MODEL REDUCTION FOR PARABOLIC CONTROL SYSTEMS BASED ON BALANCED TRUNCATION

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Outline

- Parabolic PDEs as infinite-dimensional systems
- Model reduction based on balancing
  - balanced truncation
  - LQG balanced truncation
- Error bounds
- Solving large-scale matrix equations
  - ADI method for Lyapunov equations
  - Newton-ADI method for algebraic Riccati equations
- Numerical results
- Conclusions and open problems

Distributed Parameter Systems

Given Hilbert spaces

$\mathcal{X}$ – state space,

$\mathcal{U}$ – control space,

$\mathcal{Y}$ – output space,

and operators

$$\begin{aligned} \mathbf{A}: &\quad \text{dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}, \\ \mathbf{B}: &\quad \mathcal{U} \to \mathcal{X}, \\ \mathbf{C}: &\quad \mathcal{X} \to \mathcal{Y}. \end{aligned}$$

Then, a linear distributed parameter system in abstract form is given by

$$\Sigma : \left\{ \begin{array}{rcl} \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} & = & \mathbf{C}\mathbf{x}, \end{array} \right. \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}.$$

Parabolic Systems

Parabolic PDE in domain $\Omega \in \mathbb{R}^d$ (heat equation, convection-diffusion equation)

$$\frac{\partial x}{\partial t} = \sum_{i,j=1}^{d} \frac{\partial x}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial x}{\partial \xi_j}\right) + \sum_{i=1}^{d} b_i(\xi)\frac{\partial x}{\partial \xi_i} + c(\xi)x + B_{pc}(\xi)u(t), \quad \xi \in \Omega,\ t > 0$$

with initial and boundary conditions

$$\begin{aligned} \alpha(\xi)x(\xi,t) + \beta(\xi)\frac{\partial}{\partial \eta}x(\xi,t) &= B_{bc}(\xi)u(t), \qquad \xi \in \partial\Omega, \\ x(\xi,0) &= x_0(\xi), \quad \xi \in \Omega, \\ y &= Cx, \qquad t \geq 0 \end{aligned}$$

- $B_{pc} = 0 \Longrightarrow$ boundary control problem
- $B_{bc} = 0 \Longrightarrow$ point control problem

Weak formulation, test space $\mathcal{V} = H^1(\Omega) \rightsquigarrow$ distributed parameter system.

Infinite-Dimensional Systems

Assume

- $\mathbf{A}$ generates C_0-semigroup $T(t)$ on $\mathcal{X}$,
- $(\mathbf{A}, \mathbf{B})$ is exponentially stabilizable, i.e., there exists $\mathbf{F} : \text{dom}(\mathbf{A}) \mapsto \mathcal{U}$ such that $\mathbf{A} + \mathbf{BF}$ generates an exponentially stable C_0-semigroup $\mathbf{S}(\mathbf{t})$;
- $(\mathbf{A}, \mathbf{C})$ is exponentially detectable, i.e., $(\mathbf{A}^*, \mathbf{C}^*)$ is exponentially stabilizable;
- $\mathbf{B}, \mathbf{C}$ are compact, e.g., $\mathcal{U} = \mathbb{R}^m$, $\mathcal{Y} = \mathbb{R}^p$.

Then the system $\Sigma(A, B, C)$ has a transfer function $\mathbf{G} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \in L_\infty$.

If, in addition, $\mathbf{A}$ is exponentially stable, $\mathbf{G}$ is in the Hardy space H_∞.

Weaker assumptions:

$\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is Pritchard-Salomon system, allows for certain unboundedness of $\mathbf{B}, \mathbf{C}$.

Stable Systems

$\mathbf{G}$ is the Laplace transform of

$$\mathbf{h}(t) := \mathbf{C}T(t)\mathbf{B}$$

and the symbol of the Hankel operator $\mathbf{H} : L_2(0,\infty;\mathbb{R}^m) \mapsto L_2(0,\infty;\mathbb{R}^p)$,

$$(\mathbf{H}\mathbf{u})(t) := \int_0^\infty \mathbf{h}(t+\tau)u(\tau)\,d\tau.$$

$\mathbf{H}$ is compact with countable many singular values σ_j, $j = 1,\ldots,\infty$, called the Hankel singular values of $\mathbf{G}$. Moreover,

$$\sum_{j=1}^{\infty} \sigma_j < \infty.$$

The 2-induced operator norm is the H_∞ norm; here,

$$\|\mathbf{G}\|_{H_\infty} = \sum_{j=1}^{\infty} \sigma_j.$$

Balanced Realization

For $\mathbf{G} \in H_\infty$, $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a balanced realization of $\mathbf{G}$ if the controllability and observability Gramians, given by the unique self-adjoint positive semidefinite solutions of the Lyapunov equations

$$\begin{aligned} \mathbf{APz} + \mathbf{PA}^*\mathbf{z} + \mathbf{BB}^*\mathbf{z} &= 0 \quad \text{for } \mathbf{z} \in \operatorname{dom}(\mathbf{A}^*) \\ \mathbf{A}^*\mathbf{Qz} + \mathbf{QAz} + \mathbf{C}^*\mathbf{Cz} &= 0 \quad \text{for } \mathbf{z} \in \operatorname{dom}(\mathbf{A}) \end{aligned}$$

satisfy $\mathbf{P} = \mathbf{Q} = \operatorname{diag}(\sigma_j) =: \mathbf{\Sigma}$.

