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AN ITERATIVE MODEL ORDER REDUCTION SCHEME FOR A SPECIAL CLASS OF BILINEAR DESCRIPTOR SYSTEMS APPEARING IN CONSTRAINT CIRCUIT SIMULATION

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Abstract. *We focus on interpolatory-based model order reduction for a special class of bilinear descriptor systems in the $\mathcal{H}_2$-optimal framework, appearing in constraint circuit simulations. The straightforward extension of the $\mathcal{H}_2$-optimality conditions for ODE systems to descriptor systems generically may produce an unbounded error in the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm, or both. This arises due to the inappropriate use of the polynomial part of the system. To ensure bounded error, one needs to deal with the polynomial part of the systems properly. To do so, we first transform these descriptor systems into equivalent ODE systems by means of oblique projectors, as it is widely done in the literature for linear index-2 ODEs. This enables us to employ bilinear iterative rational Krylov algorithm (*B-IRKA*) which provides us locally $\mathcal{H}_2$-optimal reduced-order systems on convergence, if it converges. Unfortunately, the direct implementation of* B-IRKA *on equivalent ODEs requires the expensive explicit computation of the oblique projectors. Therefore, as one of our contributions, we show how to apply* B-IRKA *to the equivalent bilinear ODE system without an explicit computation of the projectors. We demonstrate the efficiency of the proposed technique by means of several constraint circuit examples and compare the quality of the reduced-order systems with the ones obtained by using the projection matrices determined by applying* IRKA *on the corresponding linear part.*

1 Introduction

We discuss the interpolatory-based model order reduction technique for bilinear descriptor systems which are of the form

$$\begin{aligned} E_{11}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + \sum_{k=1}^{m} N_k x_1(t) u_k(t) + B_1 u(t), \\ 0 &= A_{21}x_1(t) + B_2 u(t), \\ y(t) &= C_1 x_1(t) + C_2 x_2(t) + Du(t), \end{aligned} \tag{1}$$

where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$ are the generalized states, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$ are the output and input of the system, respectively, and all the matrices are of appropriate dimensions. It is assumed that E_{11} and $A_{21}E_{11}^{-1}A_{12}$ are invertible. This implies that the dynamical system (1) is a Hessenberg index-2 differential algebraic system [1] in case of $N_k = 0$. Generally, these special bilinear systems (1) arise from semi-discretization of Navier-Stokes equations or constraint RLC circuits. As a motivating example, we consider a constraint transmission circuit as shown in Figure 1.

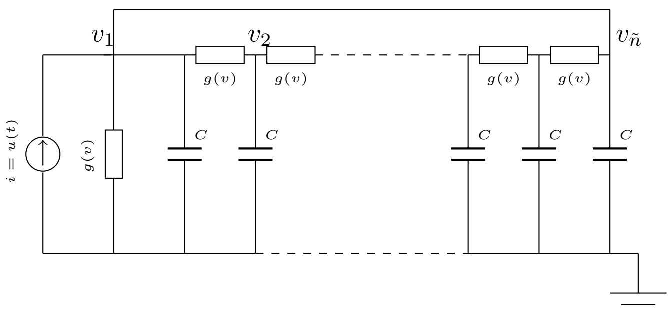

Figure 1: A constraint nonlinear transmission line.

The above transmission circuit contains nonlinear diodes, $g(v) = e^{40v_D} + v_D - 1$, where v_D is the voltage difference across the nodes. Using Kirchhoff's current law, we can model the dynamics of the circuit as a quadratic-bilinear descriptor system, having an index-2 matrix pencil $\lambda E - A$ (details are presented in Section 5). Nonetheless, such quadratic systems, having an index-2 matrix pencil, can be approximated as bilinear systems via Carleman bilinearization. The approximated bilinear systems have a similar structure as (1). We refer to Section 5 for a couple of more constraint circuit examples, having the same structure as well. Therefore, there is a need to develop efficient model reduction techniques for such bilinear structured systems.

To have high accuracy in modeling of physical phenomena or for a better understanding of the underlying process, the governing differential equations are discretized very fine over the spatial domain. As a consequence, we obtain large-scale complex dynamical systems whose numerical simulations, optimization and control study become a huge numerical burden and inefficient. Thus, model order reduction (MOR) brings forth a solution to the immense numerical demand for such large-scale systems. MOR aims to replace these large-scale systems by small surrogate systems of much smaller dimensions, while capturing the important dynamical characteristics of the original system.

MOR for linear systems has proven to be successful and efficient, see, cf. [2, 3, 4]. The extension to the bilinear systems has drawn significant attention in recent times. Many of MOR techniques have been extended from linear systems to bilinear systems. For example, Gramian-based MOR such as balanced truncation has been extended to bilinear systems in [5] and interpolatory-based methods are extended to bilinear systems in [6, 7, 8, 9]. Obviously, Gramian-based MOR requires the solutions of two generalized Lyapunov equations which are computationally cumbersome, although efficient methods are proposed to solve these generalized Lyapunov equations in [10, 11].

On the other hand, interpolatory-based MOR seeks to determine the projection matrices such that each subsystem of the reduced bilinear system interpolates the corresponding subsystem of the original system at predefined interpolation points. For the first time in [12], the problem related to the $\mathcal{H}_2$-optimal MOR for bilinear systems was considered, wherein the Gramian-based Wilson conditions [13] for the $\mathcal{H}_2$ optimality for linear systems are extended to bilinear systems. A similar problem, but in a rather different way, was again considered in [7], where the first-order necessary conditions for $\mathcal{H}_2$-optimality are derived by taking the derivative of the $\mathcal{H}_2$-norm of the error system with respect to the elements of the reduced-order system's realization. Based on these conditions, an iterative method, the so-called bilinear iterative rational Krylov algorithm *(B-IRKA)* is proposed, extending the iterative rational Krylov algorithm *(IRKA)* for linear systems to bilinear systems.

Notwithstanding, there are many up-front questions when it comes to MOR of bilinear descriptor systems. The Gramian-based MOR for bilinear descriptor systems is still an open question, but interpolatory conditions can be readily extended to descriptor systems by just replacing I by E. However, as discussed for linear descriptor systems in [14] that straightaway extending interpolation conditions for ODE systems to descriptor systems can lead to poor reduced-order systems due to mismatch of the polynomial parts of the systems. This may give rise to an unbounded error in the $\mathcal{H}_2$-norm. This holds for bilinear systems as well. Therefore, a special attention to the polynomial part is required to ensure a bounded error. Considering a special class of bilinear descriptor systems whose kth order subsystem has a constant polynomial part, the modified interpolation conditions are proposed in [15]. Therein, a special attention to the polynomial parts of the bilinear systems is paid. Later, the problem of obtaining locally $\mathcal{H}_2$-optimal reduced-order systems of such descriptor systems is considered in [16].

In this paper, we focus on MOR for bilinear descriptor systems (1), having an index-2 matrix pencil, in the $\mathcal{H}_2$-optimal framework. In this regard, we first transform the system (1) into an equivalent bilinear ODE system by means of projections as it is done in [17] for Stokes-type quadratic-bilinear descriptor systems. This allows us to employ the version of *B-IRKA* which is extended in [18] from the $E = I$ case to the $E \neq I$ case. However, the direct implementation of *B-IRKA* requires the explicit computation of the projectors which is highly undesirable. Therefore, as one of the main contributions in this paper, we show how to apply *B-IRKA* without explicit computation of the projectors.

