New Hamiltonian Eigensolvers with Applications in Control

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Outline

- (Skew-)Hamiltonian Eigenproblems
- Algorithms for (Skew-)Hamiltonian Eigenproblems
- HAPACK
- Applications
- Conclusions

(Skew-)Hamiltonian Matrices

$$\text{Hamiltonian:} \quad \mathcal{H} := \left\{ \begin{bmatrix} A & G \\ Q & -A^T \end{bmatrix} \in \mathbb{R}^{2n,2n}, \quad G, Q \text{ symmetric} \right\}$$

$$\text{skew-Hamiltonian:} \quad \mathcal{SH} := \left\{ \begin{bmatrix} A & G \\ Q & A^T \end{bmatrix} \in \mathbb{R}^{2n,2n}, \quad G, Q \text{ skew-symmetric} \right\}$$

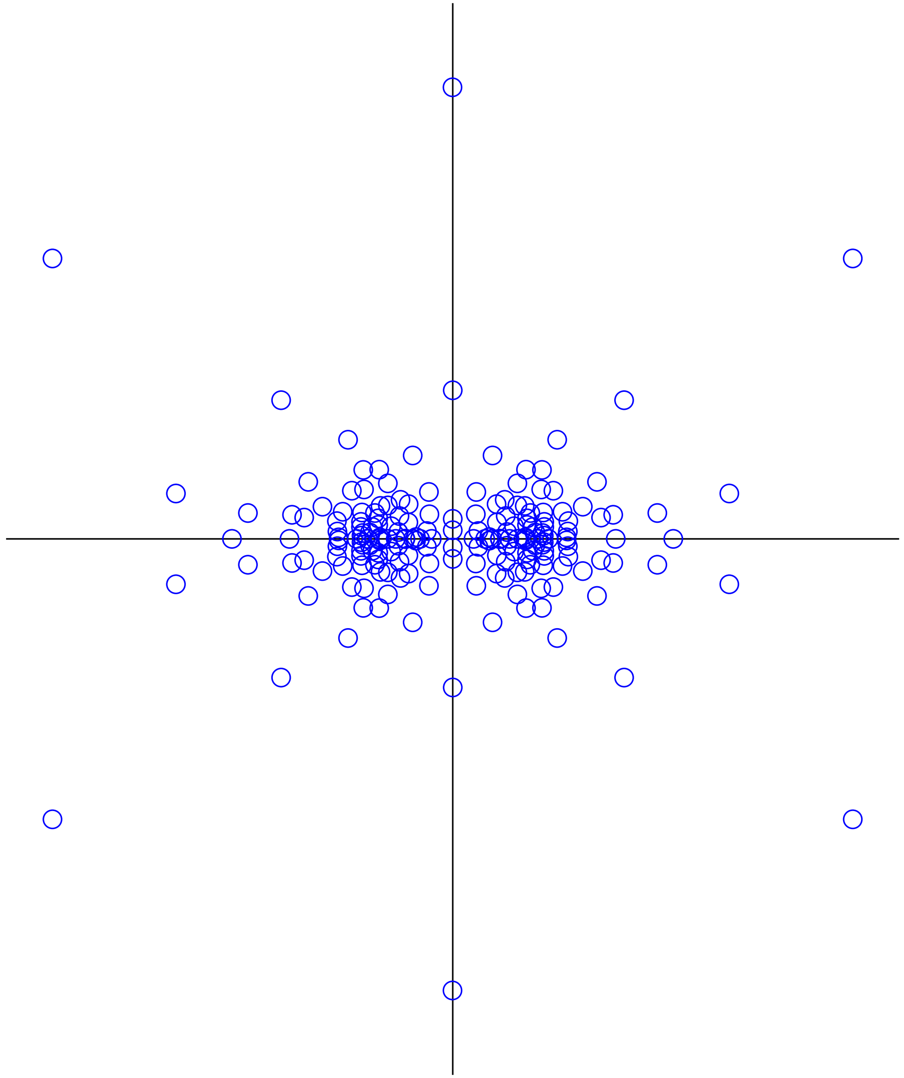
Fundamental properties:

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \Rightarrow \begin{array}{rcl} \mathcal{H} & = & \{H : (JH)^T = JH\} \\ \mathcal{SH} & = & \{W : (JW)^T = -JW\} \end{array}$$

$$\mathcal{H}^2 = \mathcal{SH}$$

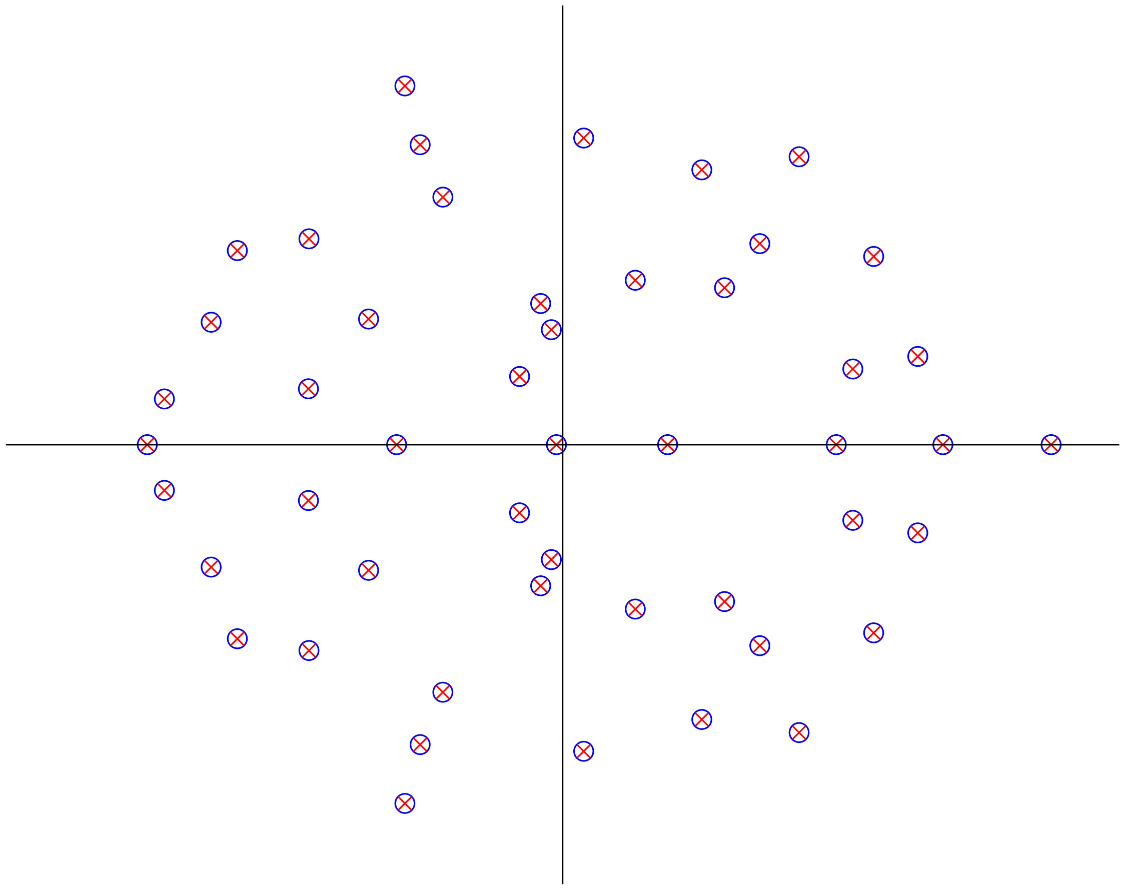
Spectra of Hamiltonian Matrices

$\lambda \in \mathrm{spec}(H) \implies$
$\overline{\lambda}, -\lambda, -\overline{\lambda} \in \mathrm{spec}(H)$



Spectra of Skew-Hamiltonian Matrices

$$\lambda \in \mathrm{spec}(W) \implies$$
$$\bar{\lambda}, \lambda, \bar{\lambda} \in \mathrm{spec}(W)$$



Applications

- LQR/LQG control, solution of algebraic Riccati equations
- Computation of stability radius and H_∞/L_∞ norm
- H_∞ control
- Gyroscopic systems, quadratic eigenvalue problems
- Model reduction: stochastic balanced truncation, truncation based on positive-real or bounded-real balancing, LQG balancing.

Preservation of spectral symmetries is essential for all applications!

General-purpose algorithms (QR, Jacobi, Jacobi-Davidson, Arnoldi, ...) do not preserve symmetries in finite precision arithmetic.

