Parametrische Modellreduktion mit dünnen Gittern

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Modellreduktion spielt eine wichtige Rolle in der Simulation, Kontrolle und Optimierung von komplexen, dynamischen Systemen. Häufig beinhalten diese Systeme zusätzliche Parameter, z.B. um Änderungen am physikalischen Modell, sei es eine geometrische Variation oder eine Veränderung der Umgebungsbedingungen, zu ermöglichen. Daher ist es erforderlich, Modelle reduzierter Ordnung zu generieren, die diese zusätzlichen Parameter beinhalten.

Wir betrachten lineare Systeme der Form

$$\begin{array}{rcl} \dot{x}(t) & = & A(p)\,x(t) + B(p)\,u(t) \\ y(t) & = & C^T x(t) \end{array} \qquad (1)$$

wobei die Systemmatrizen $A(p) \in \mathbb{R}^{n\times n}$, $B(p) \in \mathbb{R}^{n\times m}$ von einem Parametervektor $p \in \mathbb{R}^d$ abhängen. Die Ausgangsmatrix $C \in \mathbb{R}^{n\times p}$ setzen wir als konstant voraus.

Zur Modellreduktion für (1) haben wir ein Verfahren entwickelt, welches auf einer Kopplung der Methode des balancierten Abschneidens mit Interpolation beruht. Dabei wird der Parameterraum mit einem möglichst groben Gitter überzogen und die Größe der somit an den Gitterpunkten aufstellbaren, parameterunabhängigen Systeme mit dem Verfahren des balancierten Abschneidens reduziert. Um diese Technik auch auf Parameterräume mit höheren Dimensionen anwenden zu können, wurden dünne Gitter [2, 3, 4] zur Diskretisierung des Parametergebietes verwendet, die im Vergleich zu vollen Gittern signifikant weniger Punkte benötigen. Das gesamte System reduzierter Ordnung, welches alle ursprünglichen Parameterabhängigkeiten besitzt, wird durch Interpolation erhalten, siehe auch [1].

[1] Beattie, C.; Benner, P.; Gugercin, S.: Interpolatory Projection Methods for Parameterized Model Reduction, in preparation.

[2] Bungartz, H.: Dünne Gitter und deren Anwendung bei der adaptiven Lösung der dreidimensionalen Poisson-Gleichung, Dissertation, Inst. f. Informatik, TU München, June 1992.

[3] Klimke, A.: Sparse Grid Interpolation Toolbox – user's guide, Technical Report IANS report 2007/017, University of Stuttgart, 2007.

[4] Zenger, C.: Sparse grids. In Parallel algorithms for partial differential equations (Kiel, 1990), volume 31 of Notes Numer. Fluid Mech., pages 241–251, Vieweg, Braunschweig, 1991.

Parametrische Modellreduktion mit dünnen Gittern

(Parametric model reduction with sparse grids)

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Projekt mit J. Korvink, IMTEK (Universität Freiburg)
Automatic, Parameter-Preserving Model Reduction for Applications in Microsystems Technology

Outline

Parametric model reduction with sparse grids

- Parametric linear systems
- Balanced Truncation/Interpolatory MOR
- Use of sparse grids
- Numerical results
- Outlook

Motivation - parametric model reduction

- Parametric systems appear in many applications,
  e.g. MEMS design - optimization of geometry and topology

  Example from Oberwolfach Benchmark collection
  `www.imtek.de/simulation/benchmark`

  thermal conduction model:
  film coefficients $\{p_i\}_{i=1}^3$ describe heat exchange on interfaces

$$E\dot{x}(t) = (A_0 + \sum_{i=1}^{3} p_i A_i)x(t) + Bu(t), \quad y(t) = C^T x(t)$$

- **preserve the parameters in reduced-order system!**
- recent approaches:
  multivariate moment matching, e.g.
  [*Benner/Feng 07, Bond/Daniel 05, Daniel et al. 04, Eid et al. 07, Farle et al. 06, Feng 05, Moosmann/Korvink 06, Weile et al. 99*]

Parametric linear systems

Consider parametric linear systems with $p \in \mathbb{R}^d$:

$$\begin{aligned}\dot{x}(t) &= A(p)\,x(t) + B(p)\,u(t)\\ y(t) &= C(p)^T x(t)\end{aligned}$$

$$(A(p), B(p), C(p)) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times q} \text{ and } m, q \ll n$$