As mentioned before, the application of MOR for bilinear systems can also be seen in the reduction of quadratic-bilinear systems [8, 19]. This is done by first determining an approximate bilinearized system by using Carleman bilinearization process and then employing MOR techniques for bilinear systems. This way, one loses the quadratic-bilinear structure of the system and blows up the dimension of the state vector but it is possible to achieve a reduced bilinear system of much smaller dimension. The Carleman bilinearization process for ODE systems has been widely studied; see [19] and it is recently extended to descriptor systems in [20], having an index-1 matrix pencil $\lambda E - A$. In this paper, we extend the Carleman bilinearization process for quadratic-bilinear descriptor systems, with an index-2 matrix pencil, to obtain approximate bilinearized systems, under the assumption that $x_2(t)$ in (1) is a scalar variable. We also show that the bilinearized systems preserve the index of the matrix pencil.

In the following section, we briefly outline *B-IRKA* for bilinear ODE systems. In Section 3, we present the transformation of the bilinear descriptor systems (1) into equivalent ODE systems and apply *B-IRKA* to obtain reduced-order systems. Therein, we also discuss the computational issues and present implementation details. In Section 4, the Carleman bilinearization process for quadratic-bilinear descriptor systems, having index-2 of the matrix pencil $\lambda E - A$, is presented. In Section 5, we demonstrate the efficiency of the proposed MOR technique by means of several constraint electrical circuit examples.

2 $\mathcal{H}_2$-Optimal Model Reduction of Bilinear Systems

In this section, we briefly review MOR for bilinear ODE systems in the $\mathcal{H}_2$-optimal framework. We consider bilinear systems in the following form:

$$\Sigma := \begin{cases} E\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} N_k x(t) u_k(t) + Bu(t), \\ \quad y(t) = Cx(t) + Du(t), \end{cases}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the input and output of the system, respectively, and all other matrices are of appropriate sizes. Here, E is considered to be nonsingular. The $\mathcal{H}_2$-norm expression for bilinear systems (2) is defined in [12] and can be given as follows:

$$\|\Sigma\|_{\mathcal{H}_2} = \operatorname{tr}\left(\sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \sum_{l_1,\dots,l_k=1}^{m} h_k^{(l_1,\dots,l_k)} (h_k^{(l_1,\dots,l_k)})^T dt_1 dt_2 \cdots dt_k \right),$$

in which $h_k^{(l_1,\dots,l_k)} = Ce^{\tilde{A}t_k}\tilde{N}_{l_1}e^{\tilde{A}t_{k-1}}\tilde{N}_{l_2}\cdots\tilde{N}_{l_k}e^{\tilde{A}t_1}\tilde{b}_{l_k}$ with $\tilde{A} = E^{-1}A$, $\tilde{N}_i = E^{-1}N_i$, $\tilde{B} = E^{-1}B$ and $\tilde{b}_i$ is the ith column of the matrix $\tilde{B}$. Assuming the bilinear system is bounded input bounded output stable, the $\mathcal{H}_2$-norm can also be computed in terms of the Gramians associated with the bilinear system, i.e.,

$$\|\Sigma\|_{\mathcal{H}_2}^2 = \operatorname{tr}(CPC^T) = \operatorname{tr}(B^T Q B),$$

where the reachability Gramian P and the observability Gramian Q are the solutions to the following generalized Lyapunov equations:

$$APE^T + EPA^T + \sum_{k=1}^{m} N_k P N_k^T + BB^T = 0$$

and

$$A^T QE + E^T QA + \sum_{k=1}^{m} N_k^T Q N_k + C^T C = 0,$$

respectively. The main goal of $\mathcal{H}_2$-optimal MOR is to determine an r-order reduced system, satisfying

$$\Sigma_r = \arg \min_{\|\Sigma_r\|_{\mathcal{H}_2} < \infty} \|\Sigma - \Sigma_r\|_{\mathcal{H}_2}.$$

As noted, this problem for $E = I$ is investigated in [7] and an iterative scheme is proposed to determine reduced-order systems, satisfying $\mathcal{H}_2$-optimality conditions. This is extended to $E \neq I$ in [18]. In the following, we sketch *B-IRKA* to obtain reduced-order systems, satisfying the necessary conditions for $\mathcal{H}_2$-optimality.

Algorithm 1 MOR for Bilinear ODE Systems.

1: **Input:** E, A, N_k, B, C.
2: Make an initial guess of $\hat{E}, \hat{A}, \hat{N}_k, \hat{B}, \hat{C}$.
3: **while** no convergence **do**
4: Determine nonsingular S, R such that $S\hat{A}R = \Lambda$ and $S\hat{E}R = \hat{I}$, where $\hat{I}$ is the identity matrix.
5: Compute $\tilde{B} = \hat{B}^T S^T, \quad \tilde{C} = \hat{C}R$ and $\tilde{N}_k = R^T \hat{N}_k^T S^T$.
6: Determine the projection matrices V and W:
$\operatorname{vect}(V) = -(\Lambda \otimes E + \hat{I} \otimes A + \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k)^{-1}(\tilde{B}^T \otimes B)\operatorname{vect}(I_m)$,
$\operatorname{vect}(W) = -(\Lambda \otimes E + \hat{I} \otimes A + \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k)^{-T}(\tilde{C}^T \otimes C^T)\operatorname{vect}(I_p)$,
where $I_q \in \mathbb{R}^{q\times q}$ is the identity matrix, m and p are the numbers of inputs and outputs, respectively, and the operator $\operatorname{vect}(\cdot)$ yields a column vector by stacking columns of the matrix on top of each other.
7: Compute the reduced-order system matrices:
$\hat{E} = W^T E V, \quad \hat{A} = W^T A V, \quad \hat{N}_k = W^T N_k V, \quad \hat{B} = W^T B, \quad \hat{C} = CV.$
8: **end while**
9: **Output:** $\hat{E}^{opt} = \hat{E}, \quad \hat{A}^{opt} = \hat{A}, \quad \hat{N}_k^{opt} = \hat{N}_k, \quad \hat{B}^{opt} = \hat{B}$ and $\hat{C}^{opt} = \hat{C}$.

3 MOR for Bilinear Descriptor Systems

In this section, we focus on MOR for bilinear descriptor systems (1), having an index-2 matrix pencil, in the $\mathcal{H}_2$-optimal framework. We begin with the case $B_2 = 0$, i.e.,

$$E_{11}\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + \sum_{k=1}^{m} N_k x_1(t)u_k(t) + B_1 u(t), \tag{2a}$$

$$0 = A_{21}x_1(t), \quad x_1(0) = 0, \tag{2b}$$

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + Du(t), \tag{2c}$$

where the dimensions of the matrices are the same as in (1). We assume that E_{11} and $A_{21}E_{11}^{-1}A_{12}$ are invertible, and E_{11} is symmetric. As a first step, we transform the bilinear descriptor system into an equivalent bilinear ODE system. Following the same steps as for the Stokes-type quadratic-bilinear descriptor system in [17], we obtain the following system:

$$E_{11}\dot{x}_1(t) = \Pi A_{11}x_1(t) + \sum_{k=1}^{m} \Pi N_k x_1(t)u_k(t) + \Pi B_1 u(t), \quad x_1(0) = 0, \tag{3a}$$

$$y(t) = \mathcal{C}x_1(t) + \sum_{k=1}^{m} \mathcal{C}_N^{(k)} x_1(t)u_k(t) + \mathcal{D}u(t), \tag{3b}$$

where

$$\mathcal{C} = C_1 - C_2(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}E_{11}^{-1}A_{11}, \quad \mathcal{C}_N^{(k)} = -C_2(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}E_{11}^{-1}N_k,$$
$$\mathcal{D} = D - C_2(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}E_{11}^{-1}B_1$$

and

$$\Pi = I - A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}E_{11}^{-1}. \tag{4}$$