Structure-Preserving Algorithms

Advantages of exploiting the structure in numerical algorithms:

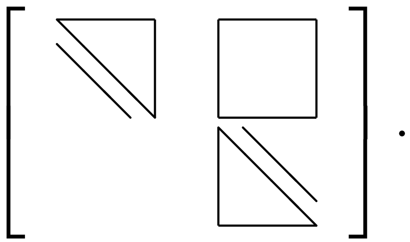
- Faster computations as complexity is reduced.
- Structured condition numbers (sometimes) smaller than unstructured ones.
- (Sometimes) more accurate results.
- Physically meaningful results. Unstructured algorithms lead to non-physical results, e.g.
  - applying `eig` (QR algorithm) to Hamiltonian matrices may yield $n+k$ stable and k unstable computed eigenvalues, where $1 \leq k \leq n$,
  - purely imaginary eigenvalues of Hamiltonian matrices cannot be detected in general ($\rightarrow$ stability radius, H_∞-norm, . . .)
- More fun.

Structured Algorithm for Skew-Hamiltonian Matrices (Van Loan '84)

Based on: If U is orthogonal ($U^T U = I$) and symplectic ($U^T J U = J$)
$\Rightarrow$ $U^T W U$ is skew-Hamiltonian.

1. Compute orthogonal symplectic U s.t.

$$U^T W U = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{11}^T \end{bmatrix} =$$



2. Apply standard QR algorithm to W_{11}.

Structured Algorithm for Hamiltonian Matrices (B./Mehrmann/Xu'97)

Based on: H^2 is skew-Hamiltonian, but H^2 is not formed explicitely.

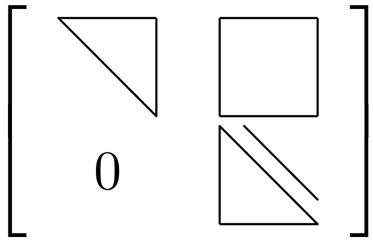
If U, V are orthogonal symplectic, then

$$U^T H^2 U = (U^T H V)(V^T H U) = U^T H V J (U^T H V)^T J$$

$\Rightarrow$ $U^T H V$ can be used to reveal eigenvalues of H.

1. Compute orthogonal symplectic U, V s.t.

$$U^T H V =: R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & -R_{22}^T \end{bmatrix} =$$



.

2. Apply periodic QR to $R_{11} \cdot R_{22}$ $\Rightarrow$ $\Lambda(R_{11} R_{22}) = \Lambda(H)^2$.

HAPACK Routines for Skew-Hamiltonian Matrices

FORTRAN 77 routines:

DSHBAL	symplectic balancing
DSHPVB	blocked reduction to PVL form (step 1)
DSHSEN	specified invariant subspaces
DSHSNA	structured eigenvalue and eigenvector condition numbers/estimates
DSH***	various driver routines

MATLAB routines (based on MEX interfaces):

shbal	symplectic scaling and balancing
sheig	eigenvalues and eigenvectors
shschur	skew-Hamiltonian Schur form
shinv	specified invariant subspaces
shcondeig	structured eigenvalue and eigenvector condition numbers/estimates

HAPACK Routines for Hamiltonian Matrices

FORTRAN 77 routines:

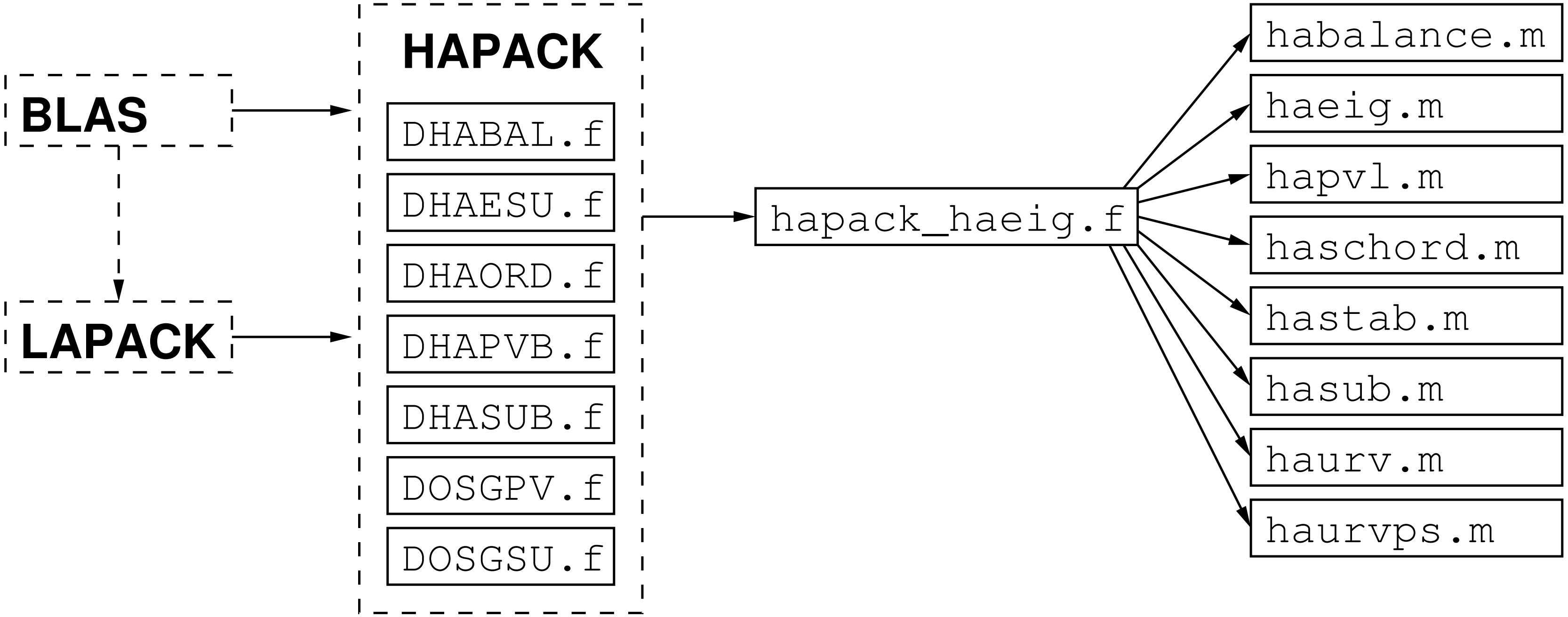
DHABAL	symplectic balancing
DHAPVB	blocked reduction to PVL form
DHAESU	blocked reduction to symplectic URV form (step 1)
DHGPQR	multi-shift periodic QR for $A \cdot B$ (step 2)
DHAORD	reorder the Hamiltonian Schur decomposition
DHASUB	specified invariant subspaces

MATLAB routines (based on MEX interfaces):

habalance	symplectic balancing
haeig	eigenvalues (supports complex Hamiltonian matrices)
hapvl	PVL decomposition
haschord	reorders Hamiltonian Schur form
hastab	stable/unstable subspaces
hasub	selected stable/unstable subspaces
haurv	symplectic URV decomposition
haurvps	symplectic URV/periodic Schur decomposition of a general matrix