See [CURTAIN/GLOVER 1986, CURTAIN 2003].

Balanced Truncation

Model reduction by truncation:

choose r with $\sigma_r > \sigma_{r+1}$ and partition $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ according to

$$\mathbf{P}_r = \mathbf{Q}_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r),$$

so that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_r & * \\ * & * \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_r \\ * \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_r & * \end{bmatrix},$$

then the reduced-order model is the stable system $\Sigma_r(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$ with transfer function $\mathbf{G}_r$ satisfying

$$\|\mathbf{G} - \mathbf{G}_r\|_{H_\infty} \leq 2 \sum_{j=r+1}^{\infty} \sigma_j.$$

See [Glover/Curtain/Partington 1988, Curtain 2003].

LQG Balanced Realization

Previous theory only applicable for *stable* systems. Now: unstable systems.

For $\mathbf{G} \in L_\infty$, $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is an LQG-balanced realization of $\mathbf{G}$ if the unique self-adjoint, positive semidefinite, stabilizing solutions of the operator Riccati equations

$$\begin{aligned} \mathbf{APz} + \mathbf{PA}^*\mathbf{z} - \mathbf{PC}^*\mathbf{CPz} + \mathbf{BB}^*\mathbf{z} &= 0 \quad \text{for } \mathbf{z} \in \mathrm{dom}(\mathbf{A}^*) \\ \mathbf{A}^*\mathbf{Qz} + \mathbf{QAz} - \mathbf{QBB}^*\mathbf{Qz} + \mathbf{C}^*\mathbf{Cz} &= 0 \quad \text{for } \mathbf{z} \in \mathrm{dom}(\mathbf{A}) \end{aligned}$$

are bounded and satisfy $\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\gamma_j) =: \mathbf{\Gamma}$.

($\mathbf{P}$ stabilizing $\Leftrightarrow$ $\mathbf{A} - \mathbf{PC}^*\mathbf{C}$ generates exponentially stable C_0-semigroup.)

See [Curtain 2003].

LQG Balanced Truncation

Model reduction by truncation:

choose r with $\gamma_r > \gamma_{r+1}$ and partition $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ according to

$$\mathbf{P}_r = \mathbf{Q}_r = \mathrm{diag}(\gamma_1, \ldots, \gamma_r),$$

so that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_r & * \\ * & * \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_r \\ * \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_r & * \end{bmatrix},$$

then the reduced-order model is the LQG balanced system $\Sigma_r(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$ with transfer function $\mathbf{G}_r$ satisfying

$$\text{``}\|\mathbf{G} - \mathbf{G}_r\|_{L_\infty}\text{''} \leq 2 \sum_{j=r+1}^{\infty} \frac{\gamma_j}{\sqrt{1+\gamma_j^2}}.$$

See [CURTAIN 2003].

Computation of Reduced-Order Systems

Spatial discretization (FEM, FDM) $\rightsquigarrow$ finite-dimensional system

$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) = x_0,\\ y &= Cx,\end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, with corresponding algebraic Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

and algebraic Riccati equations (AREs)

$$\begin{aligned}0 &= \mathcal{R}_f(P) := AP + PA^T - PC^TCP + B^TB,\\ 0 &= \mathcal{R}_c(Q) := A^TQ + QA - QBB^TQ + C^TC.\end{aligned}$$

Convergence of Gramians [Curtain 2003]

Under given assumptions for $\mathbf{\Sigma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, the stabilizing solutions of finite-dimensional Lyapunov and Riccati equations converge in the nuclear norm to the solutions of the corresponding operator equations and transfer functions converge if the n-dimensional approximations satisfy the assumptions:

- $\mathcal{X}_n \subset \mathcal{X}$ with $\dim \mathcal{X}_n = n$ and orthogonal projector $\Pi_n : \mathcal{X} \mapsto \mathcal{X}_n$ such that

$$\Pi_n \mathbf{z} \rightarrow \mathbf{z} \ (n \rightarrow \infty) \quad \forall \mathbf{z} \in \mathcal{X}, \quad B = \Pi_n \mathbf{B}, \qquad C = \mathbf{C}|_{\mathcal{X}_n}.$$

- For all $\mathbf{z} \in \mathcal{X}$ and $n \rightarrow \infty$,

$$e^{At}\Pi_n \mathbf{z} \rightarrow T(t)\mathbf{z}, \qquad (e^{At})^* \Pi_n \mathbf{z} \rightarrow T(t)^* \mathbf{z},$$

  uniformly in t on bounded intervals.

- A is uniformly exponentially stable.
  (A, B, C) is uniformly exponentially stabilizable and detectable.

