{n-1} &= \frac{1}{\delta} \left(\chi_{\mathcal{A}_n^a} \mathcal{M} \chi_{\mathcal{A}_n^a} u_a + \chi_{\mathcal{A}_n^b} \mathcal{M} \chi_{\mathcal{A}_n^b} u_a \right), \end{aligned} \quad (51a)$$

where

$$\mathcal{A}_n^a = \{x \in \Omega : u - u_a < 0\}, \quad \mathcal{A}_n^b = \{x \in \Omega : u - u_b > 0\}, \quad \text{and } \mathcal{A}_n = \mathcal{A}_n^a \cup \mathcal{A}_n^b.$$

Similarly, we now derive an a posteriori error estimate for the Moreau-Yosida regularized optimization problem (47).

Lemma 4.1 *Let (y, u, p) and (Y_h, U_h, P_h) be the solutions of (48) and (49), respectively. Then, we have the following estimate*

$$\|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2 \leq C \left(\eta_u^M \right)^2 + \|\widetilde{P}_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2, \quad (52)$$

where

$$\begin{aligned} \eta_u^M &= \sum_{n=1}^{N_T} \int_{t_{n-1}}^{t_n} \int_{\Omega_U} \left(\alpha(U_h - \widehat{u}_d) + B^* \widetilde{P}_h + \frac{1}{\delta} (\chi_{\mathcal{A}_n^a} (U_h - u_a) + \chi_{\mathcal{A}_n^b} (U_h - u_b)) \right) dt \\ &\quad + \|\alpha(u_d - \widehat{u}_d)\|_{L^2(0,T;L^2(\Omega_U))}^2 \end{aligned} \quad (53)$$

and the auxiliary functions, i.e., $y(U_h)$ and $p(U_h)$, are defined as in (24).

Proof. By using the inequalities (48c), (49c) and (50c), we obtain

$$\begin{aligned} \alpha \|u - U_h\|_{L^2(0,T;L^2(\Omega_U))}^2 &= \int_0^T (\alpha u_d - B^* p - \sigma, u - U_h)_U dt - \int_0^T (\alpha U_h, u - U_h)_U dt \\ &= \underbrace{\int_0^T (\alpha(U_h - \widehat{u}_d) + B^* \widetilde{P}_h + \widehat{\mathfrak{G}}_h, U_h - u)_U dt}_{M_1} + \underbrace{\int_0^T (\widehat{\mathfrak{G}}_h - \sigma, u - U_h)_U dt}_{M_2} \\ &\quad + \underbrace{\int_0^T (B^* (\widetilde{P}_h - p(U_h)), u - U_h)_U dt}_{M_3} + \underbrace{\int_0^T (B^* (p(U_h) - p), u - U_h)_U dt}_{M_4} \\ &\quad + \underbrace{\int_0^T (\alpha(u_d - \widehat{u}_d), u - U_h)_U dt}_{M_5}. \end{aligned} \quad (54)$$

We only derive a bound for M_2 in detail, since the estimation of the other terms is similar to the procedure in Lemma 3.1. Recall that the following inequalities

$$\begin{aligned}\|\min\{0, a\} - \min\{0, b\}\|_{L^2(\Omega)} &\leq \|a - b\|_{L^2(\Omega)}, \\ \|\max\{0, a\} - \max\{0, b\}\|_{L^2(\Omega)} &\leq \|a - b\|_{L^2(\Omega)}\end{aligned}$$

hold. We here assume that the regularization parameter δ is fixed as done in [15, 36], then we obtain

$$\begin{aligned}M_2 &= \int_0^T \frac{1}{\delta} (\max\{0, U_h - u_b\} - \max\{0, u - u_b\}, u - U_h) dt \\ &\quad + \int_0^T \frac{1}{\delta} (\min\{0, U_h - u_a\} - \min\{0, u - u_a\}, u - U_h) dt \\ &\lesssim \frac{1}{\delta} \|u - U_h\|_{L^2(0, T; L^2(\Omega_U))}^2.\end{aligned}\tag{55}$$

□

Similarly, we have the following a posteriori error estimate for the regularized optimization problem (47) from Lemma 4.1, 3.5 and 3.6.