Throughout this paper, for simplicity we assume $A_{21} = A_{12}^T$. However, $A_{21} \neq A_{12}^T$ can be treated in the current bilinear framework as well, if one extends the arguments used in [14]. By $A_{12}^T x_1(t) = 0$, we have that $\Pi^T x_1(t) = x_1(t)$, cf., e.g., [21]. Replacing $x_1(t)$ by $\Pi^T x_1(t)$ in (3) and premultiplying by Π, we obtain the following system:

$$\Pi E_{11}\Pi^T \dot{x}_1(t) = \Pi A_{11}\Pi^T x_1(t) + \sum_{k=1}^{m} \Pi N_k \Pi^T x_1(t) u_k(t) + \Pi B_1 u(t), \tag{5a}$$

$$y(t) = \mathcal{C}\Pi^T x_1(t) + \sum_{k=1}^{m} \mathcal{C}_N^{(k)} \Pi^T x_1(t) u_k(t) + \mathcal{D}u(t), \qquad x_1(0) = 0. \tag{5b}$$

As stated in [21], the dynamical system (5) evolves in the $n_1 - n_2$ dimensional subspace $\ker(\Pi)$. Therefore, with the decomposition of Π,

$$\Pi = \phi_1 \phi_2^T \tag{6a}$$

with $\phi_1, \phi_2 \in \mathbb{R}^{n_1 \times n_1 - n_2}$ assuring

$$\phi_1^T \phi_2 = I, \tag{6b}$$

the system is completely described through $\tilde{x}_1(t) = \phi_1^T x_1(t)$ which satisfies

$$\phi_2^T E_{11}\phi_2 \dot{\tilde{x}}_1(t) = \phi_2^T A_{11}\phi_2 \tilde{x}_1(t) + \sum_{k=1}^{m} \phi_2^T N_k \phi_2 \tilde{x}_1(t) u_k(t) + \phi_2^T B_1 u(t), \tag{7a}$$

$$y(t) = \mathcal{C}\phi_2 \tilde{x}_1(t) + \sum_{k=1}^{m} \mathcal{C}_N^{(k)} \phi_2 \tilde{v} u(t) + \mathcal{D}u(t), \quad \tilde{x}_1(0) = 0. \tag{7b}$$

Thus, MOR of the system (7) is equivalent to MOR of the system (2). However, the advantage of the system (7) is that $\phi_2^T E_{11}\phi_2$ is nonsingular, allowing us to employ Algorithm 1, if bilinear terms are neglected in the output equation. This leads a locally $\mathcal{H}_2$-optimal reduced-order system, provided it converges. Unfortunately, to determine the system matrices of (7), we require the explicit computation of the basis matrix ϕ_2 which is not readily available. Moreover, it might also appear that the realization of the system (7) becomes dense after multiplication with ϕ_2 which makes the computation of the reduced-order systems expensive. To overcome this, in what follows, we show how to avoid the explicit computation of ϕ_2 in the application of *B-IRKA*.

Remark 3.1. *In this paper, we neglect the nonlinear terms and the control part in the output equation as far as the computation of the projection matrices is concerned. We focus on the linear relation between the state vector and the output. Nonetheless, the bilinear terms in the output equation are projected afterwards.*

Computational issues

We consider the following associated bilinear ODE system to compute the projection matrices $\mathcal{V}$ and $\mathcal{W}$:

$$\phi_2^T E_{11}\phi_2 \dot{\tilde{x}}_1(t) = \phi_2^T A_{11}\phi_2 \tilde{x}_1(t) + \sum_{k=1}^{m} \phi_2^T N_k \phi_2 \tilde{x}_1(t) u_k(t) + \phi_2^T B_1 u(t), \tag{8a}$$

$$\tilde{y}(t) = \mathcal{C}\phi_2 \tilde{x}_1(t), \quad \tilde{x}_1(0) = 0. \tag{8b}$$

In the view of resolving the computational issues, we aim to determine $\mathcal{V}$ and $\mathcal{W}$ such that the system matrices $E_{11}, A_{11}, N_k, B_1, \mathcal{C}$ and $\mathcal{C}_N$ can be directly reduced using the projection matrices as shown in Algorithm 2.

Algorithm 2 MOR for Bilinear DAEs, having an Index-2 Matrix Pencil (Involving Projector).

1: **Input:** $E_{11}, A_{11}, N_k, B_1, \mathcal{C}, \mathcal{C}_N^{(k)}$.
2: Make an initial guess of $\hat{E}, \hat{A}, \hat{N}_k, \hat{B}, \hat{\mathcal{C}}$.
3: **while** no convergence **do**
4: Compute nonsingular matrices Y and Z such that $Y\hat{A}Z = \Lambda$ and $Y\hat{E}Z = \hat{I}$.
5: Define $\tilde{B} = \hat{B}^T Y^T$, $\tilde{C} = \hat{C}Z$ and $\tilde{N}_k = Z^T \hat{N}_k^T Y^T$.
6: Determine
$$L = -(\hat{I} \otimes \phi_2)\left(\Lambda \otimes (\phi_2^T E_{11}\phi_2) + \hat{I} \otimes (\phi_2^T A_{11}\phi_2) + \sum_{k=1}^{m} \tilde{N}_k^T \otimes (\phi_2^T N_k \phi_2)\right)^{-1} (\hat{I} \otimes \phi_2^T).$$
7: Determine the projection matrices $\mathcal{V}$ and $\mathcal{W}$:
$$\operatorname{vect}(\mathcal{V}) = L(\tilde{B}^T \otimes B)\operatorname{vect}(I_m),$$
$$\operatorname{vect}(\mathcal{W}) = L^T(\tilde{C}^T \otimes \mathcal{C}^T)\operatorname{vect}(I_p).$$
8: Compute the reduced-order system matrices:
$$\hat{E} = \mathcal{W}^T E_{11}\mathcal{V}, \quad \hat{A} = \mathcal{W}^T A_{11}\mathcal{V}, \quad \hat{N}_k = \mathcal{W}^T N_k \mathcal{V},$$
$$\hat{B} = \mathcal{W}^T B_1, \quad \hat{C} = \mathcal{C}\mathcal{V}, \quad \mathcal{C}_N^{(k)} = \hat{\mathcal{C}}_N^{(k)}\mathcal{V}.$$
9: **end while**
10: **Output:**
$$\hat{E}^{opt} = \hat{E}, \quad \hat{A}^{opt} = \hat{A}, \quad \hat{N}_k^{opt} = \hat{N}_k^{opt}, \quad \hat{B}^{opt} = \hat{B}, \quad \hat{\mathcal{C}}^{opt} = \hat{\mathcal{C}}, \quad \hat{\mathcal{C}}_N^{(k)opt} = \hat{\mathcal{C}}_N^{(k)}.$$

We notice that the projection matrices can be directly applied to the original system matrices, but in order to compute the projection matrices $\mathcal{V}$ and $\mathcal{W}$, we still require the matrix ϕ_2 explicitly. Therefore, our next goal is to construct the matrices $\mathcal{V}$ and $\mathcal{W}$ without resorting to ϕ_2.