Computation of Reduced-Order Systems from Gramians

Given the Gramians P, Q of the n-dimensional system from either the Lyapunov equations or AREs in factorized form

$$P = S^T S, \quad Q = R^T R,$$

compute SVD

$$SR^T = [\, U_1,\, U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

Set

$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$

Then the reduced-order model is

$$(A_r, B_r, C_r) = (W^T A V, W^T B, CV).$$

Thus, need to solve large-scale matrix equations—but need only factors!

Error Bounds

For control applications, want $\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^m)}$ or $\|\mathbf{y}(t) - y_r(t)\|_2$.

Error bound includes approximation errors caused by

- Galerkin projection/spatial FEM discretization,
- model reduction.

Goal: balance the two errors vs. each other.

Output Error Bound I

Assume $\mathbf{C}$ bounded, $C = \mathbf{C}|_{\mathcal{X}_n}$, $\mathcal{X}_n \subset \mathcal{X}$. Then:

$$
\begin{aligned}
\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^m)} &\leq \|\mathbf{y} - y\|_{L_2(0,T;\mathbb{R}^m)} + \|y - y_r\|_{L_2(0,T;\mathbb{R}^m)} \\
&= \|\mathbf{Cx} - Cx\|_{L_2(0,T;\mathbb{R}^m)} + \|y - y_r\|_{L_2(0,T;\mathbb{R}^m)} \\
&\leq \underbrace{\|\mathbf{C}\|}_{=:c} \cdot \underbrace{\|\mathbf{x} - x\|_{L_2(0,T;\mathcal{X})}}_{\text{FEM error}} + \underbrace{\|y - y_r\|_{L_2(0,T;\mathbb{R}^m)}}_{\text{model reduction error}}.
\end{aligned}
$$

Output Error Bound II

Corollary:

Balanced truncation:

$$\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^m)} \leq c\|\mathbf{x} - x\|_{L_2(0,T;\mathcal{X})} + 2\|u\|_{L_2(0,T;\mathbb{R}^p)} \sum_{j=r+1}^{n} \sigma_j.$$

LQG balanced truncation:

$$\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^m)} \leq c\|\mathbf{x} - x\|_{L_2(0,T;\mathcal{X})} + 2\|u\|_{L_2(0,T;\mathbb{R}^p)} \sum_{j=r+1}^{n} \frac{\gamma_j}{\sqrt{1+\gamma_j^2}}.$$

Large-Scale Algebraic Lyapunov and Riccati Equations

General form for $A, G = G^T, W = W^T \in \mathbb{R}^{n \times n}$ given and $P \in \mathbb{R}^{n \times n}$ unknown:

$$\begin{aligned} 0 &= \mathcal{L}(P) := A^T P + PA + W, \\ 0 &= \mathcal{R}(P) := A^T P + PA - PGP + W. \end{aligned}$$

In large scale applications from semi-discretized control problems for PDEs,

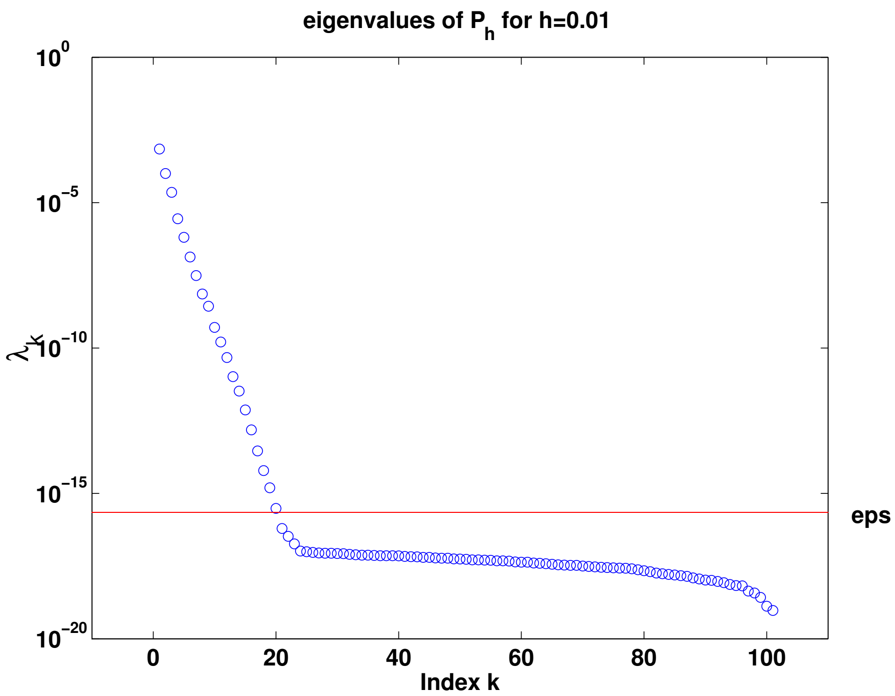
- $n = 10^3$ – 10^6 ($\Longrightarrow$ 10^6 – 10^{12} unknowns!),
- A has sparse representation ($A = -M^{-1}K$ for FEM),
- G, W low-rank with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n \times m}$, $m \ll n$, $C \in \mathbb{R}^{p \times n}$, $p \ll n$.
- Standard (eigenproblem-based) $\mathcal{O}(n^3)$ methods are not applicable!