Theorem 4.2 *Let (y, u, p) and (Y_h, U_h, P_h) be the solutions of (6) and (17), respectively. The auxiliary solutions $y(U_h)$ and $p(U_h)$ are defined in the system (24). Assume that Ω is a convex domain, then,*

$$\|u - U_h\|_{L^2(0, T; L^2(\Omega_U))}^2 + \|y - Y_h\|_{L^2(0, T; L^2(\Omega))}^2 + \|p - P_h\|_{L^2(0, T; L^2(\Omega))}^2 \lesssim (\eta_u^M)^2 + \sum_{i=1}^{14} \eta_i^2.$$

5 Numerical Implementation

In this section, we present numerical results to demonstrate the performance of the estimators proposed in Sections 3 and 4. The state, the adjoint, and the control variables are discretized by using piecewise linear polynomials, i.e., $(x, y, 1 - x - y)$. The initial guess for the control variable is equal to zero for all discretization levels in Algorithm 1. The penalty parameter within SIPG is chosen as $\sigma = 6$ on the interior edges and 12 on the boundary edges as in [30]. The Moreau-Yosida regularization parameter δ is equal to 10^{-6} . We use uniform time steps and time-step size is $k = 1/50$. Further, we take $\Omega = \Omega_U$ and $B = I$. Our adaptive strategy can be briefly described as follows:

Adaptive Algorithm

(Input) Given an initial mesh partition \mathcal{T}_h , refinement parameter θ , a tolerance parameter Tol .

Step 1. (Solve) Solve the optimality system (20) obtained by the primal-dual active set (PDAS) algorithm on the current mesh or the optimality system (51) obtained by applying the Moreau-Yosida regularization.

Step 2. (Estimate) Calculate the local error indicators on each element K and then sum them over the whole space-time domain.

Step 3. (Mark) The edges and elements for the refinement are specified by using the a posteriori error indicator and by choosing subsets $\mathcal{M}_K \subset \mathcal{T}_h$ such that the following bulk criterion is satisfied for the given marking parameter θ :

$$\theta \sum_{K \in \mathcal{T}_h} (\eta_K)^2 \leq \sum_{K \in \mathcal{M}_K} (\eta_K)^2.$$

Step 4. (Refine) The marked elements are refined by longest edge bisection, where the elements of the marked edges are refined by bisection.

Step 5. Return to Step 1 on the new mesh to update the solutions, until the error estimators are less than the given tolerance value Tol .

5.1 Example 1

We first consider the following transport of a rotating Gaussian pulse example given in [13] with only lower bound, i.e., $u_a = 0$. Fu et al. use this example in their analysis of the norm-residual based estimator in combination with a characteristic finite element approximation. The problem data are given by

$$\Omega = [-0.5, 0.5]^2, \quad T = 1, \quad \varepsilon = 10^{-4}, \quad \beta = (-x_2, x_1)^T, \quad \text{and} \quad \omega = 1.$$

The corresponding analytical solutions are given by

$$\begin{aligned} y(x, t) &= \frac{2\sigma_0^2}{2\sigma_0^2 + 4t\varepsilon} \exp\left(-\frac{(\bar{x}_1 - x_0)^2 + (\bar{x}_2 - y_0)^2}{2\sigma_0^2 + 4t\varepsilon}\right), \\ p(x, t) &= 0, \\ z(x, t) &= \begin{cases} 1/2, & x_1 + x_2 > 0, \\ 0, & x_1 + x_2 \leq 0, \end{cases} \\ u_d(x, t) &= \sin(\pi t/2) \sin(\pi x_1) \sin(\pi x_2) + z, \\ u(x, t) &= \max\left(0, u_d - \frac{p}{\alpha}\right), \end{aligned}$$

where the center (x_0, y_0) , the standard deviation $\sigma_0^2 = 0.0447$, and $\bar{x}_1 = x_1 \cos(t) + x_2 \sin(t)$, $\bar{x}_2 = x_2 \cos(t) - x_1 \sin(t)$. The source f and the desired state functions are taken as $f = -u$ and $y_d = y$, respectively.

The optimal control problem exhibits a strong jump (discontinuity) introduced by the desired control $u_d(x, t)$. Figure 1 shows that a high density of vertices are distributed along $x_1 + x_2 = 0$. By using the a posteriori error indicators η_u (23) or η_u^M (53) of the control variable, we pick out the discontinuity caused by the desired control $u_d(x, t)$ and construct an optimal adaptive mesh to obtain a better accuracy for the control as shown in Figure 2 with adaptive parameter $\theta = 0.35$.