Lemma 3.1. *Let ϕ_2 be the matrix as defined in* (6) *and $\mathcal{F}$ be a matrix such that $(\hat{I}\otimes\phi_2^T)\mathcal{F}(\hat{I}\otimes\phi_2)$ is invertible. Define $\mathcal{X}_\mathcal{F}^\mathcal{I}$ and $\mathcal{X}_\mathcal{F}$ as follows:*

$$\begin{aligned} \mathcal{X}_\mathcal{F}^\mathcal{I} &:= (\hat{I} \otimes \phi_2)\Big((\hat{I} \otimes \phi_2^T)\mathcal{F}(\hat{I} \otimes \phi_2)\Big)^{-1}(\hat{I} \otimes \phi_2^T), \\ \mathcal{X}_\mathcal{F} &:= (\hat{I} \otimes \Pi)\mathcal{F}(\hat{I} \otimes \Pi^T), \end{aligned} \tag{9}$$

where Π is defined in (4)*. Then the matrices $\mathcal{X}_\mathcal{F}^\mathcal{I}$ and $\mathcal{X}_\mathcal{F}$ satisfy the following relation:*

$$\mathcal{X}_\mathcal{F}^\mathcal{I}\mathcal{X}_\mathcal{F} = (\mathcal{X}_\mathcal{F}\mathcal{X}_\mathcal{F}^\mathcal{I})^T = \hat{I} \otimes \Pi^T.$$

Proof. We begin with

$$\mathcal{X}_\mathcal{F}^\mathcal{I}\mathcal{X}_\mathcal{F} = (\hat{I} \otimes \phi_2)\Big((\hat{I} \otimes \phi_2^T)\mathcal{F}(\hat{I} \otimes \phi_2)\Big)^{-1}(\hat{I} \otimes \phi_2^T)(\hat{I} \otimes \Pi)\mathcal{F}(\hat{I} \otimes \Pi^T).$$

We decompose $\Pi = \phi_1\phi_2^T$ and use properties of the Kronecker product to get

$$\begin{aligned} \mathcal{X}_\mathcal{F}^\mathcal{I}\mathcal{X}_\mathcal{F} &= (\hat{I} \otimes \phi_2)\Big((\hat{I} \otimes \phi_2^T)\mathcal{F}(\hat{I} \otimes \phi_2)\Big)^{-1}(\hat{I} \otimes \phi_2^T)(\hat{I} \otimes \phi_1)(\hat{I} \otimes \phi_2^T)\mathcal{F}(\hat{I} \otimes \Pi^T) \\ &= (\hat{I} \otimes \phi_2)\Big((\hat{I} \otimes \phi_2^T)\mathcal{F}(\hat{I} \otimes \phi_2)\Big)^{-1}(\hat{I} \otimes \phi_2^T\phi_1)(\hat{I} \otimes \phi_2^T)\mathcal{F}(\hat{I} \otimes \Pi^T). \end{aligned}$$

Since $\phi_2^T\phi_1 = I$ from (6), we obtain

$$\begin{aligned}\mathcal{X}_{\mathcal{F}}^I\mathcal{X}_{\mathcal{F}} &= (\hat{I}\otimes\phi_2)\Big((\hat{I}\otimes\phi_2^T)\mathcal{F}(\hat{I}\otimes\phi_2)\Big)^{-1}(\hat{I}\otimes\phi_2^T)\mathcal{F}(\hat{I}\otimes\phi_2)(\hat{I}\otimes\phi_1^T)\\ &= (\hat{I}\otimes\phi_2)(\hat{I}\otimes\phi_1^T) = \hat{I}\otimes\Pi^T.\end{aligned}$$

A similar argument can be given for the other equality. □

Using Lemma 3.1 and properties of the Kronecker product, we observe that the projection matrices $\mathcal{V}$ and $\mathcal{W}$, computed in step 7 of Algorithm 2, satisfy:

$$(\hat{I}\otimes\Pi)\mathcal{F}(\hat{I}\otimes\Pi^T)\,\mathrm{vect}(\mathcal{V}) = (\hat{I}\otimes\Pi)(\tilde{B}^T\otimes B), \tag{10a}$$
$$(\hat{I}\otimes\Pi)\mathcal{F}^T(\hat{I}\otimes\Pi^T)\,\mathrm{vect}(\mathcal{W}) = (\hat{I}\otimes\Pi)(\tilde{C}^T\otimes\mathcal{C}), \tag{10b}$$

where $\mathcal{F} = -\Big(\Lambda\otimes E_{11} + \hat{I}\otimes A_{11} + \sum_{k=1}^m \tilde{N}_k^T\otimes N_k\Big)$. Note that $\Pi_\otimes = \hat{I}\otimes\Pi$ is also an oblique projector. It can be verified that $(\Pi_\otimes)^2 = \Pi_\otimes$, $\ker(\Pi_\otimes)$ = $\mathrm{range}\Big(\hat{I}\otimes A_{12}\Big)$, and $\mathrm{range}(\Pi_\otimes)$ = $\ker\Big(\hat{I}\otimes A_{12}^TE_{11}^{-1}\Big)$. Using these properties, it can be shown that

$$(\hat{I}\otimes A_{12}^T)Z = 0 \qquad \text{if and only if} \qquad (\hat{I}\otimes\Pi^T)Z = Z. \tag{11}$$

In the following lemma, we show the way of circumventing the explicit computation of Π to solve (10) for $\mathrm{vect}(\mathcal{V})$ or $\mathrm{vect}(\mathcal{W})$ and reveal the connection between the solution of (10) and saddle point problems.

Lemma 3.2. *Consider $Z = (\hat{I}\otimes\Pi^T)Z$ and $(\hat{I}\otimes\Pi)\mathcal{F}(\hat{I}\otimes\Pi^T)Z = (\hat{I}\otimes\Pi)G$. Then, the matrix Z solves*

$$\begin{bmatrix}\mathcal{F} & \hat{I}\otimes A_{12}\\ \hat{I}\otimes A_{12}^T & 0\end{bmatrix}\begin{bmatrix}Z\\ \Xi\end{bmatrix} = \begin{bmatrix}G\\ 0\end{bmatrix}. \tag{12}$$

Proof. Since $Z = (\hat{I}\otimes\Pi^T)Z$ using the properties of $\hat{I}\otimes\Pi^T$ as stated in (11), we have $(\hat{I}\otimes A_{12}^T)Z = 0$. This implies that the second block of the equation (12) is satisfied.

Moreover, $(\hat{I}\otimes\Pi)\mathcal{F}Z - (\hat{I}\otimes\Pi)G = 0$ implies that the columns of $\mathcal{F}Z - G$ lie in $\ker\Big(\hat{I}\otimes\Pi\Big)$ = $\mathrm{range}\Big(\hat{I}\otimes A_{12}\Big)$. Therefore, there exists Ξ, satisfying $\mathcal{F}Z - G = -(\hat{I}\otimes A_{12})\Xi$ which is nothing but the first block of the equation (12). This concludes the proof. □

Therefore, using Lemma 3.2, we can determine $\mathrm{vect}(\mathcal{V})$ and $\mathrm{vect}(\mathcal{W})$ without explicitly computing Π by solving the corresponding saddle point problems. All these theoretical analysis gives rise to Algorithm 3 for MOR of the system (7).

Remark 3.2. *As discussed in [21], the general $B_2 \neq 0$ index-2 problems can be brought back to a problem with $B_2 = 0$ type by decomposing $x_1(t)$ as follows:*

$$x_1(t) = x_0(t) + x_u(t), \tag{13}$$

Algorithm 3 MOR for Bilinear DAEs, having an Index-2 Matrix Pencil.