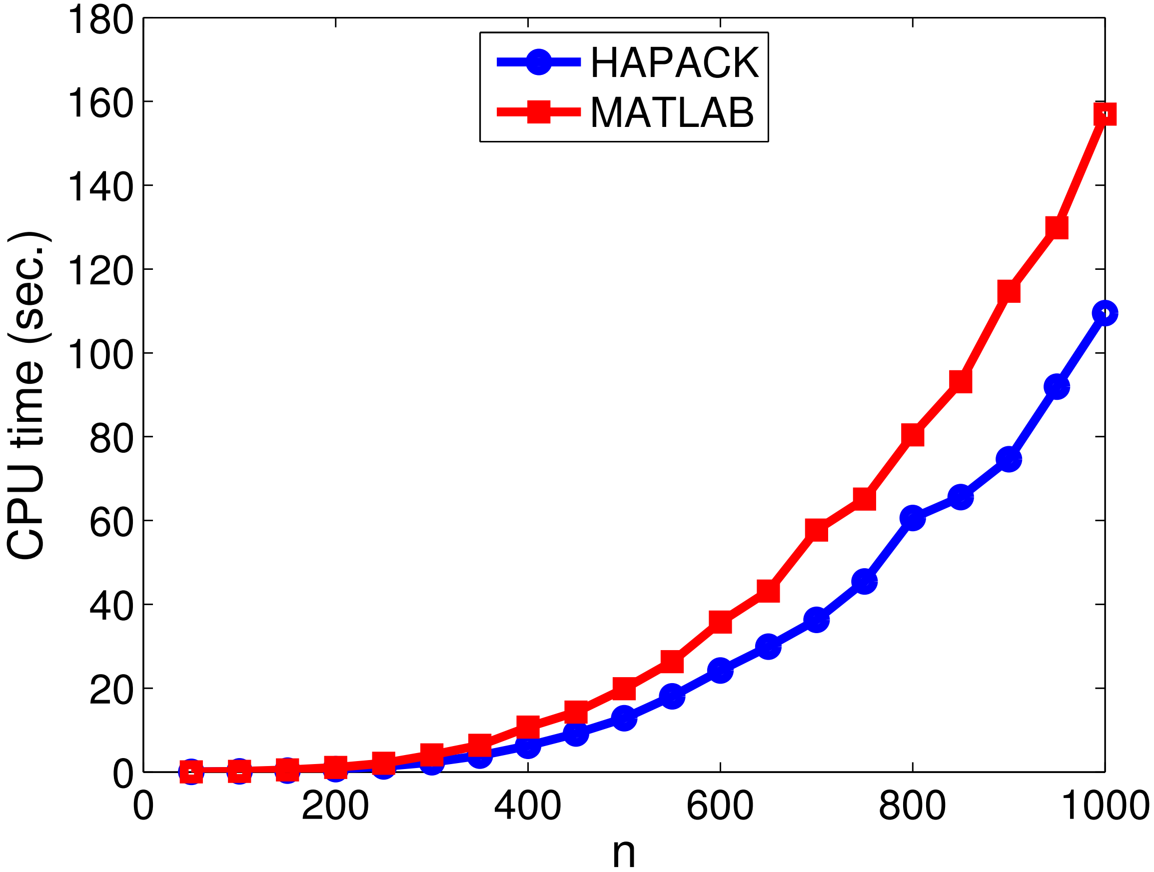
Structure of HAPACK

HAPACK MATLAB functions are called via the MEX gateway `hapack_haeig.f`.



Performance

Random Hamiltonian matrices, CPU times for haeig (HAPACK) and eig (MATLAB).



Applications

Linear continuous-time systems with constant coefficients in state-space form,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

Related issues:

1. algebraic Riccati equations
2. LQ(R,G) control (see paper)
3. stability radius computation
4. model reduction

Stability Radius Computation

Stability radius of A: $\gamma(A) := \min\{\|E\|_2 \,:\, \lambda(A+E) \cap \imath\mathbb{R} \neq \emptyset\}$.

Bisection method for computing $\gamma(A)$ can be based on the following observation:

[Byers '88, Hinrichsen/Pritchard '86]

if $\alpha \geq 0$, then the Hamiltonian matrix

$$H(\alpha) = \begin{bmatrix} A & -\alpha I_n \\ \alpha I_n & -A^T \end{bmatrix}$$

has an eigenvalue on the imaginary axis if and only if $\alpha \geq \gamma(A)$.

HAPACK Implementation

For lower and upper bounds $\beta \geq 0$ and $\delta > \gamma(A)$, obtain HAPACK implementation:

```
nA = norm(A+A','fro') / 2;
beta  = 0;
delta = nA;
e = ones(n,1);
QG = full(spdiags([e -e], 0:1, n, n+1));
while (delta-beta) > 100*eps*nA,
   e = haeig(A,(beta + delta)/2 * QG);
   if isempty( find(real(e)==0) ),
      beta  = (beta + delta)/2;
   else
      delta = (beta + delta)/2;
   end
end
```

Quadratically convergent variants [Bruinsma/Steinbuch '90, Boyd/Balakrishnan '90] can also be based on HAPACK, will be available in SLICOT!