Low-Rank Approximation

Consider spectrum of ARE solution (analogous for Lyapunov equations).

Example:

- Linear 1D heat equation with point control,
- $\Omega = [\,0,\ 1\,]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \Longrightarrow n = 101.$



Idea:

$$P = P^T \geq 0 \quad \Longrightarrow \quad P = ZZ^T = \sum_{k=1}^{n} \lambda_k z_k z_k^T \approx Z^{(r)}(Z^{(r)})^T = \sum_{k=1}^{r} \lambda_k z_k z_k^T .$$

ADI Method for Lyapunov Equations

- For $F \in \mathbb{R}^{n\times n}$ stable, $W \in \mathbb{R}^{n\times w}$ $(w \ll n)$, consider Lyapunov equation

$$F^T X + XF = -BB^T.$$

- ADI Iteration: [Wachspress 1988]

$$\begin{aligned}(F^T + p_k I)X_{(j-1)/2} &= -BB^T - X_{k-1}(F - p_k I)\\ (F^T + \overline{p_k} I){X_k}^T &= -BB^T - X_{(j-1)/2}(F - \overline{p_k} I)\end{aligned}$$

with parameters $p_k \in \mathbb{C}^-$ and $p_{k+1} = \overline{p_k}$ if $p_k \notin \mathbb{R}$.

- For $X_0 = 0$ and proper choice of p_k: $\lim_{k\to\infty} X_k = X$ superlinear.
- Re-formulation using $X_k = Y_k Y_k^T$ yields iteration for Y_k...

Factored ADI Iteration

[Penzl 1997, Li/White 2002, B./Li/Penzl 1999/2005]

Set $X_k = Y_k Y_k^T$, some algebraic manipulations $\Longrightarrow$

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(F^T + p_1 I)^{-1}B, \qquad Y_1 \leftarrow V_1$$

FOR $j = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(I - (p_k + \overline{p_{k-1}})(F^T + p_k I)^{-1}\right) V_{k-1}, \quad Y_k \leftarrow \left[\begin{array}{cc} Y_{k-1} & V_k \end{array}\right]$$

$$\Downarrow$$

$$Y_{k_{\max}} = \left[\begin{array}{ccc} V_1 & \ldots & V_{k_{\max}} \end{array}\right]$$

where

$$V_k = \Big[\;\Big] \in \mathbb{C}^{n \times w}$$

and

$$Y_{k_{\max}} Y_{k_{\max}}^T \approx X$$

Note: Implementation in real arithmetic possible by combining two steps.

Newton's Method for AREs

– Consider $0 = \mathcal{R}(P) = C^T C + A^T P + PA - PBB^T P.$

– Frechét derivative of $\mathcal{R}(P)$ at P: $\mathcal{R}'_P : Z \to (A - BB^T P)^T Z + Z(A - BB^T P).$

– Newton-Kantorovich method: $P_{j+1} = P_j - \left(\mathcal{R}'_{P_j}\right)^{-1} \mathcal{R}(P_j), \quad j = 0, 1, 2, \ldots$

$\Longrightarrow$ Newton's method (with line search) for AREs (for given $P_0 = P_0^T$ stabilizing):

FOR $j = 0, 1, \ldots$

1. $A_j \leftarrow A - BB^T P_j =: A - BK_j$.
2. Solve the Lyapunov equation $A_j^T N_j + N_j A_j = -\mathcal{R}(P_j).$
3. $P_{j+1} \leftarrow P_j + t_j N_j$.

END FOR j

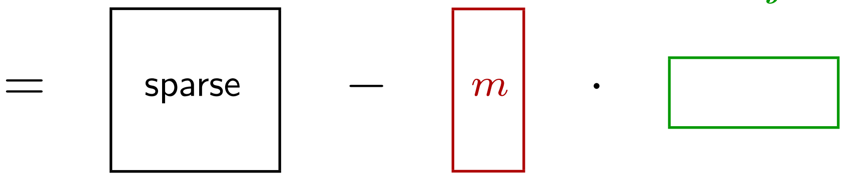
[Kleinman '68, Mehrmann '91, Lancaster/Rodman '95, B./Byers '94/'98, B. '97, Guo/Laub '99]

Properties and Implementation

- Convergence for $\Lambda\left(A - BK_0\right)$ stabilizing:
  - $A_j = A - BK_j = A - BB^T P_j$ is stable $\forall\ j \geq 1$.
  - $\lim_{j\to\infty} \|\mathcal{R}(P_j)\|_F = 0$ (monotonically).
  - $\lim_{j\to\infty} P_j = P_* \geq 0$ (locally quadratic).

- Need large-scale Lyapunov solver; here, ADI iteration [WACHSPRESS 1988]: linear systems with dense, but "sparse+low rank" coefficient matrix $F = A_j$:

$$A_j \quad = \quad A \quad - \quad B \quad \cdot \quad K_j$$



$m \ll n \Longrightarrow$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j)^{-1} = (I_n + A^{-1}B(I_m - K_jA^{-1}B)^{-1}K_j)A^{-1}.$$

- BUT: $P = P^T \in \mathbb{R}^{n\times n} \Longrightarrow n(n+1)/2$ unknowns!