Input: $E_{11}, A_{11}, N_k, B_1, \mathcal{C}, \mathcal{C}_N^{(k)}$

Make an initial choice of $\hat{E}, \hat{A}, \hat{N}_k, \hat{B}, \hat{C}$.

while no convergence **do**

Compute nonsingular matrices Y and Z such that $Y\hat{A}Z = \Lambda$ and $Y\hat{E}Z = \hat{I}$.

Define $\tilde{B} = \hat{B}^T Y^T, \quad \tilde{C} = \hat{C}Z \quad$ and $\quad \tilde{N}_k = Z^T \hat{N}_k^T Y^T$.

Determine the projection matrices $\mathcal{V}$ and $\mathcal{W}$:

$$\begin{bmatrix} \mathcal{F} & \hat{I} \otimes A_{12} \\ \hat{I} \otimes A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} \text{vect}(\mathcal{V}) \\ \Gamma \end{bmatrix} = \begin{bmatrix} (\tilde{B}^T \otimes B)\,\text{vect}(I_m) \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{F}^T & \hat{I} \otimes A_{12} \\ \hat{I} \otimes A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} \text{vect}(\mathcal{W}) \\ \Delta \end{bmatrix} = \begin{bmatrix} (\tilde{C}^T \otimes \mathcal{C})\,\text{vect}(I_p) \\ 0 \end{bmatrix},$$

where $\mathcal{F} = -(\Lambda \otimes E_{11} + \hat{I} \otimes A_{11} + \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k)$.

Compute the reduced-order system matrices:

$$\hat{E} = \mathcal{W}^T E_{11} \mathcal{V}, \qquad \hat{A} = \mathcal{W}^T A_{11} \mathcal{V}, \qquad \hat{N}_k = \mathcal{W}^T N_k \mathcal{V},$$
$$\hat{B} = \mathcal{W}^T B_1, \qquad \hat{C} = \mathcal{C}\mathcal{V}, \qquad \mathcal{C}_N^{(k)} = \hat{\mathcal{C}}_N^{(k)} \mathcal{V}.$$

end while

Output:

$\hat{E}^{opt} = \hat{E}, \;\; \hat{A}^{opt} = \hat{A}, \;\; \hat{N}_k^{opt} = \hat{N}_k, \;\; \hat{B}^{opt} = \hat{B}, \;\; \hat{\mathcal{C}}^{opt} = \hat{\mathcal{C}}, \;\; \hat{\mathcal{C}}_N^{(k)opt} = \hat{\mathcal{C}}_N^{(k)}$.

where $x_u(t) = -\underbrace{E_{11}^{-1} A_{12} (A_{12}^T E_{11}^{-1} A_{12}) B_2}_{\Upsilon} u(t)$ *and* $x_0(t)$ *satisfies* $A_{12}^T x_0(t) = 0$. *After doing the algebraic calculations as done for the case* $B_2 = 0$ *case, we get*

$$\Pi E_{11} \Pi^T \dot{x}_0(t) = \Pi A_{11} \Pi^T x_0(t) + \sum_{k=1}^{m} \Pi N_k \Pi^T x_0(t) u_k(t) + \Pi \mathcal{B} \tilde{u}(t), \tag{14a}$$

$$\Pi^T x_0(0) = \Pi^T (x_0 - x_u(0)), \tag{14b}$$

$$y(t) = \mathcal{C} \Pi^T x_0(t) + \sum_{k=1}^{m} \mathcal{C}_N^{(k)} \Pi^T x_0(t) u_k(t) + \mathcal{D} \tilde{u}(t) - C_2 (A_{12}^T E_{11}^{-1} A_{12})^{-1} B_2 \dot{u}(t), \tag{14c}$$

where

$$\mathcal{B} = [\mathcal{B}_1, \mathcal{B}_u^{(1)}, \ldots, \mathcal{B}_u^{(m)}] \;\; \text{with} \;\; \mathcal{B}_u^{(k)} = -N_i \Upsilon, \qquad \tilde{u}(t) = \Big([1, u(t)^T] \otimes u(t)^T \Big)^T,$$

$$\mathcal{C} = C_1 - C_2 (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{11}, \qquad \mathcal{C}_N^{(k)} = -C_2 (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T E_{11}^{-1} N_k,$$

$$\mathcal{D} = \Big[D - C_1 \Upsilon - C_2 (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T E_{11}^{-1} \mathcal{B}_1, \;\; C_2 (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T E_{11}^{-1} [B_u^{(1)}, \ldots, B_u^{(m)}] \Big].$$

Although the system (14) *has terms associated with* u, $u{\cdot}u_k$ *which are functions of* u *eventually, but we treat them as different inputs of the system as far as MOR is considered. Now, it can be easily seen that determining reduced-order systems of the system* (14) *is analogous to the system* (5). *Therefore, Algorithm 3 can be readily applied to the system* (14) *to obtain locally* $\mathcal{H}_2$*-optimal reduced-order systems, having neglected bilinear terms in the output equation.*

4 Carleman Bilinearization for Quadratic-Bilinear Descriptor Systems with Index-2 Matrix Pencil

In this section, we study the Carleman bilinearization process for quadratic-bilinear descriptor systems. The Carleman bilinearization process can be applied to approximate quadratic-bilinear systems; see, e.g., [8, 19]. This process was recently extended to quadratic-bilinear descriptor systems, having an index-1 matrix pencil $\lambda E - A$ in [20]. For simplicity of notation, we consider a single-input quadratic-bilinear descriptor system in the following form:

$$\begin{aligned} E_{11}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + Hx_1(t)\otimes x_1(t) + Nx_1(t)u(t) + B_1u(t),\\ 0 &= A_{21}x_1(t) + B_2u(t),\\ y(t) &= C_1x_1(t) + C_2x_2(t), \end{aligned} \tag{15}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}$ are the generalized states, and $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}$ are the output and the input to the system, respectively. All other matrices are fixed by the size of the state vectors, the input and the output. We assume that matrices E_{11} and $A_{21}E_{11}^{-1}A_{12}$ are invertible. This implies that the matrix pencil $\lambda E - A$ is index-2 pencil. Moreover, it is assumed that the system has only one constraint due to which $x_2(t)$ is considered to be a scaler variable, rather than a vector. With these assumptions, we aim to determine an approximate bilinear system via Carleman bilinearization, preserving the properties like the index of the matrix pencil $\lambda E - A$. Without loss of generality, we can assume that $B_2 = 0$, otherwise it can be brought back to the $B_2 = 0$ case by an appropriate change of variables as stated in Remark 3.2. As a first step, we develop a differential equation for $x_1^{\otimes} = x_1(t) \otimes x_1(t)$, neglecting cubic and higher order terms:

$$\begin{aligned} (E_{11}\otimes E_{11})\frac{d}{dt}x_1^{\otimes}(t) &= E_{11}\dot{x}_1(t)\otimes E_{11}x_1(t) + E_{11}x_1(t)\otimes E_{11}\dot{x}_1(t)\\ &= \mathcal{L}(A_{11},E_{11})x_1^{\otimes}(t) + \mathcal{L}(A_{12},E_{11})x_1(t)x_2(t)\\ &\quad + \mathcal{L}(B_1,E_{11})x_1(t)u(t) + \mathcal{L}(N,E_{11})x_1^{\otimes}u(t) \end{aligned} \tag{16}$$

