Control Theory

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Given a dynamical system described by the ordinary differential equation (ODE)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \ \mathbf{x}(t_0) = \mathbf{x}^0,$$

where $\mathbf{x}$ is the state of system and $\mathbf{u}$ serves as input, the major problem in control theory is to steer the state from $\mathbf{x}^0$ to some desired state, i.e., for a given initial value $\mathbf{x}(t_0) = \mathbf{x}^0$ and target $\mathbf{x}^1$, can we find a piecewise continuous or L_2 (i.e., square-integrable, Lebesgue measurable) control function $\hat{\mathbf{u}}$ such that there exists $t_1 \geq t_0$ with $\mathbf{x}(t_1; \hat{\mathbf{u}}) = \mathbf{x}^1$ where $\mathbf{x}(t; \hat{\mathbf{u}})$ is the solution trajectory of the ODE given above for $\mathbf{u} \equiv \hat{\mathbf{u}}$? Often, the target is $\mathbf{x}^1 = 0$, in particular if $\mathbf{x}$ describes the deviation from a nominal path. A weaker demand is to asymptotically stabilize the system, i.e., to find an admissible control function $\hat{\mathbf{u}}$ (i.e., a piecewise continuous or L_2 function $\hat{\mathbf{u}} : [t_0, t_1] \mapsto \mathcal{U}$) such that $\lim_{t\to\infty} \mathbf{x}(t; \hat{\mathbf{u}}) = 0$.
Another major problem in control theory arises from the fact that often, not all states are available for measurements or observations. Thus we are faced with the question: given partial information about the states, is it possible to reconstruct the solution trajectory from the measurements/observations? If this is the case, the states can be estimated by state observers. The classical approach leads to the Luenberger observer, but nowadays most frequently the famous Kalman-Bucy filter [KB61] is used as it can be considered as an optimal state observer in a least-squares sense and allows for stochastic uncertainties in the system.
Analyzing the above questions concerning controllability, observability, etc. for general control systems is beyond the scope of linear algebra. Therefore we will mostly focus on linear time-invariant systems which can be analyzed with tools relying on linear algebra techniques. For further reading, see, e.g., [Lev96, Mut99, Son98].
Once the above questions are settled, it is interesting to ask how the desired control objectives can be achieved in an optimal way. The linear-quadratic regulator (LQR) problem is equivalent to a dynamic optimization problem for linear differential equations. Its significance for control theory was fully discovered first by Kalman in 1960 [Kal60]. One of its main applications is to steer the solution of the underlying linear differential equation to a desired reference trajectory with minimal cost given full information on the states. If full information is not available, then the states can be estimated from the measurements or observations using a Kalman-Bucy filter. This leads to the linear-quadratic Gaussian (LQG) control problem. The latter problem and its solution were first described in the classical papers [Kal60, KB61] and are nowadays contained in any textbook on control theory.
In the past decades, the interest has shifted from optimal control to robust control: the question raised is whether a given control law is still able to achieve a desired performance in the presence of uncertain disturbances. In this sense, the LQR control law has some robustness, while the LQG design cannot be considered to be robust [Doy78]. The H_∞ control problem aims at minimizing the worst-case error that can occur if the system is perturbed by exogenous perturbations. It is thus one example of a *robust control* problem. We will only introduce the standard H_∞ control problem, though there exist many other robust control problems and several variations of the H_∞ control problem, see [GL95, PUA00, ZDG96].
Many of the above questions lead to methods that involve the solution of linear and nonlinear matrix equations, in particular Lyapunov, Sylvester, and Riccati equations. For instance, stability, controllability, observability of LTI systems can be related to solutions of Lyapunov equations; see, e.g., [LT85, Section 13] and [HJ91], while the LQR, LQG, and H_∞ control problems lead to the solution of algebraic Riccati equations, see, e.g., [AKFIJ03, Dat04, LR95,

Meh91, Sim96]. Therefore, we will provide the most relevant properties of these matrix equations.

The concepts and solution techniques contained in this section and many other control-related algorithms are implemented in the MATLAB Control System Toolbox, the Subroutine Library in Control SLICOT [BMS+99], and many other computer-aided control systems design tools.

Finally, we note that all concepts described in this section are related to *continuous-time* systems. Analogous concepts hold for *discrete-time systems* whose dynamics are described by difference equations, see, e.g., [Kuc91].

1 BASIC CONCEPTS

Definitions:

Given vector spaces $\mathcal{X}$ (the **state space**), $\mathcal{U}$ (the **input space**), and $\mathcal{Y}$ (the **output space**) and measurable functions $\mathbf{f}, \mathbf{g} : [t_0, t_f] \times \mathcal{X} \times \mathcal{U} \mapsto \mathbb{R}^n$, a **control system** is defined by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \\ \mathbf{y}(t) &= \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)), \end{aligned}$$

where the differential equation is called the **state equation**, the second equation is called the **observer equation**, and $t \in [t_0, t_f]$ ($t_f \in [\,0, \infty\,]$).

Here,

$$\begin{aligned} \mathbf{x} : [t_0, t_f] &\mapsto \mathcal{X} \text{ is the } \textbf{state (vector)}\ , \\ \mathbf{u} : [t_0, t_f] &\mapsto \mathcal{U} \text{ is the } \textbf{control (vector)}\ , \\ \mathbf{y} : [t_0, t_f] &\mapsto \mathcal{Y} \text{ is the } \textbf{output (vector)}\ . \end{aligned}$$

A control system is called **autonomous** (**time-invariant**) if

$$\mathbf{f}(t, \mathbf{x}, \mathbf{u}) \equiv \mathbf{f}(\mathbf{x}, \mathbf{u}) \text{ and } \mathbf{g}(t, \mathbf{x}, \mathbf{u}) \equiv \mathbf{g}(\mathbf{x}, \mathbf{u}).$$

The number of state-space variables n is called the **order** or **degree** of the system.

