A Robust Numerical Method for Optimal H_∞ Control

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***Abstract*— We present a new numerical method for the solution of the optimal H_∞ control problem. The method is based on the γ-iteration as well as the state-space solution to the (sub)optimal H_∞ control problem, but re-formulates all steps in order to achieve better robustness in the presence of rounding errors than any implementation of the textbook formulae. It remains robust in the presence of rounding errors even as γ approaches its optimal value.**

I. INTRODUCTION

The optimal infinite-horizon output (or measurement) feedback H_∞ control problem is one of the central tasks in robust control, see, e.g., [1], [2], but the development of robust numerical methods for the H_∞ control is unusually difficult [3] and remains a major open problem [4].

Consider the linear time-invariant system

$$\begin{array}{rcll} \dot{x} & = & Ax + B_1 w + B_2 u, & x(0) = x^0, \\ z & = & C_1 x + D_{11} w + D_{12} u, & \\ y & = & C_2 x + D_{21} w + D_{22} u, & \end{array} \quad (1)$$

where $A \in \mathbb{R}^{n,n}$, $B_i \in \mathbb{R}^{n,m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$ for $i,j = 1,2$. (By $\mathbb{R}^{n,k}$ we denote the set of real $n \times k$ matrices.) Let $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right]$ denote the corresponding transfer function such that

$$\left[\begin{array}{c} Z \\ Y \end{array}\right] = \left[\begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array}\right] \left[\begin{array}{c} W \\ U \end{array}\right],$$

where Y, Z, U, W denote the Laplace transforms of y, z, u, w. The optimal H_∞ control problem then is to determine a dynamic compensator

$$\begin{array}{rcl} \dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}y, \\ u & = & \hat{C}\hat{x} + \hat{D}y, \end{array} \quad (2)$$

with $\hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$ and transfer function $K(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$ such that the resulting closed-loop system

1) is *internally stable*, i.e., the solution of the system with $w \equiv 0$ is asymptotically stable, and
2) the transfer function from disturbance inputs to error signals, represented by the transfer function

$$T_{zw} = G_{22} + G_{21}K(I - G_{11}K)^{-1}G_{12},$$

is minimized in the H_∞ norm.

The well-known state-space solution to the H_∞ control problem [5], relating H_∞ control to algebraic Riccati equations, provides a way to solve the H_∞ optimal control problem. Under the usual assumptions that (A, B_i) is stabilizable and (A, C_i) is detectable for $i = 1, 2$, and for brevity, $D_{11} = 0$, $D_{22} = 0$, as well as

$$D_{12}^T \left[\begin{array}{cc} C_1 & D_{12} \end{array}\right] = \left[\begin{array}{cc} 0 & I \end{array}\right], \quad \left[\begin{array}{c} B_1 \\ D_{21} \end{array}\right] D_{21}^T = \left[\begin{array}{c} 0 \\ I \end{array}\right],$$

the result states that $K(s)$ internally stabilizing with $\|T_{zw}\|_\infty < \gamma$ exists if and only if the algebraic Riccati equations (AREs)

$$0 = C_1^T C_1 + AX + XA^T + X(\tfrac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T)X \quad (3)$$

$$0 = B_1^T B_1 + A^T Y + YA + Y(\tfrac{1}{\gamma^2} C_1 C_1^T - C_2 C_2^T)Y \quad (4)$$

both have positive semidefinite stabilizing solutions X_∞ and Y_∞, respectively, and the spectral radius condition

$$\rho(XY) < \gamma^2 \quad (5)$$

is satisfied.

These conditions allow for a bisection-type iteration, called γ-iteration, to compute the optimal value γ_{opt}. Unfortunately, there are severe numerical difficulties involved in using the γ-iteration in the form implied by the above result which often lead to failure of the procedure [2], [3]. Some of the difficulties are: primary is the fact that often as γ approaches γ_{opt}, one of the Riccati solutions X_∞ or Y_∞ either diverges to ∞ or becomes highly ill-conditioned. That is, tiny errors in forming the coefficients may lead to large errors in the solutions. Ill-conditioned or diverging Riccati solutions make it difficult or impossible to check the conditions numerically. Frequently, the closed-loop spectrum associated to either (3) or (4) will approach the imaginary axis if γ approaches γ_{opt}. Most numerical methods for solving AREs face severe problems in this situation; particularly if the symmetry properties of the associated Hamiltonian eigenproblems are not respected. But even if this difficulty is not encountered, already rounding errors and cancellation effects resulting from computing the coefficients of the constant and quadratic terms in the AREs may cause such a procedure to deliver erroneous results. The situation becomes more severe if the simplifying assumptions on the D_{ij} in (1) are not satisfied. In this case, certain matrix inverses and factorizations have to be computed, see [6] for a detailed discussion.