in which $\mathcal{L}(\mathcal{A},\mathcal{B}) = \mathcal{A}\otimes\mathcal{B} + \mathcal{B}\otimes\mathcal{A}$. We also get additional constraints as

$$0 = (E_{11}\otimes A_{21})x_1^{\otimes}(t). \tag{17}$$

Combining (15),(16) and (17), we get the following bilinear system as an approximation to (15):

$$\begin{aligned} \tilde{E}_{11}\dot{\tilde{x}}_1(t) &= \tilde{A}_{11}\tilde{x}_1(t) + \tilde{A}_{12}\tilde{x}_2(t) + \tilde{N}\tilde{x}_1(t)u(t) + \tilde{B}_1(t),\\ 0 &= \tilde{A}_{21}\tilde{x}_1(t),\\ \tilde{y}(t) &= \tilde{C}_1\tilde{x}_1(t) + \tilde{C}_2\tilde{x}_2(t), \end{aligned}$$

where

$$\tilde{E}_{11} = \begin{bmatrix} E_{11} & 0\\ 0 & \mathcal{L}(E_{11},E_{11}) \end{bmatrix},\ \tilde{A}_{11} = \begin{bmatrix} A_{11} & H\\ 0 & \mathcal{L}(A_{11},E_{11}) \end{bmatrix},\ \tilde{A}_{12} = \begin{bmatrix} A_{12} & 0\\ 0 & \mathcal{L}(A_{12},E_{11}) \end{bmatrix},$$
$$\tilde{A}_{21} = \begin{bmatrix} A_{21} & 0\\ 0 & (A_{21}\otimes E_{11}) \end{bmatrix},\ \tilde{N} = \begin{bmatrix} N & 0\\ \mathcal{L}(B_1,E_{11}) & \mathcal{L}(N,E_{11}) \end{bmatrix},$$
$$\tilde{B}_1 = \begin{bmatrix} B_1\\ 0 \end{bmatrix},\ \hat{C}_1 = \begin{bmatrix} C_1^T\\ 0 \end{bmatrix}^T,\ \tilde{C}_2 = \begin{bmatrix} C_2^T\\ 0 \end{bmatrix}^T,\ \tilde{x}_1 = \begin{bmatrix} x_1\\ x_1^{\otimes} \end{bmatrix},\ \tilde{x}_2 = \begin{bmatrix} x_2\\ x_1x_2 \end{bmatrix}.$$

Next, we show that the matrix pencil $\lambda\tilde{E} - \tilde{A}$ is also an index-2 pencil. For this, we need to prove that $\tilde{A}_{21}\tilde{E}_{11}^{-1}\tilde{A}_{12}$ is a full rank matrix. Therefore, we consider the following:

$$\begin{aligned}\tilde{A}_{21}\tilde{E}_{11}^{-1}\tilde{A}_{12} &= \begin{bmatrix} A_{21} & 0 \\ 0 & (A_{21}\otimes E_{11})\end{bmatrix}\begin{bmatrix} E_{11}^{-1} & 0 \\ 0 & \mathcal{L}(E_{11}^{-1}, E_{11}^{-1})\end{bmatrix}\begin{bmatrix} A_{12} & 0 \\ 0 & \mathcal{L}(A_{12}, E_{11})\end{bmatrix} \\ &= \begin{bmatrix} A_{21}E_{11}^{-1}A_{12} & 0 \\ 0 & E_{11}\left(\left(A_{21}E_{11}^{-1}A_{12}\right)I + E_{11}^{-1}A_{12}A_{21}\right)\end{bmatrix}.\end{aligned}$$

Since $E_{11}^{-1}A_{12}$ and A_{21} both are vectors, this implies that the eigenvalues of $E_{11}^{-1}A_{12}A_{21}$ are zero and $\left(A_{21}E_{11}^{-1}A_{12}\right)$ with multiplicities $n-1$ and 1, respectively. Also, we know that if σ_i are the eigenvalues of matrix P, then the matrix $Q := I + P$ has the eigenvalues $1+\sigma_i$. This shows that the matrix $\left(\left(A_{21}E_{11}^{-1}A_{12}\right)I + E_{11}^{-1}A_{12}A_{21}\right)$ has eigenvalues $A_{21}E_{11}^{-1}A_{12}$ and $2\cdot A_{21}E_{11}^{-1}A_{12}$ with multiplicities $n-1$ and 1, respectively. Hence, $\tilde{A}_{21}\tilde{E}_{11}^{-1}\tilde{A}_{12}$ is invertible and so, the bilinearized system has an index-2 matrix pencil.

5 Numerical Experiments

In this section, we investigate the efficiency of the proposed iterative algorithm for bilinear descriptor systems, having an index-2 matrix pencil and compare the quality of the determined reduced-order systems with the ones obtained by using the projection matrices determined by linear *IRKA* [14, Algo. 6.2]. The stopping criterion for Algorithm 3 is based on the relative change in the eigenvalues of the reduced-order system. If this change is below the square root of the machine precision, then the iteration is stopped. We randomly select the initial guess of the reduced-order matrices in Algorithm 3, and also choose a scaling factor γ as suggested in [7] for a smooth convergence of *B-IRKA*. All the simulations are done on a CPU 2x Intel Xeon E5620, 12 MB Cache, 48 GB DDR3 RAM, MATLAB® Version 7.11.0.584 (R2010b) 64-bit(glnxa64).

5.1 Nonlinear RC Circuit

Nonlinear transmission line circuits are considered to be standard test cases the community of interpolation-based MOR; see, e.g., [22] for the unconstrained RC circuit and [15] for the constraint RC circuit. Here, we consider a variant of the constraint transmission line circuit as shown in Figure 1 (in Introduction), where it is assumed that the voltages at the first and last nodes are the same. The electrical component i.e., diode I-V, has nonlinear characteristics $g(v_D) = e^{40v_D} + v_D - 1$, where v_D is the voltage across the node. Using Kirchhoff's current law at each node, we get the following set of equations:

$$\begin{aligned}\dot{v}_1 &= -2v_1 + v_2 + 2 - e^{40v_1} - e^{40(v_1 - v_2)} + u(t), \\ \dot{v}_i &= -2v_i + v_{i-1} + v_{i+1} + e^{40(v_{i-1}-v_i)} - e^{40(v_i - v_{i+1})}, \quad 1 < i < \tilde{n}, \\ \dot{v}_{\tilde{n}} &= -v_{\tilde{n}} + v_{\tilde{n}-1} - 1 + e^{40(v_{\tilde{n}-1} - v_{\tilde{n}})}\end{aligned} \tag{18}$$

with a constraint

$$0 = v_1 - v_{\tilde{n}}.$$

The system of equations (18) can be written as a quadratic-bilinear descriptor system by appropriately introducing the new state variables, as shown in [22] for a nonlinear RC circuit example. The dynamics of the system, in state-space representation, is given as follows:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + G\lambda + Hx(t)\otimes x(t) + Nx(t)u(t) + Bu(t), \\ 0 &= G^T x(t),\end{aligned}$$

where $x(t) \in \mathbb{R}^{\tilde{n}}$ and λ are state vector, containing voltages at each node, and an appropriate Lagrangian multiplier, respectively. We observe the voltage at the first node. Since we have only one constraint in the system dynamics, this allows us to employ Carleman bilinearization as discussed in Section 4 to obtain an approximate bilinearized descriptor system. We set $\tilde{n} = 15$, leading to a bilinearized system of order $n = 2 \cdot \tilde{n} + 4 \cdot \tilde{n}^2 + 1 = 961$. We apply the $\mathcal{H}_2$-optimal model reduction method given in Algorithm 3 by setting the order of the reduced-order system to $r = 10$. We choose the scaling factor $\gamma = 0.01$ in order to achieve convergence of *B-IRKA*. We also determine a reduced-order system by using linear *IRKA* of the same order. In Figure 2, we compare the quality of the reduced-order systems with the original system by computing transient responses for an input $u(t) = (\sin(10\pi t) + 1)/2$.