Let $\mathbf{x}^1 \in \mathbb{R}^n$. A control system with initial value $\mathbf{x}(t_0) = \mathbf{x}^0$ is **controllable to $\mathbf{x}^1$ in time $t_1 > t_0$** if there exists an **admissible** control function $\mathbf{u}$ (i.e., a piecewise continuous or L_2 function $\mathbf{u} : [t_0, t_1] \mapsto \mathcal{U}$) such that $\mathbf{x}(t_1; \mathbf{u}) = \mathbf{x}^1$. (Equivalently, $(t_1, \mathbf{x}^1)$ is **reachable from** $(t_1, \mathbf{x}^0)$.)

A control system with initial value $\mathbf{x}(t_0) = \mathbf{x}^0$ is **controllable to $\mathbf{x}^1$** if there exists $t_1 > t_0$ such that $(t_1, \mathbf{x}^1)$ is reachable from $(t_0, \mathbf{x}^0)$.

If the control system is controllable to all $\mathbf{x}^1 \in \mathcal{X}$ for all $(t_0, \mathbf{x}^0)$ with $\mathbf{x}^0 \in \mathcal{X}$, it is **(completely) controllable.**

A control system is **linear** if $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, $\mathcal{Y} = \mathbb{R}^p$ and

$$\begin{aligned} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) &= A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t), \\ \mathbf{g}(t, \mathbf{x}, \mathbf{u}) &= C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t), \end{aligned}$$

where $A : [t_0, t_f] \mapsto \mathbb{R}^{n\times n}, B : [t_0, t_f] \mapsto \mathbb{R}^{n\times m}, C : [t_0, t_f] \mapsto \mathbb{R}^{p\times n}, D : [t_0, t_f] \mapsto \mathbb{R}^{p\times m}$ are smooth functions.

A linear **time-invariant system** (**LTI system**) has the form

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t), \\
\mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t),
\end{aligned}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$.

An LTI system is **(asymptotically) stable** if the corresponding linear homogeneous ODE $\dot{\mathbf{x}} = A\mathbf{x}$ is (asymptotically) stable. (For a definition of (asymptotic) stability confer §12.1, §12.2.)

An LTI system is **stabilizable (by state feedback)** if there exists an admissible control in the form of a **state feedback**

$$\mathbf{u}(t) = F\mathbf{x}(t), \quad F \in \mathbb{R}^{m\times n},$$

such that the unique solution of the corresponding **closed-loop** ODE

$$\dot{\mathbf{x}}(t) = (A + BF)\mathbf{x}(t) \tag{1}$$

is asymptotically stable.

An LTI system is **observable (reconstructible)** if for two solution trajectories $\mathbf{x}(t)$ and $\tilde{\mathbf{x}}(t)$ of its state equation, it holds that

$$C\mathbf{x}(t) = C\tilde{\mathbf{x}}(t) \quad \forall t \leq t_0 \ (\forall t \geq t_0)$$

implies $\mathbf{x}(t) = \tilde{\mathbf{x}}(t) \quad \forall t \leq t_0 \ (\forall t \geq t_0)$.

An LTI system is **detectable** if for any solution $\mathbf{x}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$ with $C\mathbf{x}(t) \equiv 0$ we have $\lim_{t\to\infty} \mathbf{x}(t) = 0$.

Facts:

1. For LTI systems, all controllability and reachability concepts are equivalent. Therefore, we only speak of controllability of LTI systems.

2. Observability implies that one can obtain all necessary information about the LTI system from the output equation.

3. Detectability weakens observability in the same sense as stabilizability weakens controllability: not all of $\mathbf{x}$ can be observed, but the unobservable part is asymptotically stable.

4. Observability (detectability) and controllability (stabilizability) are dual concepts in the following sense: an LTI system is observable (detectable) if and only if the dual system

$$\dot{\mathbf{z}}(t) = A^T\mathbf{z}(t) + C^T\mathbf{v}(t)$$

is controllable (stabilizable). This fact is sometimes called the **duality principle of control theory.**

Examples:

1. A fundamental problem in robotics is to control the position of a single–link rotational joint using a motor placed at the "pivot". A simple mathematical model for this is the pendulum [Son98]. Applying a torque $\mathbf{u}$ as external force, this can serve as a means to control the motion of the pendulum; see Figure 1.

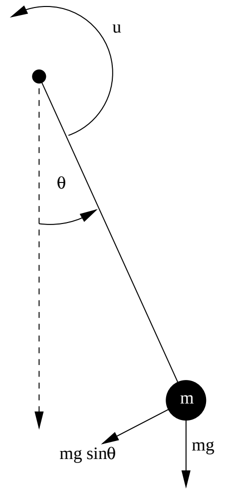

Figure 1: Pendulum as mathematical model of a single-link rotational joint

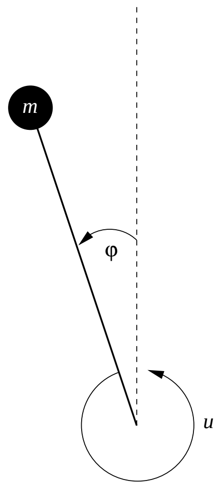

Figure 2: Inverted pendulum: apply control to move to upright position.

If we neglect friction and assume that the mass is concentrated at the tip of the pendulum, Newton's law for rotating objects

$$m\ddot{\Theta}(t) + mg\sin\Theta(t) = \mathbf{u}(t)$$

describes the counter clockwise movement of the angle between the vertical axis and the pendulum subject to the control $\mathbf{u}(t)$. This is a first example of a (nonlinear) control system if we set

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \Theta(t) \\ \dot{\Theta}(t) \end{bmatrix}, \\ \mathbf{f}(t,\mathbf{x},\mathbf{u}) &= \begin{bmatrix} x_2 \\ -mg\sin(x_1) \end{bmatrix}, \quad \mathbf{g}(t,\mathbf{x},\mathbf{u}) = x_1, \end{aligned}$$

where we assume that only $\Theta(t)$ can be measured, but not the angular velocity $\dot{\Theta}(t)$

For $\mathbf{u}(t) \equiv 0$, the stationary position $\Theta = \pi, \dot{\Theta} = 0$ is an unstable equilibrium, i.e., small perturbations will lead to unstable motion. The objective now is to apply a torque (control $\mathbf{u}$) to correct for deviations from this unstable equilibrium, i.e., to keep the pendulum in the upright position, see Figure 2.