Therefore, our aim is to replace the conditions for the existence of an H_∞ suboptimal controller by other conditions that can be checked in a numerically reliable way. This is

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achieved by replacing the AREs by associated generalized eigenproblems for which numerical methods preserving the spectral symmetries can be used. Moreover, the spectral radius condition needs to be replaced as well when the ARE solutions are no longer available. The following section gives the main result; proofs, more details, and numerical results illustrating that the new characterization yields a numerically robust method that often can compute the optimal γ when standard methods based on the Riccati approach fail, can be found in [6].

II. MAIN RESULT

In this section we present a new characterization of the existence of a suboptimal H_∞ controller. A complete proof of the result can be found in [6]. In order to state the result we need the following two matrix pencils related to the AREs (3), (4):

$$H_X - \lambda K_X = \begin{bmatrix} A & 0 & B_1 & B_2 & 0 \\ 0 & A^T & 0 & 0 & C_1^T \\ 0 & B_1^T & -\gamma^2 I_{m_1} & 0 & 0 \\ 0 & B_2^T & 0 & I_{m_2} & 0 \\ C_1 & 0 & 0 & 0 & -I_{p_1} \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (6)$$

$$H_Y - \lambda K_Y = \begin{bmatrix} A^T & 0 & C_1 & C_2 & 0 \\ 0 & A & 0 & 0 & B_1^T \\ 0 & C_1^T & -\gamma^2 I_{p_1} & 0 & 0 \\ 0 & C_2^T & 0 & I_{p_2} & 0 \\ B_1 & 0 & 0 & 0 & -I_{p_1} \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (7)$$

Now it can be shown that the AREs (3), (4) have positive semidefinite stabilizing solutions if and only if these matrix pencils have stable n-dimensional deflating subspaces, spanned by the columns of $\begin{bmatrix} P_1^T & P_2^T & P_3^T & P_4^T & P_5^T \end{bmatrix}^T$, and $\begin{bmatrix} Q_1^T & Q_2^T & Q_3^T & Q_4^T & Q_5^T \end{bmatrix}^T$, respectively, and $X_\infty = P_2 P_1^{-1}$, $Y_\infty = Q_2 Q_1^{-1}$.

Moreover, we will need the symmetric matrix

$$Z(\gamma) := \begin{bmatrix} \gamma P_2^T P_1 & P_2^T Q_2 \\ Q_2^T P_2 & \gamma Q_2^T Q_1 \end{bmatrix} \quad (8)$$

which allows to re-formulate the spectral radius condition (5). We also denote by r_X, r_Y the ranks of X_∞, Y_∞ for $\gamma > \gamma_{opt}$. (These ranks are known to be constant; see [7].)

Theorem 2.1: Under the assumptions set forth in the introduction, there exists a suboptimal controller $K(s)$ of the form (2) for the system (1) such that $\|T_{zw}\| < \gamma$ if and only if the following conditions are satisfied.

1) *The matrix pencil $H_X - \lambda K_X$ in (6) has a unique stable n-dimensional deflating subspace.*
2) *The matrix pencil $H_Y - \lambda K_Y$ in (7) has a unique stable n-dimensional deflating subspace.*
3) *The matrix $Z(\gamma)$ in (8) is positive semidefinite with rank $r_X + r_Y$.*

The above theorem suggests a new implementation of the γ-iteration based essentially on the eigenvalues of $Z(\gamma)$ which can be viewed as a scalar optimization problem [6]. An implementation of this iteration is a lot more robust than the textbook formulae in the presence of rounding errors as it uses original (unperturbed) data as much as possible, avoids unnecessary matrix products and inversions as well as the solution of ill-conditioned AREs. The rank check for $Z(\gamma)$ can be implemented using the numerically stable CS decomposition; see [6]. Moreover, using appropriate permutations, the matrix pencils in (6) and (7) can be transformed to become Hamiltonian/skew-Hamiltonian pencils for which structure preserving methods for computing the stable deflating subspaces exist (see [8], [6]). Numerical examples demonstrating the efficiency of the new method compared to existing methods can be found in [6]. It is also shown there that the simplifying assumptions on the D_{ij} are not needed.

III. CONCLUSIONS

This paper discusses the design of a robust numerical method for the H_∞ control problem. The proposed method avoids rounding errors and cancellation due to unnecessary matrix sums, products and inverses as far as possible and avoids solving potentially ill-conditioned algebraic Riccati equations by working with structured matrix pencils and its deflating subspaces. The computation of the optimal γ reduces to a one-dimensional optimization problem for which, in principle, one can apply quadratically convergent methods. The new approach may effectively increase the set of problems to which H_∞ control can be applied.

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