First: single parameter case $p \in [a,\ b]$

$$\begin{aligned}\dot{x}(t) &= A(p)\,x(t) + B\,u(t)\\ y(t) &= C^T x(t)\end{aligned}$$

with

$$A(p) = A_0 + p\,A_1$$

and

$$G(s,p) = C^T(sI_n - (A_0 + p\,A_1))^{-1}B$$

1. Choose interpolation points $p_1, \ldots, p_k \in [a,\ b]$
2. Compute reduced-order systems by balanced truncation (BT):

$$\hat{G}_j(s) = \hat{C}_j^T (sI_{r_j} - \hat{A}_j)^{-1} \hat{B}_j \quad \textit{for } j = 1, \ldots, k$$

3. Parametric reduced-order system by interpolation:

$$\hat{G}(s,p) = \sum_{j=1}^{k} l_j(p) \hat{G}_j(s)$$

with

$$\hat{G}(s, p_j) = \hat{G}_j(s) \quad \textit{for } j = 1, \ldots, k$$

2 Balanced Truncation

For

$$G_j(s) := G(s, p_j) = C^T(sI_n - (A_0 + p_j\ A_1))^{-1}B$$

compute reduced-order systems by BT:

$$\hat{G}_j(s) = \hat{C}_j^T(sI_{r_j} - \hat{A}_j)^{-1}\hat{B}_j \quad \textit{for } j = 1, \ldots, k$$

BT preserves stability, computable $\mathcal{H}_\infty$-error bound:

$$\|G_j - \hat{G}_j\|_\infty \leq 2\left(\sum_{i=r_j+1}^{n} \sigma_i\right) < tol$$

For numerical solution of Lyapunov equations:

- LR-ADI [*Penzl 00, Li/White 02*],
- Smith [*Penzl 00, Antoulas/Gugercin/Sor. 03*],
- Krylov [*Jaimoukha/Kasenally 94, Saad 90, Simoncini 07*],
- sign function method [*Benner/Quintana-Ortí 99, Baur 08*]

Parametric reduced-order system by interpolation:

$$\hat{G}(s,p) = \sum_{j=1}^{k} l_j(p)\hat{G}_j(s) = \sum_{j=1}^{k} \left(\prod_{i=1, i\neq j}^{k} \frac{p-p_i}{p_i-p_j} \right) \hat{C}_j^T (sI_{r_j} - \hat{A}_j)^{-1} \hat{B}_j$$

$$= \begin{bmatrix} \hat{C}_1(p) \\ \vdots \\ \hat{C}_k(p) \end{bmatrix}^T \begin{bmatrix} (sI_{r_1} - \hat{A}_1)^{-1} & & \\ & \ddots & \\ & & (sI_{r_k} - \hat{A}_k)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B}_1 \\ \vdots \\ \hat{B}_k \end{bmatrix}$$

But: for high-dimensional parameter space many interpolation points $\Rightarrow$ many times BT, i.e. very high complexity!

On $[0,\ 1]$ construct equidistant grid with mesh size $h_\ell = 2^{-\ell}$ and associated $(2^\ell - 1)$-dim. space S_ℓ of piecewise linear functions.

Hierarchical basis decomposition [Yse86]:

$$S_\ell = T_1 \oplus \cdots \oplus T_\ell$$

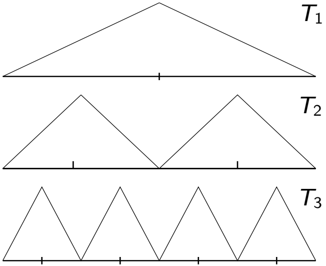
For $f \in C^2([0,\ 1])$ and interpolant $f_\mathrm{I} \in S_\ell$

$$f_\mathrm{I} = \sum_{i=1}^{\ell} f_i,\ f_i \in T_i,$$

the interpolation error is bounded

$$\|f - f_\mathrm{I}\|_\infty \le \mathcal{O}(h_\ell^2) \text{ and } \|f_i\|_\infty \le \frac{1}{2} 4^{-i} \|\frac{\partial^2 f}{\partial x^2}\|_\infty.$$

Use subspaces of S_ℓ:



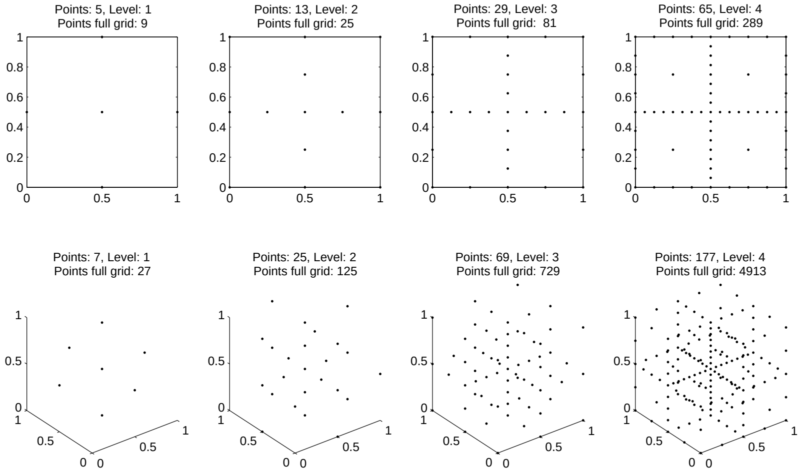
On $[0,\ 1]^d$ construct grid with mesh size $h_{\underline{\ell}}$ ($\underline{i} := (i_1, \dots, i_d) \in \mathbb{N}^d$).