2. Scaling the variables such that $m = 1 = g$ and assuming a small perturbation $\Theta - \pi$ in the inverted pendulum problem described above, we have

$$\sin\Theta = -(\Theta - \pi) + o((\Theta - \pi)^2).$$

(Here, $\mathbf{g}(x) = o(x)$ if $\lim\limits_{x\to\infty} \frac{\mathbf{g}(x)}{x} = 0$.) This allows us to linearize the control system in order to obtain a linear control system for $\varphi(t) := \Theta(t) - \pi$:

$$\ddot{\varphi}(t) - \varphi(t) = \mathbf{u}(t).$$

This can be written as an LTI system, assuming only positions can be observed, with

$$\mathbf{x} = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

Now the objective translates to: given initial values $x_1(0) = \varphi(0)$, $x_2(0) = \dot{\varphi}(0)$, find $\mathbf{u}(t)$ to bring $\mathbf{x}(t)$ to zero "as fast as possible". It is usually an additional goal to avoid overshoot and oscillating behaviour as much as possible.

2 FREQUENCY-DOMAIN ANALYSIS

So far LTI systems are treated in state-space. In systems and control theory, it is often beneficial to use the **frequency domain** formalism obtained from applying the Laplace transformation to its state and observer equations.

Definitions:
The rational matrix function

$$G(s) = C(sI - A)^{-1}B + D \in \mathbb{R}^{p\times m}[s]$$

is called the **transfer function** of the LTI system defined in §12.3.1.

In a **frequency domain analysis**, $G(s)$ is evaluated for $s = i\omega$, where $\omega \in [0, \infty]$ has the physical interpretation of a frequency and the input is considered as a signal with frequency ω.

The L_∞**-norm** of a transfer function is the operator norm induced by the frequency domain analogue of the L_2-norm which applies to Laplace transformed input functions $\mathbf{u} \in L_2(-\infty, \infty; \mathbb{R}^m)$, where $L_2(a, b; \mathbb{R}^m)$ is the Lebesgue space of square-integrable, measurable functions on the interval $(a, b) \subset \mathbb{R}$ with values in $\mathbb{R}^m$.

The $p \times m$-matrix-valued functions G for which $\|G\|_{L_\infty}$ is bounded form the space L_∞.

The subset of L_∞ containing all $p \times m$-matrix-valued functions that are analytical and bounded in the open right half complex plane form the **Hardy space** H_∞.

The H_∞**-norm** of $G \in H_\infty$ is defined as

$$\|G\|_{H_\infty} = \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \sigma_{\max}(G(i\omega)), \tag{2}$$

where $\sigma_{\max}(M)$ is the maximum singular value of the matrix M and $\operatorname{ess\,sup}_{t\in M} h(t)$ is the essential supremum of a function h evaluated on the set M, that is the function's supremum on $M \setminus L$ where L is a set of Lebesgue measure zero.

For $T \in \mathbb{R}^{n \times n}$ nonsingular, the mapping implied by

$$(A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$$

is called a **state-space transformation**.

(A, B, C, D) is called a **realization** of an LTI system if its transfer function can be expressed as $G(s) = C(sI_n - A)^{-1}B + D$.

The minimum number $\hat{n}$ so that there exists no realization of a given LTI system with $n < \hat{n}$ is called the **McMillan degree** of the system.

A realization with $n = \hat{n}$ is a **minimal realization**.

Facts:

1. If $\mathbf{X}, \mathbf{Y}, \mathbf{U}$ are the Laplace transforms of $\mathbf{x}, \mathbf{y}, \mathbf{u}$, respectively, s is the Laplace variable and $\mathbf{x}(0) = 0$, the state and observer equation of an LTI system transform to

$$\begin{aligned} s\mathbf{X}(s) &= A\mathbf{X}(s) + B\mathbf{U}(s), \\ \mathbf{Y}(s) &= C\mathbf{X}(s) + D\mathbf{U}(s). \end{aligned}$$

   Thus, the resulting input-output relation

$$\mathbf{Y}(s) = \left(C(sI - A)^{-1}B + D\right)\mathbf{U}(s) = G(s)\mathbf{U}(s) \tag{3}$$

   is completely determined by the transfer function of the LTI system.

2. As a consequence of the maximum modulus theorem, H_∞ functions must be bounded on the imaginary axis so that the essential supremum in the definition of the H_∞-norm simplifies to a supremum for rational functions G.

3. The transfer function of an LTI system is invariant w.r.t. state-space transformations:

$$D + (CT^{-1})(sI - TAT^{-1})^{-1}(TB) = C(sI_n - A)^{-1}B + D = G(s).$$

   Consequently, there exist infinitely many realizations of an LTI system.

4. Adding zero inputs/outputs does not change the transfer function, thus the order n of the system can be increased arbitrarily.

Examples:

1. The LTI system corresponding to the inverted pendulum has the transfer function

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} = \frac{1}{s^2 - 1}.$$

2. The L_∞-norm of the transfer function corresponding to the inverted pendulum is

$$\|G\|_{L_\infty} = 1.$$

3. The transfer function corresponding to the inverted pendulum is not in H_∞ as $G(s)$ has a pole at $s = 1$ and thus is not bounded in the right half plane.

3 ANALYSIS OF LTI SYSTEMS

In this section we provide characterizations of the properties of LTI systems defined in the introduction. Controllability and the related concepts can be checked using several algebraic criteria.