Low-Rank Newton-ADI for AREs

Re-write Newton's method for AREs [Kleinman 1968]

$$A_j^T N_j + N_j A_j = -\mathcal{R}(P_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(P_j + N_j)}_{=P_{j+1}} + \underbrace{(P_j + N_j)}_{=P_{j+1}} A_j = \underbrace{-C^T C - P_j B B^T P_j}_{=:-W_j W_j^T}$$

Set $P_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T \right) + \left(Z_{j+1} Z_{j+1}^T \right) A_j = -W_j W_j^T$$

$$\Downarrow$$

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and use *'sparse + low-rank'* structure of A_j. [B./Li/Penzl 1999/2005]

Application to LQR Problem

LQR problem: linear-quadratic optimization problem w/o control/state constraints is solved by feedback control law

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t) := \mathbf{B}^*\mathbf{Q}\mathbf{x}(t),$$

where $\mathbf{Q}$ is the unique selfadjoint nonnegative definite solution to the 2nd operator Riccati equation in LQG approach.

Finite-dimensional approximation is

$$u(t) = K_* x(t) := B^T Q_* x(t),$$

where Q_* is the stabilizing solution of the corresponding ARE.

Feedback Iteration

K_* can be computed by direct feedback iteration:

- jth Newton iteration:

$$K_j = B^T Z_j Z_j^T = \sum_{k=1}^{k_{\max}} (B^T V_{j,k}) V_{j,k}^T \quad \xrightarrow{j\to\infty} \quad K_* = B^T Z_* Z_*^T$$

- K_j can be updated in ADI iteration, no need to even form Z_j, need only fixed workspace for $K_j \in \mathbb{R}^{m\times n}$!

Optimal Control from Reduced-Order Model

If the reduced-order model is used to compute approximate solution to LQR solution, we have

$$u_r(t) = K_{r,*}x_r(t) := B_r Q_{r,*} x_r(t).$$

Theorem: Let K_* be the feedback matrix computed from finite-dimensional approximation to LQR problem, $K_{r,*}$ the feedback matrix obtained from the LQR problem for the LQG reduced-order model obtained using the projector VW^T, then

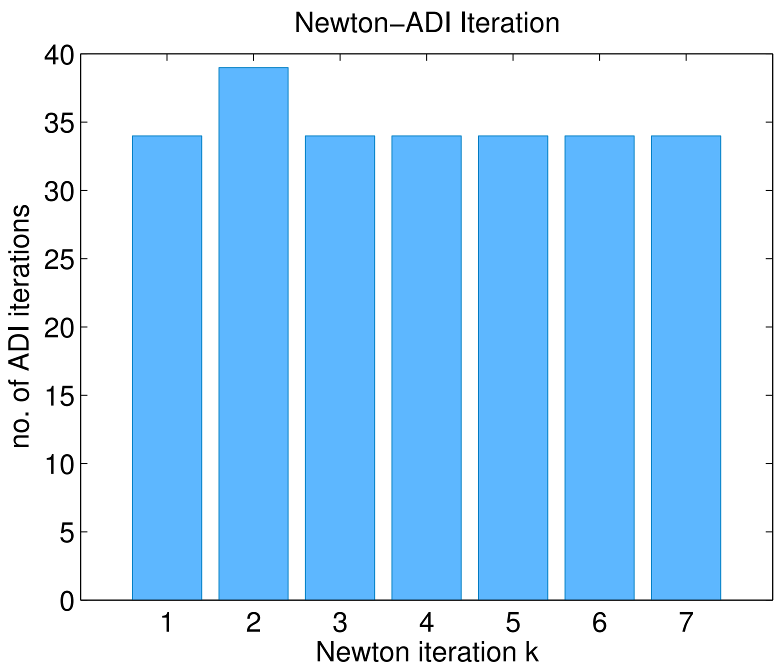
$$K_{r,*} = K_* V.$$

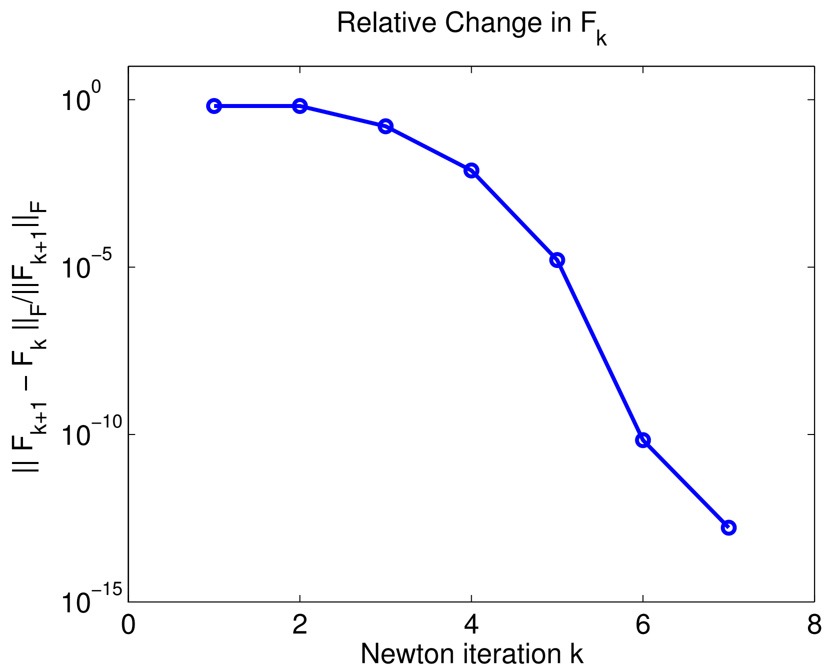
Consequence: the reduced-order optimal control can be computed as by-product in the model reduction process!