For $f : [0,\ 1]^d \to \mathbb{R}$, $\frac{\partial^{2d} f}{\partial x_1^2 \dots \partial x_d^2} \in C^0([0,\ 1]^d)$ search interpolant f_{I} in space of piecewise d-linear functions:

	full grid space	sparse grid space
	$S_\ell = \bigoplus_{i_1=1}^{\ell} \cdots \bigoplus_{i_d=1}^{\ell} T_{\underline{i}}$	$\tilde{S}_\ell = \bigoplus_{\lvert\underline{i}\rvert_1 \le \ell+d-1} T_{\underline{i}}$
dimension	$\mathcal{O}(h_\ell^{-d})$	$\mathcal{O}(h_\ell^{-1}\ (log(h_\ell^{-1}))^{d-1})$
$\lVert f - f_{\mathrm{I}} \rVert_\infty$	$\mathcal{O}(h_\ell^2)$	$\mathcal{O}(h_\ell^2\ (log(h_\ell^{-1}))^{d-1})$

We employ sparse grids for high-dimensional parameter space $p \in [0,\ 1]^d$.

MATLAB Sparse Grid Interpolation Toolbox: Clenshaw-Curtis grid



Numerical results - convection-diffusion equation

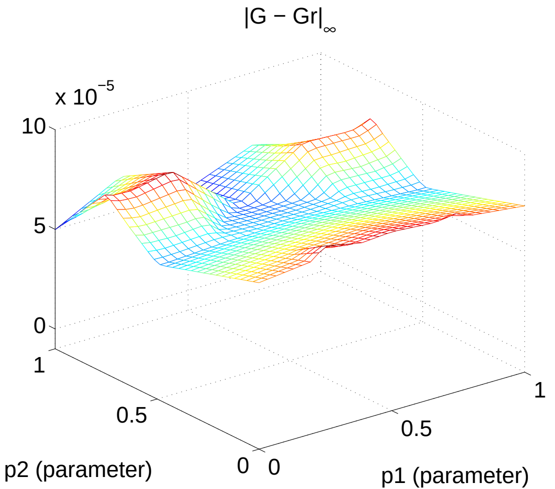
$$
\begin{aligned}
\frac{\partial \mathbf{x}}{\partial t}(t,\xi) &= \Delta \mathbf{x}(t,\xi) + \mathbf{p}\cdot\nabla \mathbf{x}(t,\xi) + b(\xi)u(t) \quad \xi \in (0,1)^2 \\
&\Downarrow \quad \textit{using FDM with } n = 100 \\
\dot{x}(t) &= (A + p_1 A_1 + p_2 A_2)\, x(t) + B\, u(t)
\end{aligned}
$$

$B, C \in \mathbb{R}^n$ random
parameter space: $p_1, p_2 \in [0,\ 1]$
MATLAB Sparse Grid Interpolation Toolbox
[*Klimke/Wohlmuth 05, Klimke 07*]

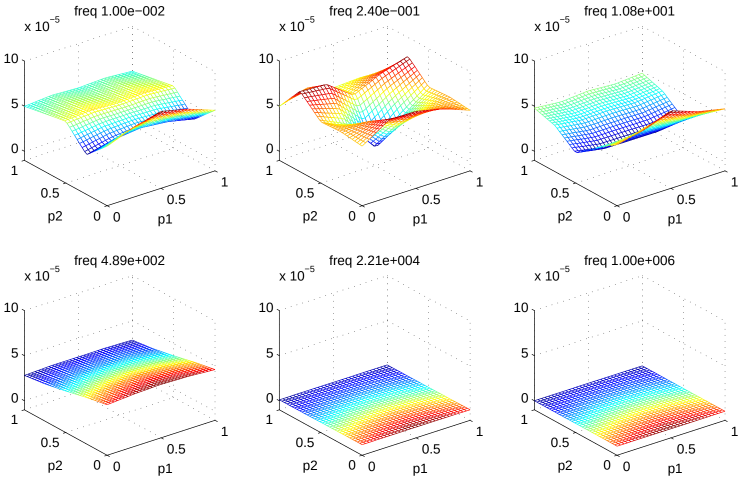
absolute tolerance for grid refinement: $10^{-4} \Rightarrow$ level $\ell = 2$
$\Rightarrow$ number of sparse grid points: $k = 13$

tolerance for BT: $10^{-4} \Rightarrow$ systems of reduced order $r_j \in \{3, 4\}$ for $j = 1, \ldots, k$