Definitions:
A matrix $A \in \mathbb{R}^{n\times n}$ is **Hurwitz** or **(asymptotically) stable** if all its eigenvalues have strictly negative real part.
The **controllability matrix** corresponding to an LTI system is

$$\mathcal{C}(A,B) = [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times n\cdot m}.$$

The **observability matrix** corresponding to an LTI system is

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{np\times n}.$$

The following transformations are **state-space transformations**

- **Change of Basis:**

$$\begin{array}{llll} \mathbf{x} \mapsto P\mathbf{x} & \text{for} & P \in \mathbb{R}^{n\times n} & \text{nonsingular,} \\ \mathbf{u} \mapsto Q\mathbf{u} & \text{for} & Q \in \mathbb{R}^{m\times m} & \text{nonsingular,} \\ \mathbf{y} \mapsto R\mathbf{y} & \text{for} & R \in \mathbb{R}^{p\times p} & \text{nonsingular.} \end{array}$$

- **Linear state feedback:** $\mathbf{u} \mapsto F\mathbf{x} + \mathbf{v}, F \in \mathbb{R}^{m\times n}$, $\mathbf{v} : [t_0, t_f] \mapsto \mathbb{R}^m$.

- **Linear output feedback:** $\mathbf{u} \mapsto G\mathbf{y} + \mathbf{v}, G \in \mathbb{R}^{m+p}$, $\mathbf{v} : [t_0, t_f] \mapsto \mathbb{R}^m$.

The **Kalman decomposition** of (A, B) is

$$V^TAV = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad V^TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad V \in \mathbb{R}^{n\times n} \text{ orthogonal,}$$

where (A_1, B_1) is controllable.
The **observability Kalman decomposition** of (A, C) is,

$$W^TAW = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, \quad CW = [C_1 \ 0], \quad W \in \mathbb{R}^{n\times n} \text{ orthogonal,}$$

where (A_1, C_1) is observable.

Facts:

1. An LTI system is asymptotically stable if and only if A is Hurwitz.
2. For a given LTI system, the following are equivalent.
   a) The LTI system is controllable.
   b) The controllability matrix corresponding to the LTI system has full (row) rank, i.e., $\operatorname{rank}\mathcal{C}(A,B) = n$.
   c) **(Hautus-Popov test)** If $\mathbf{p}$ is a left eigenvector of A, then $\mathbf{p}^*B \neq 0$.
   d) $\operatorname{rank}([\lambda I - A, B]) = n \ \forall \lambda \in \mathbb{C}$.

   The essential part of the proof of the above characterizations (which is "d)⇒b)") is an application of the Cayley-Hamilton theorem.
3. For a given LTI system, the following are equivalent:
   a) The LTI system is stabilizable, i.e., $\exists F \in \mathbb{R}^{m\times n}$ such that $A + BF$ is Hurwitz.
   b) **(Hautus-Popov test)** If $\mathbf{p} \neq 0$, $\mathbf{p}^*A = \lambda \mathbf{p}^*$ and $\mathrm{Re}(\lambda) \geq 0$, then $\mathbf{p}^*B \neq 0$.
   c) $\operatorname{rank}([A - \lambda I, B]) = n \quad \forall \lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$.
   d) In the Kalman decomposition of (A, B), A_3 is Hurwitz.
4. Using the change of basis $\tilde{\mathbf{x}} = V^T\mathbf{x}$ implied by the Kalman decomposition we obtain

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}_1 &= A_1\tilde{\mathbf{x}}_1 + A_2\tilde{\mathbf{x}}_2 + B_1\mathbf{u}\\ \dot{\tilde{\mathbf{x}}}_2 &= A_3\tilde{\mathbf{x}}_2.\end{aligned}$$

   Thus, $\tilde{\mathbf{x}}_2$ is not controllable. The eigenvalues of A_3 are therefore called **uncontrollable modes**.
5. For a given LTI system, the following are equivalent:
   a) The LTI system is observable.
   c) The observability matrix corresponding to the LTI system has full (column) rank, i.e., $\operatorname{rank}\mathcal{O}(A, C) = n$.
   d) **(Hautus-Popov test)** $A\mathbf{p} = \lambda\mathbf{p} \Longrightarrow C^T\mathbf{p} \neq 0$.
   e) $\operatorname{rank}\begin{bmatrix}\lambda I - A\\ C\end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}$.
6. For a given LTI system, the following are equivalent:
   a) The LTI system is detectable.
   b) The dual system $\dot{\mathbf{z}} = A^T\mathbf{z} + C^T\mathbf{v}$ is stabilizable.
   c) **(Hautus-Popov test)** $A\mathbf{p} = \lambda\mathbf{p}, \mathrm{Re}(\lambda) \geq 0 \Longrightarrow C^T\mathbf{p} \neq 0$.
   d) $\operatorname{rank}\begin{bmatrix}\lambda I - A\\ C\end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$.

e) In the observability Kalman decomposition of (A, C), A_3 is Hurwitz.

7. Using the change of basis $\tilde{\mathbf{x}} = W^T\mathbf{x}$ implied by the observability Kalman decomposition we obtain

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}_1 &= A_1\tilde{\mathbf{x}}_1 + B_1\mathbf{u}, \\ \tilde{\mathbf{x}}_2 &= A_2\tilde{\mathbf{x}}_1 + A_3\tilde{\mathbf{x}}_2 + B_2\mathbf{u}, \\ \mathbf{y} &= C_1\tilde{\mathbf{x}}_1 \end{aligned}$$

   Thus, $\tilde{\mathbf{x}}_2$ is not observable. The eigenvalues of A_3 are therefore called **unobservable modes**.

8. The characterizations of observability and detectability are proved using the duality principle and the characterizations of controllability and stabilizability.

9. If an LTI system is controllable (observable, stabilizable, detectable), then the corresponding LTI system resulting from a state-space transformation is controllable (observable, stabilizable, detectable).