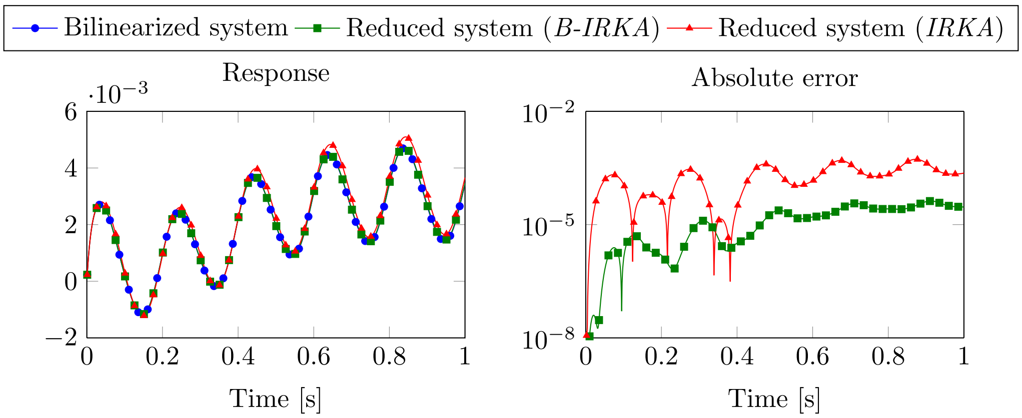

Figure 2: Comparison of the transient response of the systems for an input $u(t) = (\sin(10\pi t) + 1)/2$.

We observe that the reduced-order system obtained by modified *B-IRKA* captures the dynamics of the system better as compared to the reduced-order system obtained by linear *IRKA*.

5.2 Resistance-varying RC circuit

As our second example, we consider the RC circuit as shown in Figure 3 in which the ith node is connected to the $(i-1)$st and the $(i+1)$st nodes via resistances, and connected to the ground via capacitors. Moreover, the first node is connected to the ground via a variable resistance, and the voltage at the first node is influenced by the current (the input u_1). We also add an extra control u_2, controlling the voltage difference between the first and last nodes. Now, we apply Kirchhoff's current law at each node to obtain the following set of ODEs:

$$\begin{aligned} C\dot{v}_1(t) &= \tfrac{1}{R}(-v_1 + v_2) + \tfrac{1}{R_v}(0 - v_1) + u(t), \\ C\dot{v}_i(t) &= \tfrac{1}{R}(-2v_i + v_{i-1} + v_{i+1}), \qquad (2 \leq i \leq n-1), \\ C\dot{v}_n(t) &= \tfrac{1}{R}(-v_n + v_{n-1}) \end{aligned}$$

along with a constraint

$$0 = v_1 - v_n - u_2(t).$$

We set all the capacitors (C) and the constant resistance (R) equal to 1, and consider that the variable resistance R_v varies with respect to the parameter δ as follows:

$$R_v = \frac{R}{1+\delta}.$$

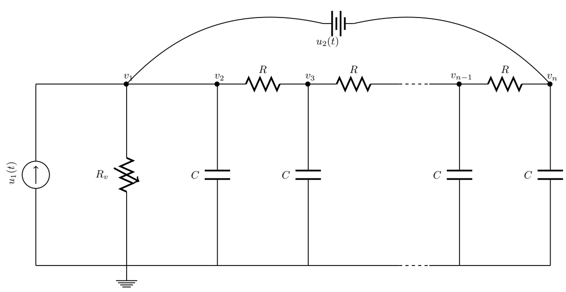

Figure 3: A constraint RC-circuit diagram.

Combining all these equations together, we obtain the dynamics of the RC circuit which are described by the following descriptor system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + G^T\lambda(t) + \delta N x(t) + B_1 u_1(t), \\ 0 &= Gx(t) + B_2 u_2(t), \\ y(t) &= Cx(t), \end{aligned} \tag{19}$$

where $x(t) \in \mathbb{R}^n$ is the state vector containing the voltage at each node, $\lambda \in \mathbb{R}$ is the Lagrange multiplier, $G = [1, 0, \dots, 0, -1]$ is a constraint matrix, $B_1 = [1, 0, \dots, 0]$ and $B_2 = 1$. The voltage at the second node is the output of interest, and it yields $C = [0, 1, 0, \dots, 0]$. For this example, we first transform the system (19) into an equivalent system with $\mathcal{B}_2 = 0$ as suggested in Remark 3.2, leading to the following system:

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + \delta N\tilde{x}(t) + G^T\lambda(t) + \mathcal{B}\tilde{u}(t), \\ 0 &= G\tilde{x}(t), \\ y(t) &= C\tilde{x}(t) + D\tilde{u}(t), \end{aligned} \tag{20}$$

where $\mathcal{B} = [B_1,\ A\mathcal{G},\ N\mathcal{G}]$ and $D = [0,\ C\mathcal{G},\ 0]$ in which $\mathcal{G} = -G^T(GG^T)^{-1}B_2$, and $\tilde{u}(t) = [u_1(t),\ u_2(t),\ \delta u_2(t)]$. Now, the system (20) can be seen as a linear parameter-varying system in the parameter δ. It is shown in [23] that the special class of parametric systems is closely related to MOR of bilinear systems. Therefore, we reformulate the linear system (20) appropriately as a bilinear system with four inputs and one output as follows:

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + \sum_{i=1}^{4} N_i\tilde{x}(t)u_i(t) + G^T\lambda(t) + \mathcal{B}_b\tilde{u}_b(t), \\ 0 &= G\tilde{x}(t), \\ y(t) &= C\tilde{x}(t) + D_b\tilde{u}_b(t), \end{aligned} \tag{21}$$

where $\begin{bmatrix}N_1, N_2, N_3, N_4\end{bmatrix} = \begin{bmatrix}0, 0, 0, N\end{bmatrix}$, $\mathcal{B}_b = [\mathcal{B}, 0]$ and $D_b = [D, 0]$ with inputs $\tilde{u}_b(t) = [\tilde{u}^T(t), \delta]^T$. We consider $n = 1000$, leading to the order of the system $\tilde{n} = 1001$. Next, we determine reduced bilinear systems of order $r = 15$ by employing the $\mathcal{H}_2$-optimal model reduction method given in Algorithm 3 and by using linear *IRKA*. These reduced bilinear systems again can be rewritten into reduced linear parametric systems. This allows us to determine the quality of the reduced-order systems by comparing the relative H_∞-norm $\frac{\|H-\hat{H}\|_{\mathcal{H}_\infty}}{\|H\|_{\mathcal{H}_\infty}}$ of the error system varying parameter values δ which is shown in Figure 4.