Numerical results - convection-diffusion equation



Numerical results - convection-diffusion equation

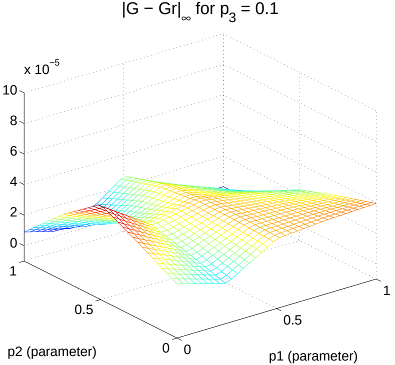


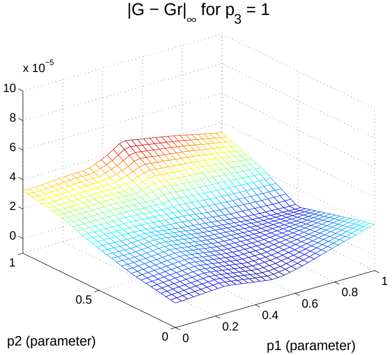
$$\dot{x}(t) = (p_3 A + p_1 A_1 + p_2 A_2)\, x(t) + B\, u(t), \quad p_1, p_2 \in [0,\ 1],\ p_3 \in [0.1,\ 1]$$

absolute tolerance for grid refinement: $10^{-4} \Rightarrow$ level $\ell = 2$

number of sparse grid points: $k = 25$

tolerance for BT: $10^{-4} \Rightarrow$ systems of reduced order $r_j \in \{3, \dots, 8\}$





Conclusions

Summary:

- We have developed a Balanced Truncation/Interpolatory method for parametric model reduction.
- The method can be applied to higher dimensional parameter spaces.

Next steps:

- only function for evaluation of reduced-order system, search for explicit description of TFM, state-space model;
- derive global error bound by combination of error estimates for BT, interpolation;
- use sparse grids also for other interpolatory methods as proposed in [*Beattie/Benner/Gugercin 08*];
- combine sparse grid interpolation with $\mathcal{H}_2$-optimal model reduction.

- Beattie, C.; Benner, P.; Gugercin, S.: Interpolatory Projection Methods for Parameterized Model Reduction, in preparation.
- B. Bond and L. Daniel, *Parameterized model order reduction of nonlinear dynamical systems*, in Proceedings of the IEEE Conference on Computer-Aided Design, 2005.
- L. Daniel, O. Siong, L. Chay, K. Lee, and J. White, *Multiparameter Moment Matching Model Reduction Approach for Generating Geometrically Parameterized Interconnect Performance Models*, Trans. on Computer Aided Design of Integrated Circuits, 2004.
- R. Eid, B. Salimbahrami, B. Lohmann, E. B. Rudnyi, and J. G. Korvink, *Parametric order reduction of proportionally damped second-order systems*, Sensors and Materials, 2007.
- O. Farle, V. Hill, P. Ingelström, and R. Dyczij-Edlinger, *Ordnungsreduktion linearer zeitinvarianter Finite-Elemente-Modelle mit multivariater polynomieller Parametrierung,* at-Automatisierungstechnik, 2006.

References

L. Feng, *Parameter independent model order reduction*, Math. Comput. Simulation, 2005.

D. S. Weile, E. Michielssen, E. Grimme, and K. Gallivan, *A method for generating rational interpolant reduced order models of two-parameter linear systems*, Appl. Math. Lett., 1999.

Bungartz, H.: *Dünne Gitter und deren Anwendung bei der adaptiven Lösung der dreidimensionalen Poisson-Gleichung*, Dissertation, TU München, 1992.

J. Garcke, *Sparse grid tutorial*, TU Berlin, 2007.

A. Klimke and B. Wohlmuth, *Algorithm 847: spinterp: Piecewise multilinear hierarchical sparse grid interpolation in MATLAB*, ACM Transactions on Mathematical Software, 2005.

Klimke, A.: *Sparse Grid Interpolation Toolbox – user's guide*, Technical Report, University of Stuttgart, 2007.

Zenger, C.:*Sparse grids. In Parallel algorithms for partial differential equations*, Notes Numer. Fluid Mech., 1991.