10. For $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ there exist $P \in \mathbb{R}^{n\times n}, Q \in \mathbb{R}^{m\times m}$ orthogonal such that

$$PAP^T = \begin{array}{cc} \left[\begin{array}{ccccc|c} A_{11} & & & & A_{1,s-1} & A_{1,s} \\ A_{21} & \ddots & & & \vdots & \vdots \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & & & & \\ 0 & \cdots & 0 & A_{s-1,s-2} & A_{s-1,s-1} & A_{s-1,s} \\ 0 & \cdots & 0 & 0 & 0 & A_{ss} \end{array}\right] & \begin{array}{c} n_1 \\ n_2 \\ \\ \\ n_{s-1} \\ n_s \end{array} \\ \begin{array}{cccccc} n_1 & & & n_{s-2} & n_{s-1} & n_s \end{array} & \end{array},$$

$$PBQ = \begin{array}{cc} \left[\begin{array}{cc} B_1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array}\right] & \begin{array}{c} n_1 \\ n_2 \\ \vdots \\ n_s \end{array} \\ \begin{array}{cc} n_1 & m - n_1 \end{array} & \end{array},$$

   where $n_1 \geq n_2 \geq \ldots \geq n_{s-1} \geq n_s \geq 0$, $n_{s-1} > 0$, $A_{i,i-1} = [\Sigma_{i,i-1}\ 0] \in \mathbb{R}^{n_1\times n_{i-1}}$, $\Sigma_{i,i-1} \in \mathbb{R}^{n_i\times n_i}$ nonsingular for $i = 1, \ldots, s-1$, $\Sigma_{s-1,s-2}$ is diagonal, and B_1 is nonsingular.

   Moreover, this transformation to **staircase form** can be computed by a finite sequence of singular value decompositions.

11. An LTI system is controllable if in the staircase form of (A, B), $n = 0$.

12. An LTI system is observable if $n_s = 0$ in the staircase form of (A^T, C^T).

13. An LTI system is stabilizable if in the staircase form of (A, B), A_{ss} is Hurwitz.

14. An LTI system is detectable if in the staircase form of (A^T, C^T), A_{ss} is Hurwitz.

15. In case $m = 1$, the staircase form of (A, B) is given by

$$PAP^T = \begin{bmatrix} a_{11} & \dots & \dots & a_{1,n} \\ a_{21} & & & \vdots \\ & \ddots & & \vdots \\ & & a_{n,n-1} & a_{n,n} \end{bmatrix}, PB = \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and is called the **controllability Hessenberg form**. The corresponding staircase from of (A^T, C^T) in case $p = 1$ is called the **observability Hessenberg form**.

Examples:

1. The LTI system corresponding to the inverted pendulum problem is not asymptotically stable as A is not Hurwitz: $\sigma(A) = \{\pm 1\}$.
2. The LTI system corresponding to the inverted pendulum problem is controllable as the controllability matrix
$$\mathcal{C}(A, B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
has full rank. Thus, it is also stabilizable.
3. The LTI system corresponding to the inverted pendulum problem is observable as the observability matrix
$$\mathcal{O}(A, C) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
has full rank. Thus, it is also detectable

4 MATRIX EQUATIONS

A fundamental role in many tasks in control theory is played by matrix equations. We therefore review their most important properties. More details can be found in [AKFIJ03, HJ91, LR95, LT85].

Definitions:

A linear matrix equation of the form

$$AX + XB = W, \qquad A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times m}, W \in \mathbb{R}^{n \times m},$$

is called **Sylvester equation**.

A linear matrix equation of the form

$$AX + XA^T = W, \qquad A \in \mathbb{R}^{n \times n}, W = W^T \in \mathbb{R}^{n \times n},$$

is called **Lyapunov equation**.

A quadratic matrix equation of the form

$$0 = Q + A^T X + XA - XGX, \qquad A \in \mathbb{R}^{n\times n}, G = G^T, Q = Q^T \in \mathbb{R}^{n\times n},$$

is called **algebraic Riccati equation (ARE)**.

Facts:

1. The Sylvester equation is equivalent to the linear system of equations

$$\left[(I_m \otimes A) + (B^T \otimes I_n)\right] \operatorname{vec}(X) = \operatorname{vec}(W),$$

   where $\otimes$ and vec denote the Kronecker product and the vec-operator defined in §2.5.4. Thus, the Sylvester equation has a unique solution if and only if $\sigma(A) \cap \sigma(-B) = \emptyset$.

2. The Lyapunov equation is equivalent to the linear system of equations

$$[(I_m \otimes A) + (A \otimes I_n)] \operatorname{vec}(X) = \operatorname{vec}(W).$$

   Thus, it has a unique solution if and only if $\sigma(A) \cap \sigma(-A^T) = \emptyset$. In particular, this holds if A is Hurwitz.

3. If G and Q are positive semidefinite with (A, G) stabilizable and (A, Q) detectable, then the ARE has a unique positive semidefinite solution X_* with the property that $\sigma(A-GX_*)$ is Hurwitz.

4. If the assumptions given above are not satisfied, there may or may not exist a stabilizing solution with the given properties. Besides, there may exist a continuum of solutions, a finite number of solutions, or no solution at all. The solution theory for AREs is a vast topic by itself; see the monographs [AKFIJ03, LR95] and [Ben99, Dat04, Meh91, Sim96] for numerical algorithms to solve these equations.

Examples:

1. For

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

   a solution of the Sylvester equation is

$$X = \frac{1}{4}\begin{bmatrix} -3 & 3 \\ 1 & -3 \end{bmatrix}.$$

   Note that $\sigma(A) = \sigma(B) = \{1, 1\}$ so that $\sigma(A) \cap \sigma(-B) = \emptyset$. Thus, this Sylvester equation has the unique solution X given above.

2. For
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$
the stabilizing solution of the associated ARE is
$$X_* = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
and the spectrum of the closed-loop matrix
$$A - GX_* = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$
is $\{-1, \ -1\}$.

3. Consider the ARE
$$0 = C^TC + A^TX + XA - XBB^TX$$
corresponding to an LTI system with
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix}, \quad D = 0.$$
For this ARE, $X = \begin{bmatrix} -1+\sqrt{3} & 0 \\ 0 & \xi \end{bmatrix}$ is a solution for all $\xi \in \mathbb{R}$. It is positive semidefinite for all $\xi \geq 0$, but this ARE does not have a stabilizing solution as the LTI system is neither stabilizable nor detectable.

5 STATE ESTIMATION

In this section we present the two most famous approaches to state observation, that is, finding a function $\hat{\mathbf{x}}(t)$ that approximates the state $\mathbf{x}(t)$ of a given LTI system if only its inputs $\mathbf{u}(t)$ and outputs $\mathbf{y}(t)$ are known. While the the first approach (the Luenberger observer) assumes a deterministic system behaviour, the Kalman-Bucy filter allows for uncertainty in the system, modelled by white-noise, zero-mean stochastic processes.