We observe that the reduced parametric system obtained from *B-IRKA* captures the dynamics of the original system for a wide parameter range much better as compared to the one obtained

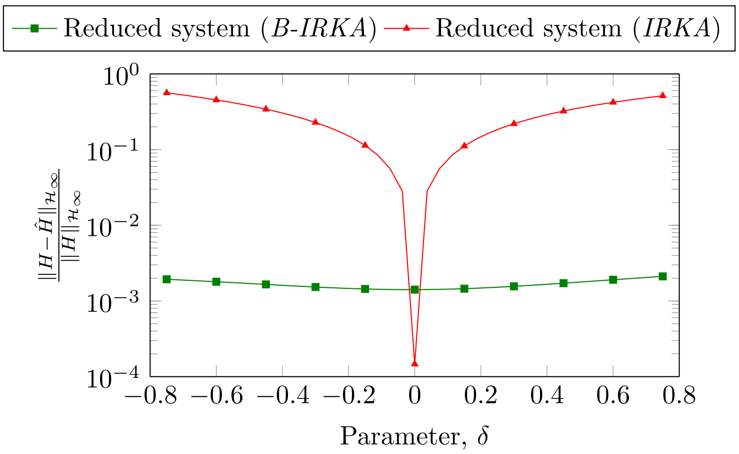

Figure 4: Relative H_∞ error versus the parameter δ for the reduced linear parametric systems obtained from *B-IRKA* and *IRKA*.

by using linear *IRKA*. However, one can see the drop in the relative H_∞ error in Figure 4. This is due to an obvious reason that the projection matrices obtained by employing *IRKA* capture the dynamics of the system quite accurately for $\delta = 0$, but fail to capture the dynamics of the system as the parameter δ moves away from $\delta = 0$. On the contrary the reduced parametric system obtained from *B-IRKA* performs quite well over a wide parameter range.

5.3 Parameter dependant RLC circuit

Lastly, we consider an RLC circuit as shown in Figure 5. The governing equations of the RLC circuit can be written as follows:

$$\begin{aligned} C\tfrac{d}{dt}v_j &= i_j - i_{j-1}, \quad j = 1,\dots,g-1,\\ C\tfrac{d}{dt}v_g &= i_g,\\ L\tfrac{d}{dt}i_j + Ri_j &= v_{j-1} - v_j \quad j = 2,\dots,g, \end{aligned}$$

where v_j and i_j are the voltage at the jth node and the current passing through the $(j-1)$st inductor, respectively. Also, $V(t)$ is a control voltage source of the system and i_1 is the current passing through this voltage source. Since the voltage source is connected to the first node via ground, this leads to a constraint $0 = v_1 - V(t)$. We set all capacitors and inductors to 1, and consider variable resistances, depending linearly on the parameter p as follows:

$$R = 1 + p.$$

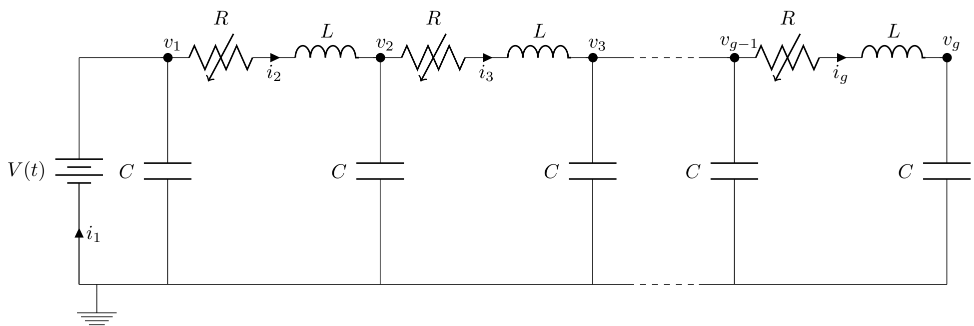

Figure 5: A variable resistance RLC circuit.

With these relations, we can write the system in state-space form as:

$$\begin{aligned}\frac{d}{dt}x_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + pNx_1(t),\\ 0 &= A_{12}^T x_1(t) + B_2 u(t),\end{aligned} \tag{22}$$

where $x_1(t)$ contains the voltages at each node and the currents passing through each inductor, and $x_2(t)$ contains the current though the voltage source. The voltage at the last node is observed. We choose $g = 500$ which results in the order of the system (22) $n = 1001$. As a first step, we convert system (22) to an equivalent system by using an appropriate change of the state variable so that the constraint equation becomes independent of the input, leading to the following system:

$$\begin{aligned}\frac{d}{dt}\tilde{x}_1(t) &= A_{11}\tilde{x}_1(t) + \tilde{A}_{12}x_2(t) + pN\tilde{x}_1(t) + \tilde{B}u(t),\\ 0 &= A_{12}^T\tilde{x}_1(t),\\ y(t) &= C_1\tilde{x}_1(t).\end{aligned}$$

Next, we treat the above system as a bilinear system by considering the parameter p as an input to the system. We determine reduced bilinear systems of order $r = 10$, by employing the proposed *B-IRKA* and *IRKA*, and then convert back to have linear parametric reduced-order systems. In order to compare the quality of the reduced-order systems, we plot the relative H_∞-norm of the error system in Figure 6.

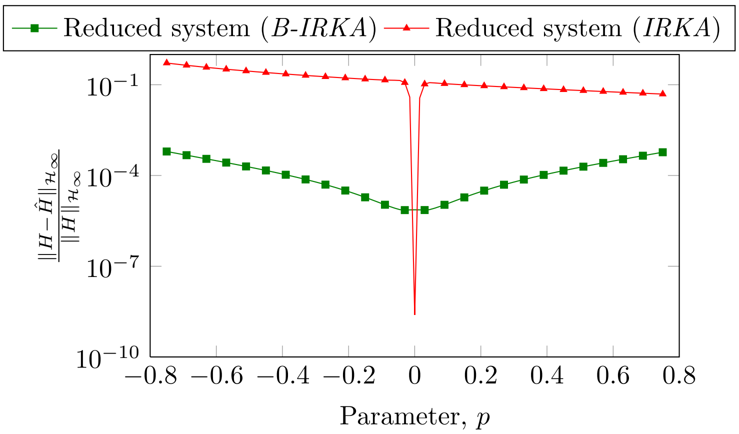

Figure 6: Relative H_∞ error versus the parameter p for the reduced linear parametric systems obtained from *B-IRKA* and *IRKA*.

The similar phenomenon, as observed in Example 5.2, can be seen in Figure 6, in particular a drop in the relative H_∞ error for *IRKA* at $p = 0$. Nevertheless, the reduced-order system, obtained by using *B-IRKA*, outperforms the one obtained by using *IRKA* for a wide range of the parameter.

6 Conclusions

We have proposed an iterative algorithm for MOR of bilinear descriptor systems, having an index-2 matrix pencil. This gives rise to locally $\mathcal{H}_2$-optimal reduced-order systems on convergence, if it converges. First we have transformed the original bilinear descriptor system into an equivalent bilinear ODE system by means of projectors. This enabled us to employ bilinear iterative rational Krylov algorithm *(B-IRKA)*. Next, in the view of implementation, we have proposed a modified *B-IRKA* which does not require the undesirable explicit computation of the spectral projector in order to compute reduced-order systems. We have also extended the Carleman bilinearization process for quadratic-bilinear descriptor systems, having an index-2 matrix pencil and only one constraint. It is shown that the bilinearized systems preserve the index-2 of the matrix pencil. Finally, we have illustrated the efficiency of the proposed *B-IRKA* using various constraint electrical circuit examples, showing that reduced-order systems, obtained by using modified *B-IRKA*, replicate the dynamics of the original system much better as compared to the reduced-order systems obtained by using linear *IRKA*.

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