Similar result for LQG controller.

Numerical Results

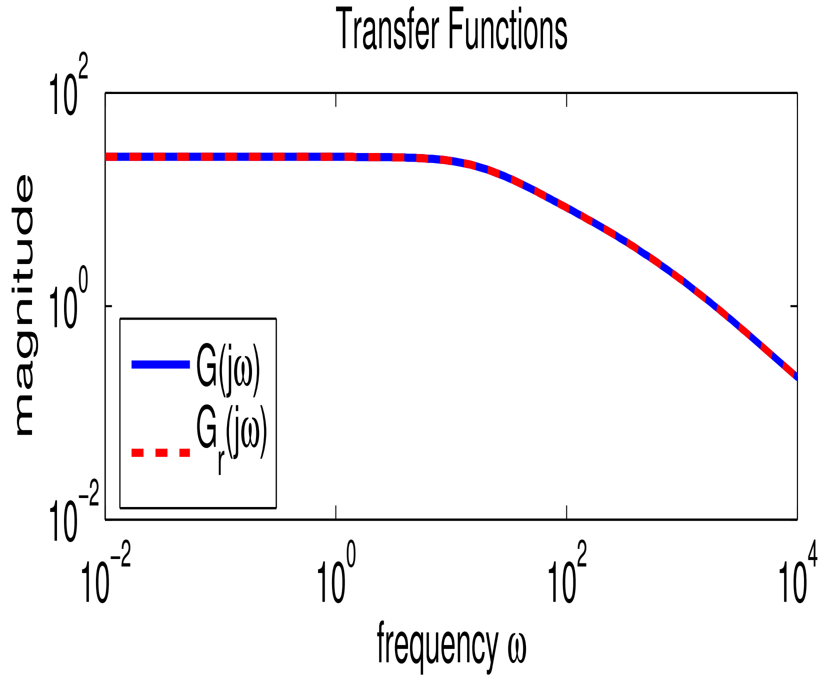
- Linear 2D heat equation with homogeneous Dirichlet boundary and point control/observation.
- FD discretization on uniform 150×150 grid.
- $n = 22.500$, $m = p = 1$, 10 shifts for ADI iterations.
- Convergence of large-scale matrix equation solvers:

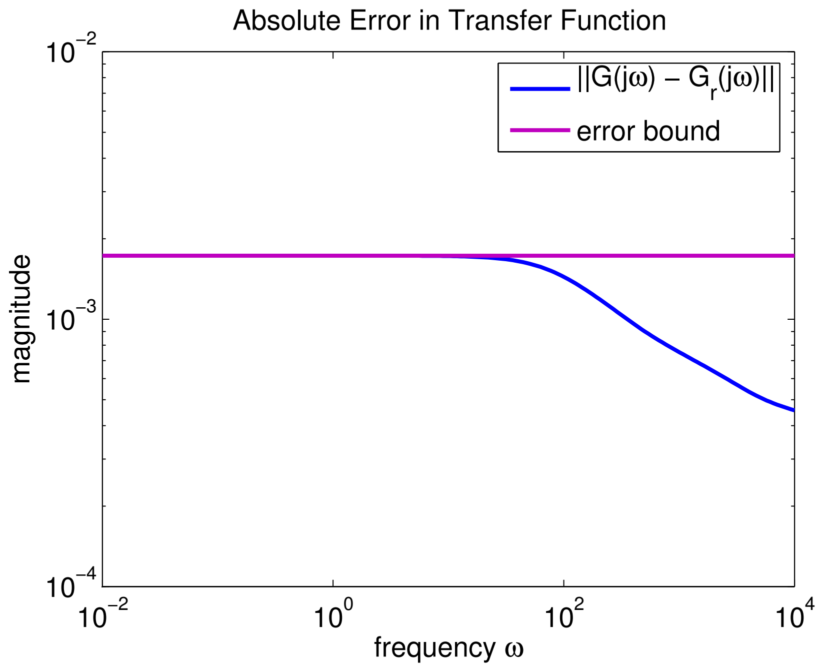




Numerical Results (2D heat equation ctd.)