Definitions:
Given an LTI system with $D = 0$, a **state observer** is a function
$$\hat{\mathbf{x}} : [0, \infty) \mapsto \mathbb{R}^n$$
such that for some nonsingular matrix $Z \in \mathbb{R}^{n \times n}$ and $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - Z\mathbf{x}(t)$, we have
$$\lim_{t \to \infty} \mathbf{e}(t) = 0.$$
Given an LTI system with stochastic disturbances
$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) + \tilde{B}\mathbf{w}(t), \\ \mathbf{y}(t) &= C\mathbf{x}(t) + \mathbf{v}(t), \end{aligned}$$

where A, B, C are as before, $\tilde{B} \in \mathbb{R}^{n\times\tilde{m}}$, and $\mathbf{w}(t), \mathbf{v}(t)$ are white-noise, zero-mean stochastic processes with corresponding covariance matrices $W = W^T \in \mathbb{R}^{\tilde{m}\times\tilde{m}}$ (positive semidefinite), $V = V^T \in \mathbb{R}^{p\times p}$ (positive definite), the problem to minimize the mean square error

$$E\left[\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\|_2^2\right]$$

over all state observers is called the **optimal estimation problem**. (Here, $E[r]$ is the **expected value** of r.)

Facts:

1. A state observer, called the **Luenberger observer**, is obtained as the solution of the dynamical system
$$\dot{\hat{\mathbf{x}}}(t) = H\hat{\mathbf{x}}(t) + F\mathbf{y}(t) + G\mathbf{u}(t),$$
where $H \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times p}$ are chosen so that H is Hurwitz and the **Sylvester observer equation**
$$HX - XA + FC = 0$$
has a nonsingular solution X. Then $G = XB$ and the matrix Z in the definition of the state observer equals the solution of X of the Sylvester observer equation.

2. Assuming that
   - $\mathbf{w}$ and $\mathbf{v}$ are uncorrelated stochastic processes,
   - the initial state $\mathbf{x}^0$ is a Gaussian zero-mean random variable, uncorrelated with $\mathbf{w}$ and $\mathbf{v}$,
   - (A, B) is controllable and (A, C) is observable,

   the solution to the optimal estimation problem is given by the **Kalman-Bucy** filter, defined as the solution of the linear differential equation
$$\dot{\hat{\mathbf{x}}}(t) = (A - Y_* C^T V^{-1} C)\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + Y_* C^T V^{-1}\mathbf{y}(t),$$
where Y_* is the unique stabilizing solution of the **filter ARE**
$$0 = \tilde{B}W\tilde{B}^T + AY + YA^T - YC^T V^{-1} CY.$$

3. Under the same assumptions as above, the stabilizing solution of the filter ARE can be shown to be symmetric positive definite.

Examples:

1. A Luenberger observer for the LTI system corresponding to the inverted pendulum problem can be constructed as follows: choose $H = \mathrm{diag}(-2, -\frac{1}{2})$ and $F = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$. Then the Sylvester observer equation has the unique solution
$$X = \frac{1}{3}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}.$$
Note that X is nonsingular. We thus get $G = XB = \frac{1}{3}\begin{bmatrix} -2 & 4 \end{bmatrix}$.

2. Consider the inverted pendulum with disturbances v, w and $\tilde{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Assume that $V = W = 1$. The Kalman-Bucy filter is determined via the filter ARE, yielding

$$Y_* = (1 + \sqrt{2}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus, the state estimation obtained from the Kalman filter is given by the solution of

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} -1 - \sqrt{2} & 1 \\ -\sqrt{2} & 0 \end{bmatrix} \hat{\mathbf{x}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t) + (1 + \sqrt{2}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{y}(t).$$

6 CONTROL DESIGN FOR LTI SYSTEMS

This section provides the background for some of the most important control design methods.

Definitions:
A **(feedback) controller** for an LTI system is given by another LTI system

$$\begin{aligned} \dot{\mathbf{r}}(t) &= E\mathbf{r}(t) + F\mathbf{y}(t), \\ \mathbf{u}(t) &= H\mathbf{r}(t) + K\mathbf{y}(t), \end{aligned}$$

where $E \in \mathbb{R}^{N \times N}$, $F \in \mathbb{R}^{N \times p}$, $H \in \mathbb{R}^{m \times N}$, $K \in \mathbb{R}^{m \times p}$, and the "output" $\mathbf{u}(t)$ of the controller serves as the input for the original LTI system.

If E, F, H are zero matrices, a controller is called **static feedback**, otherwise it is called a **dynamic compensator**.

A static feedback control law is a **state feedback** if in the controller equations, the output function $\mathbf{y}(t)$ is replaced by the state $\mathbf{x}(t)$, otherwise it is called **output feedback**.

The **closed-loop system** resulting from inserting the control law $\mathbf{u}(t)$ obtained from a dynamic compensator into the LTI system is illustrated by the block diagram in Figure 3, where $\mathbf{w}$ is as in the definition of LTI systems with stochastic disturbances and $\mathbf{z}$ will only be needed later when defining the H_∞ control problem.

The linear-quadratic optimization (optimal control) problem

$$\min_{\mathbf{u} \in L_2(0,\infty;\mathcal{U})} \mathcal{J}(\mathbf{u}), \quad \text{where } \mathcal{J}(\mathbf{u}) = \frac{1}{2} \int_0^\infty \left(\mathbf{y}(t)^T Q \mathbf{y}(t) + \mathbf{u}(t)^T R \mathbf{u}(t) \right) dt$$

subject to the dynamical constraint given by an LTI system is called the **linear-quadratic regulator (LQR) problem**.