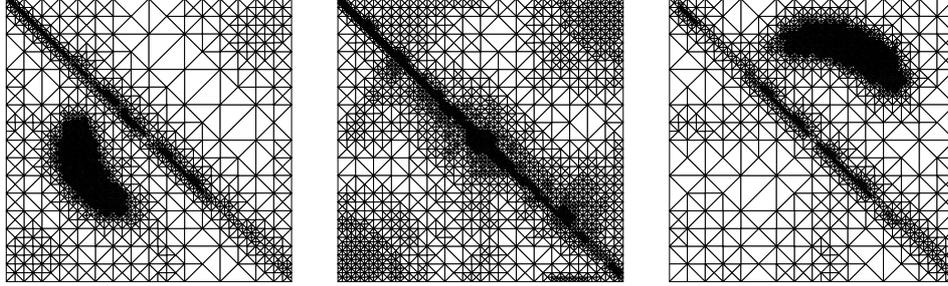


Figure 1: Example 5.1: Adaptively refined meshes for different values of (x_0, y_0) , i.e., left $(-0.25, 0)$, middle $(0, 0)$, right $(0.25, 0.25)$, at $t = 1$ using the primal-dual active set strategy. The number of refinement steps and vertices are $(9,4224)$, $(10,3615)$, and $(9,4728)$ (from left to right) with adaptive parameter $\theta = 0.35$.

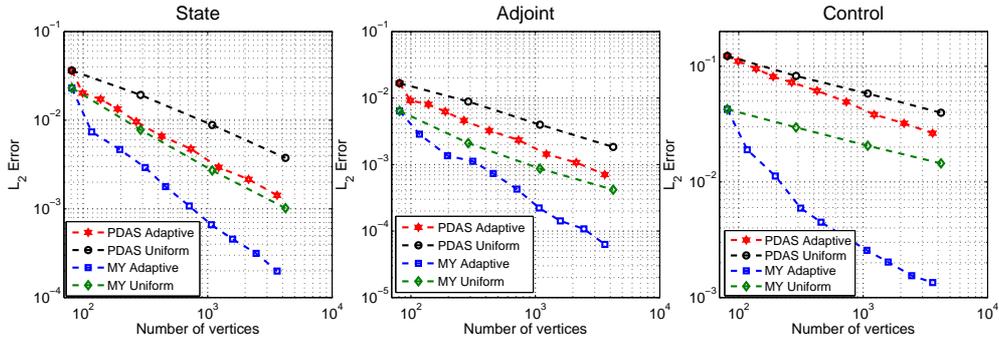


Figure 2: Example 5.1: Global errors of the state, adjoint and control in the $L^2(0, T; L^2(\Omega))$ norm with $(x_0, y_0) = (0, 0)$.

As the state y exhibits different regularity, we make experiments for different values of the center point such as $(x_0, y_0) \in \{(-0.25, 0), (0, 0), (0.25, 0.25)\}$. For all cases, we obtain a higher density of vertices in the neighborhood of $(\bar{x}_1, \bar{x}_2) = (x_0, y_0)$ as shown in Figure 1. Therefore, the adaptive meshes in Figure 1 show that the a posteriori error estimators provided in Section 3 pick out the regions well, where more refinements are needed.

Figure 2 displays the summation of the $L^2(0, T; L^2(\Omega))$ errors for each time step for the state, adjoint and control variables at $(x_0, y_0) = (0, 0)$, obtained using the primal-dual active set strategy and the Moreau-Yosida regularization. For both approaches, the errors on adaptively refined meshes are decreasing faster than the errors on uniformly refined meshes. Especially, we obtain a better convergence result for the adaptive implementation of the Moreau-Yosida approach as shown in Figure 2.

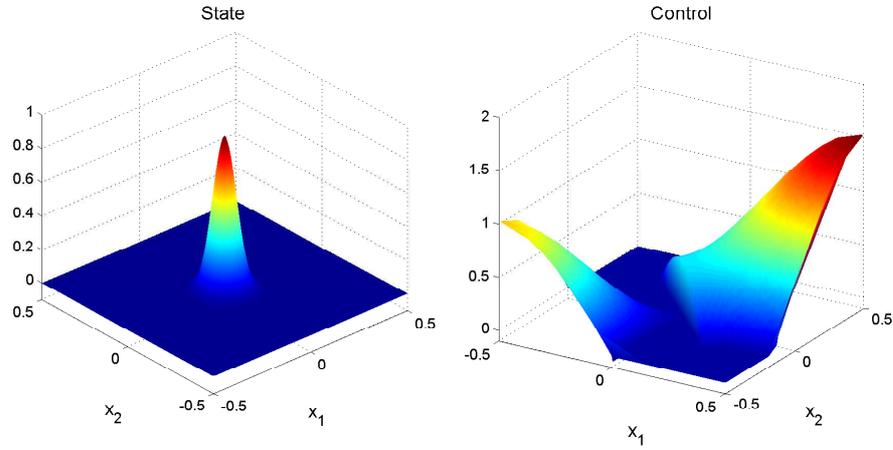


Figure 3: Example 5.1: The computed state (left) and control (right) on an adaptively refined mesh with 3,618 vertices by using the Moreau-Yosida regularization for $(x_0, y_0) = (0, 0)$ at $t = 1$ after 10 refinement steps.