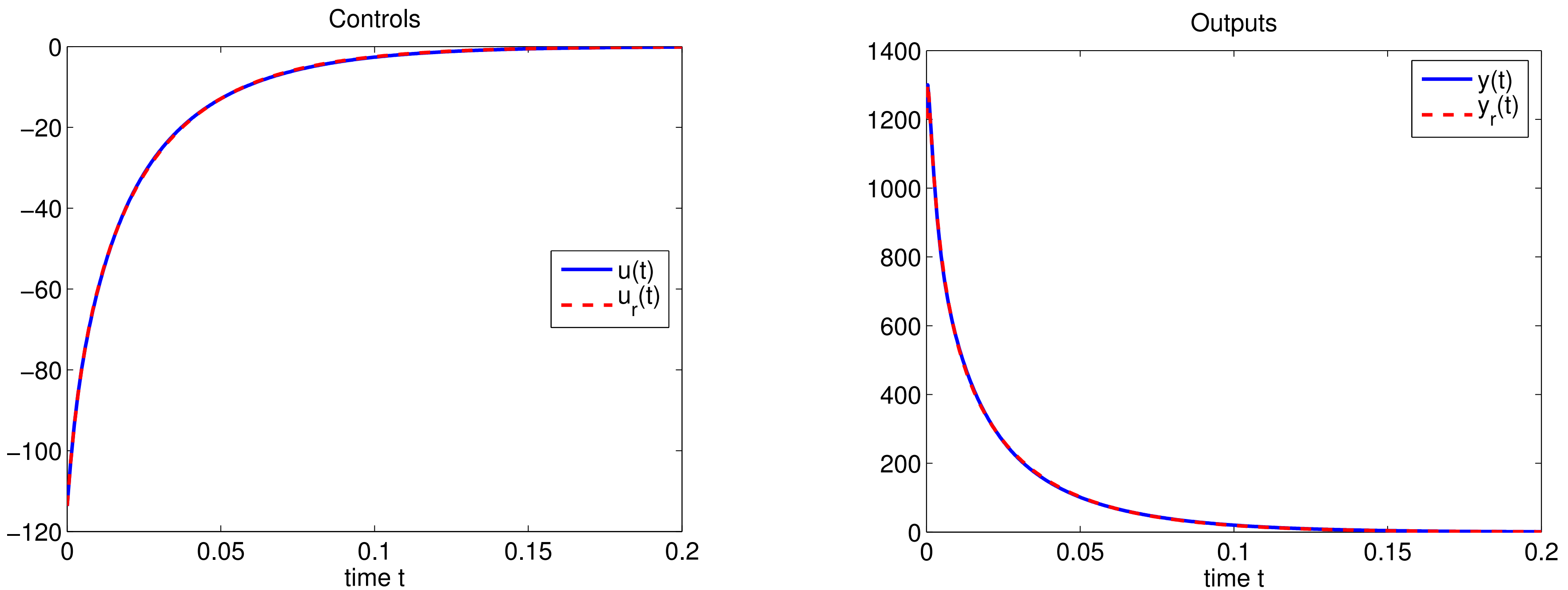
- Numerical ranks of Gramians are 31 and 26, respectively.
- Computed reduced-order model (BT): $r = 6$.
- BT error bound $\delta = 1.7 \cdot 10^{-3}$ ($\sigma_7 = 5.8 \cdot 10^{-4}$, $\sigma_{26} = 9.3 \cdot 10^{-16}$.)





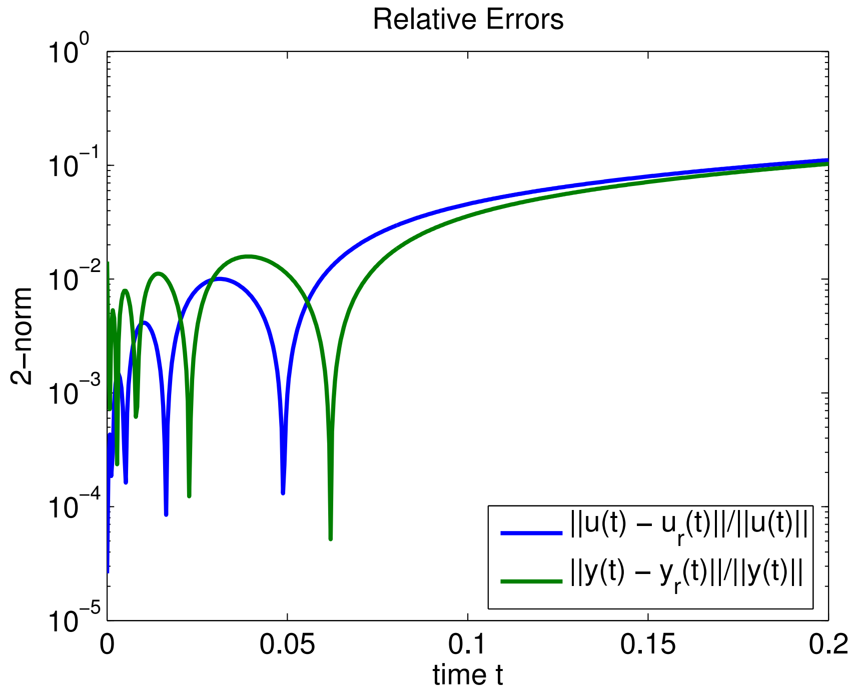
Numerical Results (2D heat equation ctd.)