The linear-quadratic optimization (optimal control) problem

$$\min_{\mathbf{u} \in L_2(0,\infty;\mathcal{U})} \mathcal{J}(\mathbf{u}), \quad \text{where } \mathcal{J}(\mathbf{u}) = \lim_{t_f \to \infty} \frac{1}{2t_f} E \left[\int_{-t_f}^{t_f} \left(\mathbf{y}(t)^T Q \mathbf{y}(t) + \mathbf{u}(t)^T R \mathbf{u}(t) \right) dt \right]$$

subject to the dynamical constraint given by an LTI system with stochastic disturbances is called the **linear-quadratic Gaussian (LQG) problem**.

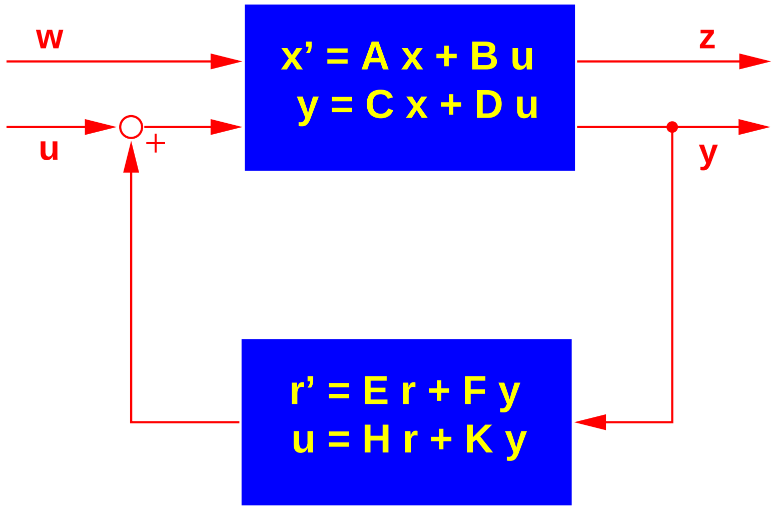

Figure 3: Closed-loop diagram of an LTI system and a dynamic compensator.

Consider an LTI system where inputs and outputs are split into two parts, so that instead of $B\mathbf{u}(t)$ we have

$$B_1\mathbf{w}(t) + B_2\mathbf{u}(t),$$

and instead of $\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$ we write

$$\begin{aligned}
\mathbf{z}(t) &= C_1\mathbf{x}(t) + D_{11}\mathbf{w}(t) + D_{12}\mathbf{u}(t),\\
\mathbf{y}(t) &= C_2\mathbf{x}(t) + D_{21}\mathbf{w}(t) + D_{22}\mathbf{u}(t),
\end{aligned}$$

where $\mathbf{u}(t) \in \mathbb{R}^{m_2}$ denotes the control input, $\mathbf{w}(t) \in \mathbb{R}^{m_1}$ is an **exogenous input** that may include noise, linearization errors and unmodeled dynamics, $\mathbf{y}(t) \in \mathbb{R}^{p_2}$ contains **measured outputs**, while $\mathbf{z}(t) \in \mathbb{R}^{p_1}$ is the **regulated output** or an **estimation error**. Let $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right]$ denote the corresponding transfer function such that

$$\begin{bmatrix} \mathbf{Z} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{U} \end{bmatrix},$$

where $\mathbf{Y}, \mathbf{Z}, \mathbf{U}, \mathbf{W}$ denote the Laplace transforms of $\mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{w}$.
The **optimal** H_∞ **control problem** is then to determine a dynamic compensator

$$\begin{aligned}
\dot{\mathbf{r}}(t) &= E\mathbf{r}(t) + F\mathbf{y}(t),\\
\mathbf{u}(t) &= H\mathbf{r}(t) + K\mathbf{y}(t),
\end{aligned}$$

with $E \in \mathbb{R}^{N\times N}$, $F \in \mathbb{R}^{N\times p_2}$, $H \in \mathbb{R}^{m_2\times N}$, $K \in \mathbb{R}^{m_2\times p_2}$ and transfer function $M(s) = H(sI - E)^{-1}F + K$ such that the resulting closed-loop system

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= (A + B_2KZ_1C_2)\mathbf{x}(t) + (B_2Z_2H)\mathbf{r}(t) + (B_1 + B_2KZ_1D_{21})\mathbf{w}(t),\\
\dot{\mathbf{r}}(t) &= FZ_1C_2\mathbf{x}(t) + (E + FZ_1D_{22}H)\mathbf{r}(t) + FZ_1D_{21}\mathbf{w}(t),\\
\mathbf{z}(t) &= (C_1 + D_{12}Z_2KC_2)\mathbf{x}(t) + D_{12}Z_2H\mathbf{r}(t) + (D_{11} + D_{12}KZ_1D_{21})\mathbf{w}(t),
\end{aligned}$$

with $Z_1 = (I - D_{22}K)^{-1}$ and $Z_2 = (I - KD_{22})^{-1}$,

- is **internally stable**, i.e., the solution of the system with $\mathbf{w}(t) \equiv 0$ is asymptotically stable, and
- the closed-loop transfer function $T_{zw}(s) = G_{22}(s) + G_{21}(s)M(s)(I - G_{11}(s)M(s))^{-1}G_{12}(s)$ from $\mathbf{w}$ to $\mathbf{z}$ is minimized in the H_∞-norm.

The **suboptimal H_∞ control problem** is to find an internally stabilizing controller so that

$$\|T_{zw}\|_{H_\infty} < \gamma,$$

where $\gamma > 0$ is a robustness threshold.

Facts:

1. If $D = 0$ and the LTI system is both stabilizable and detectable, the weighting matrix Q is positive semidefinite and R is positive definite, then the solution of the LQR problem is given by the state feedback controller

$$\mathbf{u}_*(t) \;=\; -R^{-1}B^TX_*\mathbf{x}(t), \qquad t \geq 0,$$

   where X_* is the unique stabilizing solution of the LQR ARE

$$0 = C^TQC + A^TX + XA - XBR^{-1}B^TX.$$

2. The LQR problem does not require an observer equation — inserting $\mathbf{y}(t) = C\mathbf{x}(t)$ into the cost functional, we obtain a problem formulation depending only on states and inputs:

$$\begin{aligned} \mathcal{J}(\mathbf{u}) &= \frac{1}{2}\int_0^\infty \big(\mathbf{y}(t)^TQ\mathbf{y}(t) + \mathbf{u}(t)^TR\mathbf{u}(t)\big)\,dt \\ &= \frac{1}{2}\int_0^\infty \big(\mathbf{x}(t)^TC^TQC\mathbf{x}(t) + \mathbf{u}(t)^TR\mathbf{u}(t)\big)\,dt. \end{aligned}$$

3. Under the given assumptions, it can also be shown that X_* is symmetric and the unique positive semidefinite matrix among all solutions of the LQR ARE.

