A New Test for Passivity of Descriptor Systems

Peter Benner
(joint work with Delin Chu)

Passivity is an important concept in circuit and control theory [1]. A linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= Cx(t) + Du(t), \qquad C \in \mathbb{R}^{p\times n},\ B \in \mathbb{R}^{p\times m}, \end{aligned}$$

is passive if and only if its transfer function

$$G(s) = C(sI_n - A)^{-1}B + D$$

is positive real. The positive real lemma (or Kalman-Yakubovich-Popov-Anderson lemma) states that for minimal systems this is the case if and only if the *linear matrix inequality (LMI)*

$$\begin{bmatrix} A^TX + XA & XB - C^T \\ B^TX - C & -(D + D^T) \end{bmatrix} \leq 0. \tag{1}$$

has a positive semidefinite solution $X \in \mathbb{R}^{n\times n}$. Moreover, if $D + D^T > 0$, then $G(s)$ is (strictly) positive real if the *algebraic Riccati equation (ARE)*

$$A^TX + XA + (XB - C^T)(D + D^T)^{-1}(B^TX - C) = 0,$$

has a stabilizing solution X.

Recently, a similar LMI-based criterion for testing positive realness (and thereby passivity) of descriptor systems

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \qquad A, E \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= Cx(t) + Du(t), \qquad C \in \mathbb{R}^{p\times n},\ B \in \mathbb{R}^{p\times m}, \end{aligned} \tag{2}$$

was proposed by Freund and Jarre in [3]. It states that the descriptor system (2) is positive real if the LMI (1) has a solution satisfying $E^TX = X^TE \geq 0$. A Riccati equation based test is not known in the case that E is singular.

The task of checking passivity of descriptor systems arises, e.g., when validating models of passive devices generated by automatic modeling tools. It also plays an important role in model order reduction techniques for the large-scale dynamical systems that arise in the simulation of VLSI circuits. Most of the methods in use for this purpose do not compute a reduced-order model that can be guaranteed to be passive. Unfortunately, the complexity of solving the semidefinite program related to the LMI arising in the positive real lemma for descriptor systems makes this test infeasible for many of the aforementioned applications. For single-input single-output systems, a positive realness test exclusively relying on eigenvalue computations is proposed in [2], but this does not extend to the general situation.

Here, we investigate a numerical method for testing whether a given general rational matrix is positive real. The main features of our method are:

- it exclusively relies on orthogonal restricted system equivalence transformations;
- it has the acceptable computational complexity of order n^3;

- it can be implemented in a numerically reliable manner.

The main contribution is that the positive realness test for an arbitrary rational matrix function is reduced to testing positive realness of a proper rational matrix function in a special format. Employing this special format, we can use the positive real lemma for standard systems by employing a recursive reduction procedure along the lines of the method proposed in [4].

The following lemma is the main step needed for the reduction to the case of a proper rational function.

Lemma 1. *For any regular pencil $A - \lambda E$ there exist orthogonal matrices $U, V \in \mathbb{R}^{n \times n}$ such that*

$$U(A-\lambda E)V \quad = \quad \begin{array}{cc} \begin{array}{cccc} n_1 & n_2 & n_3 & n_4 \end{array} & \\ \left[\begin{array}{cccc} A_{11}-\lambda E_{11} & A_{12}-\lambda E_{12} & A_{13}-\lambda E_{13} & A_{14}-\lambda E_{14} \\ 0 & A_{22} & A_{23}-\lambda E_{23} & A_{24}-\lambda E_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{array} \right] & \begin{array}{l} \}n_1 \\ \}n_3 \\ \}n_2 \\ \}n_4 \end{array} \end{array},$$

where $\operatorname{rank}(E_{11}) = n_1, \quad \operatorname{rank}(E_{23}) = n_3, \quad \operatorname{rank}(A_{44}) = n_4$*, and*

$$\operatorname{rank}\left(\begin{bmatrix} A_{22} & A_{23}-\lambda E_{23} \\ 0 & A_{33} \end{bmatrix} \right) = n_2 + n_3 \quad \forall \lambda \in \mathbb{C}.$$

The proof of this lemma is constructive and yields an algorithm to compute the given form. The algorithm requires a sequence of orthogonal decompositions including URV and QR factorizations and the computation of a generalized Schur form.

Our positive realness test makes use of the fact that the transfer function of (2) has an expansion at $s = \infty$ of the form

$$G(s) = C(sE-A)^{-1}B + D = \sum_{k=-\infty}^{q} s^k M_k,$$

where $M_k \in \mathbb{R}^{m \times m}$ are the *Markov parameters* of G. Positive realness can be related to the Markov parameters as follows, see [1, 3].

Proposition 1. *Given a rational matrix-valued function*

$$G(s) = G_p(s) + sM_1 + \sum_{k=2}^{q} s^k M_k,$$

where G_p is the proper part of G, then $G(s)$ is positive real if and only if

(1) *$G_p(s)$ is positive real,*
(2) *$M_1 \geq 0$,*
(3) *$M_k = 0$, $k = 2, 3, \ldots, q$.*

Applying the restricted system equivalence

$$(E, A, B, C, D) \mapsto (UEV, UAV, UB, CV, D)$$

induced by the matrices U, V from Lemma 1, we can prove the following result.

Lemma 2. *Let (E, A, B, C, D) be a minimal realization of the descriptor system 2. Then:*

a) *If the descriptor system is positive real, then $n_2 = n_3$.*
b) *If $n_2 = n_3$, then $M_k \geq 0$ for all $k \geq 2$.*

With this lemma, the positive realness test of minimal descriptor systems is reduced to checking $M_1 \geq 0$ and positive realness of the proper part of G.

We can now distinguish two cases.

Case 1: $n_2 = n_3 = 0$**::** in this case, it is easy to see that in the new coordinates induced by Lemma 1, $M_1 = 0$ and the proper part of G can be transformed via another orthogonal restricted system equivalence to

$$G_p(s) = \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix} \left(s \begin{bmatrix} \mathcal{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} + \mathcal{D}, \tag{3}$$

with $\mathcal{E}_{11}, \mathcal{A}_{22}$ nonsingular.

Case 2: $n_2 = n_3 \neq 0$**:** this case is slightly more involved, but using the structure imposed by Lemma 1, we obtain a reliable test for $M_1 \geq 0$ and we can show that $G_p(s)$ can be transformed to the same representation as in (3) using again only orthogonal transformations.

Thus, in both cases, we have reduced the passivity test for descriptor systems to testing positive realness of a proper transfer function which can be done using the Riccati equation-based criterion resulting from the standard positive real lemma together with a specially adapted version of the recursive reduction procedure given in [4].

References

[1] B.D.O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis*, Prentice-Hall, Englewood Cliffs, NJ (1973).

[2] Z. Bai and R.W. Freund, *Eigenvalue-based characterization and test for positive realness of scale transfer functions*, IEEE Trans. Automat. Control **AC-45** (2000), 2396–2401.

[3] R.W. Freund and F. Jarre, *An extension of the positive real lemma to descriptor systems*, Optim. Meth. Softw. **18** (2004), 69–87.

[4] H. Weiss, Q. Wang and J.L. Speyer. *System characterization of positive real conditions*, IEEE Trans. Automat. Control **AC-39** (1994), 541–544.