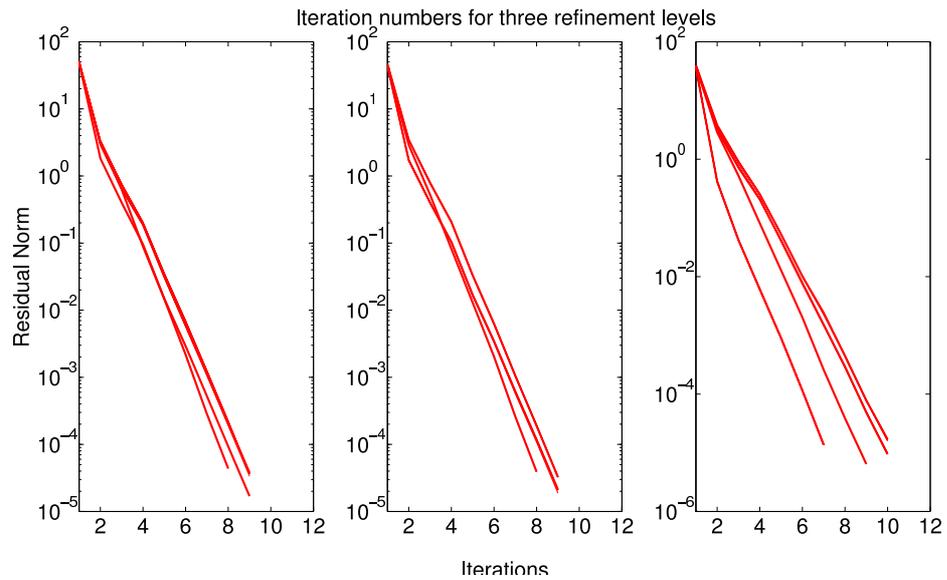


Figure 4: Example 5.1: GMRES iterations for three different refinement levels. A block-triangular preconditioner was used and the stopping criterion is set to 10^{-4} for the relative preconditioned residual.

Figure 3 shows the computed solutions on an adaptively refined mesh with 3,618 vertices by using the Moreau-Yosida regularization for $(x_0, y_0) = (0, 0)$ at $t = 1$ after 10 refinement steps. We conclude that substantial computing work can be saved by using efficient adaptive meshes for both approaches and the Moreau-Yosida technique captures the errors of the control better than the primal-dual active set strategy.

Additionally, our approach is also amendable by efficient preconditioning strategies such as the ones given in [28, 29] where an iterative method of Krylov subspace type is combined with efficient and robust Schur complement approaches. Figure 4 shows the iteration numbers of GMRES with a block-triangular preconditioner for three consecutive stages of refinement and the associated systems within the Newton method.

5.2 Example 2

We set up our second example according to

$$\Omega = [-1, 1]^2, \quad T = 0.5, \quad \varepsilon = 10^{-5}, \quad \text{and} \quad \beta = (2, 3)^T, \quad \text{and} \quad \omega = 0.1.$$

The source function f and the desired state y_d are computed by using the following analytical solutions:

$$\begin{aligned} y(x, t) &= 16 \sin(\pi t) x_1 (1 - x_1) x_2 (1 - x_2) \\ &\quad \times \left(\frac{1}{2} + \frac{1}{\pi} \arctan \left[\frac{2}{\sqrt{\varepsilon}} \left(\frac{1}{16} - \left(x_1 - \frac{1}{2} \right)^2 - \left(x_2 - \frac{1}{2} \right)^2 \right) \right] \right), \\ p(x, t) &= 0, \\ u_d(x, t) &= \sin(\pi t) \sin\left(\frac{\pi}{2} x_1\right) \sin\left(\frac{\pi}{2} x_2\right), \\ u(x, t) &= \max \left(0, \min \left(0.5, u_d - \frac{p}{\alpha} \right) \right), \end{aligned}$$

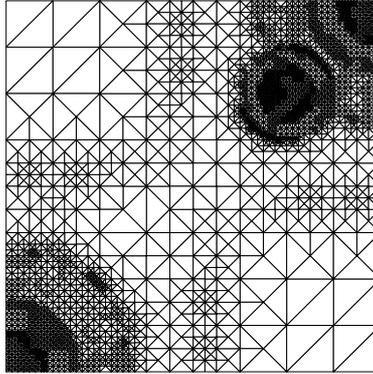


Figure 5: Example 5.2: Adaptively refined mesh with 4,417 vertices at $t = 0.5$ after 6 refinement steps with $\theta = 0.55$ by using the primal-dual active set strategy.