4. The assumptions for the feedback solution of the LQR problem can be weakened in several aspects, see, e.g., [Gee89, SSC95].

5. Assuming that
   - $\mathbf{w}$ and $\mathbf{v}$ are uncorrelated stochastic processes,
   - the initial state $\mathbf{x}^0$ is a Gaussian zero-mean random variable, uncorrelated with $\mathbf{w}$ and $\mathbf{v}$,
   - (A, B) is controllable and (A, C) is observable,

the solution to the LQG problem is given by the feedback controller

$$\mathbf{u}(t) = -R^{-1}B^T X_* \hat{\mathbf{x}}(t),$$

where X_* is the solution of the LQR ARE and $\hat{\mathbf{x}}$ is the Kalman-Bucy filter

$$\dot{\hat{\mathbf{x}}}(t) = (A - BR^{-1}B^T X_* - Y_* C^T V^{-1} C)\hat{\mathbf{x}}(t) + Y_* C^T V^{-1} \mathbf{y}(t),$$

corresponding to the closed-loop system resulting from the LQR solution with Y_* being the stabilizing solution of the corresponding filter ARE.

6. In principle, there is no restriction on the degree N of the H_∞ controller, although, smaller dimensions N are preferred for practical implementation and computation.

7. The state-space solution to the H_∞ suboptimal control problem [DGKF89] relates H_∞ control to AREs: under the assumptions that

   - (A, B_k) is stabilizable and (A, C_k) is detectable for $k = 1, 2$,
   - $D_{11} = 0$, $D_{22} = 0$, and

   $$D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

   a suboptimal H_∞ controller exists if and only if the AREs

   $$\begin{aligned} 0 &= C_1^T C_1 + AX + XA^T + X(\frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T)X \\ 0 &= B_1^T B_1 + A^T Y + YA + Y(\frac{1}{\gamma^2} C_1 C_1^T - C_2 C_2^T)Y \end{aligned}$$

   both have positive semidefinite stabilizing solutions X_∞ and Y_∞, respectively, satisfying the spectral radius condition

   $$\rho(XY) < \gamma^2.$$

8. The solution of the optimal H_∞ control problem can be obtained by a bisection method (or any other root-finding method) minimizing γ based on the characterization of a H_∞ suboptimal controller given in 7., starting from γ_0 for which no suboptimal H_∞ controller exists and γ_1 for which the above conditions are satisfied.

9. The assumptions made for the state-space solution of the H_∞ control problem can mostly be relaxed.

10. The robust numerical solution of the H_∞ control problem is a topic of ongoing research—the solution via AREs may suffer from several difficulties in the presence of roundoff errors and should be avoided if possible. One way out is a reformulation of the problem using structured generalized eigenvalue problems, see [BBMX99b, CS92, GL97].

11. Once a (sub-)optimal γ is found, it remains to determine a realization of the H_∞ controller. One possibility is the **central (minimum entropy) controller** [ZDG96]:

$$\begin{aligned} E &= A + \frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T X_\infty - Z_\infty Y_\infty C_2^T C_2, \\ F &= Z_\infty Y_\infty C_2^T, \quad K \;=\; -B_2^T X_\infty, \quad H \;=\; 0, \end{aligned}$$

where

$$Z_\infty = (I - \frac{1}{\gamma^2} Y_\infty X_\infty)^{-1}.$$

Examples:

1. The cost functional in the LQR and LQG problems values the energy needed to reach the desired state by the weighting matrix R on the inputs. Thus, usually

$$R = \mathrm{diag}(\rho_1, \ldots, \rho_m).$$

The weighting on the states or outputs in the LQR or LQG problems is usually used to penalize deviations from the desired state of the system and is often also given in diagonal form. Common examples of weighting matrices are $R = \rho I_m$, $Q = \gamma I_p$ for $\rho, \gamma > 0$.

2. The solution to the LQR problem for the inverted pendulum with $Q = R = 1$ is given via the stabilizing solution of the LQR ARE which is

$$X_* = \begin{bmatrix} 2\sqrt{1+\sqrt{2}} & 1+\sqrt{2} \\ 1+\sqrt{2} & \sqrt{2}\sqrt{1+\sqrt{2}} \end{bmatrix},$$

resulting in the state feedback law

$$\mathbf{u}(t) = -\begin{bmatrix} 1+\sqrt{2} & \sqrt{2}\sqrt{1+\sqrt{2}} \end{bmatrix} \mathbf{x}(t).$$

The eigenvalues of the closed-loop system are (up to four digits) $\sigma(A - BR^{-1}B^T X_*) = \{-1.0987 \pm 0.4551i\}$.

3. The solution to the LQG problem for the inverted pendulum with Q, R as above and uncertainties $\mathbf{v}, \mathbf{w}$ with $\tilde{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is obtained by combining the LQR solution derived above with the Kalman-Bucy filter obtained as in the examples part of the previous section.

Thus we get the LQG control law

$$\mathbf{u}(t) = -\begin{bmatrix} 1+\sqrt{2} & \sqrt{2}\sqrt{1+\sqrt{2}} \end{bmatrix} \hat{\mathbf{x}}(t),$$

where $\hat{\mathbf{x}}$ is the solution of

$$\dot{\hat{\mathbf{x}}}(t) = -\begin{bmatrix} 1+\sqrt{2} & -1 \\ 1+2\sqrt{2} & \sqrt{2}\sqrt{1+\sqrt{2}} \end{bmatrix} \mathbf{x}(t) + (1+\sqrt{2}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{y}(t).$$

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