Example: The Demmel Matrix

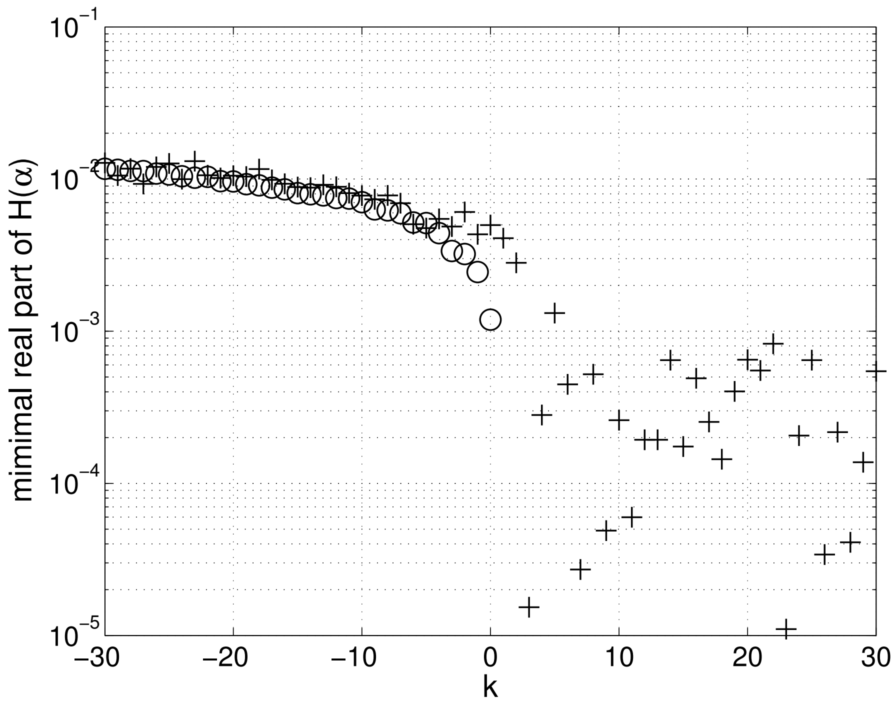
$$A = Q^T \begin{bmatrix} -0.05 & -10 & -10^2 & -10^3 & -10^4 \\ 0 & -0.05 & -10 & -10^2 & -10^3 \\ 0 & 0 & -0.05 & -10 & -10^2 \\ 0 & 0 & 0 & -0.05 & -10 \\ 0 & 0 & 0 & 0 & -0.05 \end{bmatrix} Q, \quad Q \text{ orthogonal, randomly generated.}$$

$\gamma(A) \approx 3.6 \times 10^{-11}$!

Right: minimum absolute values for real parts of spec$(H(\alpha))$, computed by `eig` ('$+$') and `haeig` ('$\circ$').

$\alpha = 1.005^k \times \gamma(A)$, $k \in [-30, 30]$.

For $k > 0$, `haeig` computes exact zeroes which are not shown in the figure.



Model Reduction

Passivity-preserving model reduction for linear systems can be based on truncating a positive real balanced realization, obtained from balancing the Gramians given as solutions of the dual algebraic Riccati equations

$$\begin{aligned} 0 &= \hat{A}P + P\hat{A}^T + PC^TR^{-1}CP + BR^{-1}B^T, \\ 0 &= \hat{A}^TQ + Q\hat{A} + QBR^{-1}B^TQ + C^TR^{-1}C, \end{aligned}$$

where

$$\begin{aligned} R &= D + D^T, \\ \hat{A} &= A - BR^{-1}C. \end{aligned}$$

HAPACK Implementation

Solving positive real AREs with HAPACK requires only one call to `hastab` as P, Q can be obtained from stable and anti-stable invariant subspaces:

```
R = chol(D+D');
B = B / R;
C = R' \ C;
A = A - B*C;
H = [ A  -B*B'; C'*C -A'];
[X,Y,e] = hastab(H);
P = -Y(1:n,:) / Y(n+1:2*n,:);
Q = -X(n+1:2*n,:) / X(1:n,:);
```

Conclusions

- HAPACK provides efficient Fortran 77 and MATLAB routines for Hamiltonian eigenproblems and related control problems.

- In contrast to other MATLAB software for the considered control problems, HAPACK provides reliable and physically meaningful results.

- Software is freely available at

  www.tu-chemnitz.de/mathematik/hapack/

  and will (partially) be included in SLICOT (www.slicot.net).

- Most HAPACK routines are described in Algorithm 8xx: Fortran 77 Subroutines for Computing the Eigenvalues of Hamiltonian Matrices II
  PETER BENNER, DANIEL KRESSNER, *ACM Transactions on Mathematical Software*, to appear.

... Thanks for your attention!