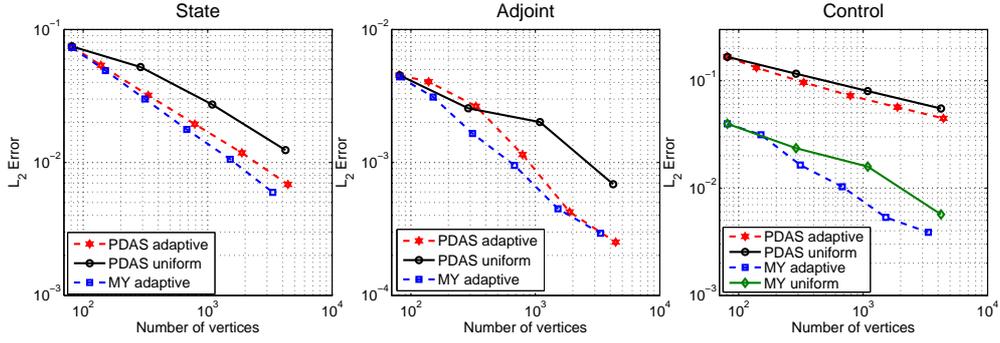


Figure 6: Example 5.2: Global errors of the state, adjoint and control in the $L^2(0, T; L^2(\Omega))$ norm.

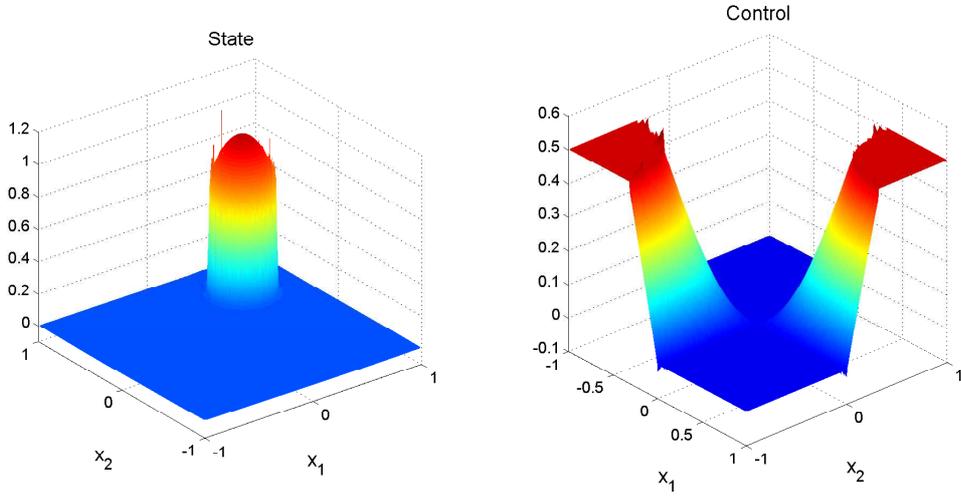


Figure 7: Example 5.2: The computed state (left) and control (right) on an adaptively refined mesh with 3.340 vertices by using the Moreau-Yosida regularization at $t = 0.5$ after 6 refinement steps with $\theta = 0.55$.

The optimal state exhibits an interior layer depending on the diffusion parameter ε . Also, it is a hump changing its height in the course of the time. Figure 5 shows a high density of vertices being distributed along the interior layer and contact set. It again demonstrates that the error indicators proposed in Section 3 work well.

The global $L^2(0, T; L^2(\Omega))$ errors of the state, adjoint, and the control variables, obtained using both approaches, are given in Figure 6. We here only present the results of the primal-dual active set strategy on the uniform meshes for the state and adjoint, since the results for both approaches are quite similar. The Moreau-Yosida approach especially produces better

convergence results for the control variable.

Finally, Figure 7 exhibits the computed solutions of the state and control, obtained by using the Moreau-Yosida approach, on an adaptive mesh with 3.340 vertices after 6 refinement steps with the adaptive parameter $\theta = 0.55$.

6 Conclusions

We discuss the optimal control problem governed by transient convection diffusion equations, discretized by the symmetric interior penalty Galerkin (SIPG) method in space and the standard backward Euler in time. In order to handle control constraints, we apply the primal-dual active set strategy and the Moreau-Yosida-based regularization. For both approaches, we propose error estimators to guide the mesh refinement. Numerical results show that substantial computing work can be saved by using efficient adaptive meshes for both approaches and the Moreau-Yosida technique captures the errors of the control better than the PDAS strategy.

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