- Computed reduced-order model (BT): $r = 6$, BT error bound $\delta = 1.7 \cdot 10^{-3}$.
- Solve LQR problem: quadratic cost functional, solution is linear state feedback.
- Computed controls and outputs (implicit Euler):



Numerical Results (2D heat equation ctd.)

- Computed reduced-order model (BT): $r = 6$, BT error bound $\delta = 1.7 \cdot 10^{-3}$.
- Solve LQR problem: quadratic cost functional, solution is linear state feedback.
- Errors in controls and outputs:



Numerical Results (2D heat equation ctd.)

Performance of Newton's method for accuracy $\sim 1/n$:

grid	unknowns	$\|\mathcal{R}(P)\|_F/\|P\|_F$	iter. (ADI iter.)	CPU time (sec.)
8×8	2,080	4.7e-7	2 (8)	0.47
16×16	32,896	1.6e-6	2 (10)	0.49
32×32	524,800	1.8e-5	2 (11)	0.91
64×64	8,390,656	1.8e-5	3 (14)	7.98
128×128	134,225,920	3.7e-6	3 (19)	79.46

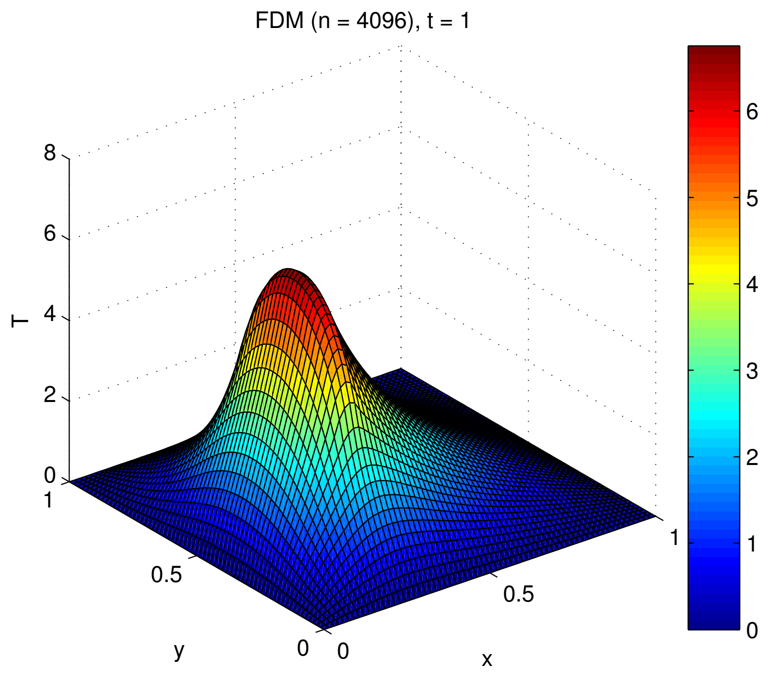
Here, $m = 1$ input and $q = 2$ outputs.

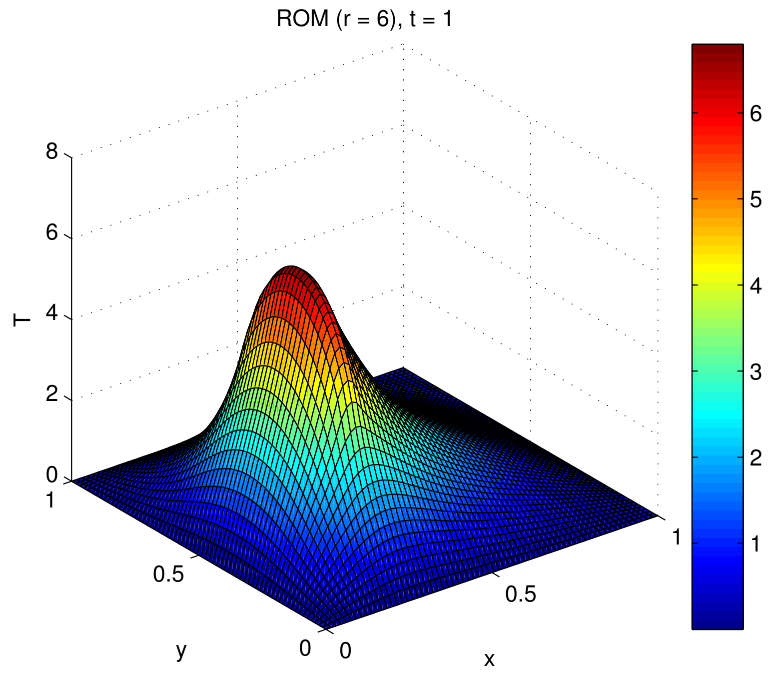
$X = X^T \in \mathbb{R}^{n \times n} \Rightarrow \frac{n(n+1)}{2}$ unknowns.

Reconstruction of the State

$$x(t) \approx V x_r(t).$$

Example: 2D heat equation with localized heat source, 64×64 grid,
$r = 6$ model by balanced truncation, simulation for $u(t) = 10\cos(t)$.

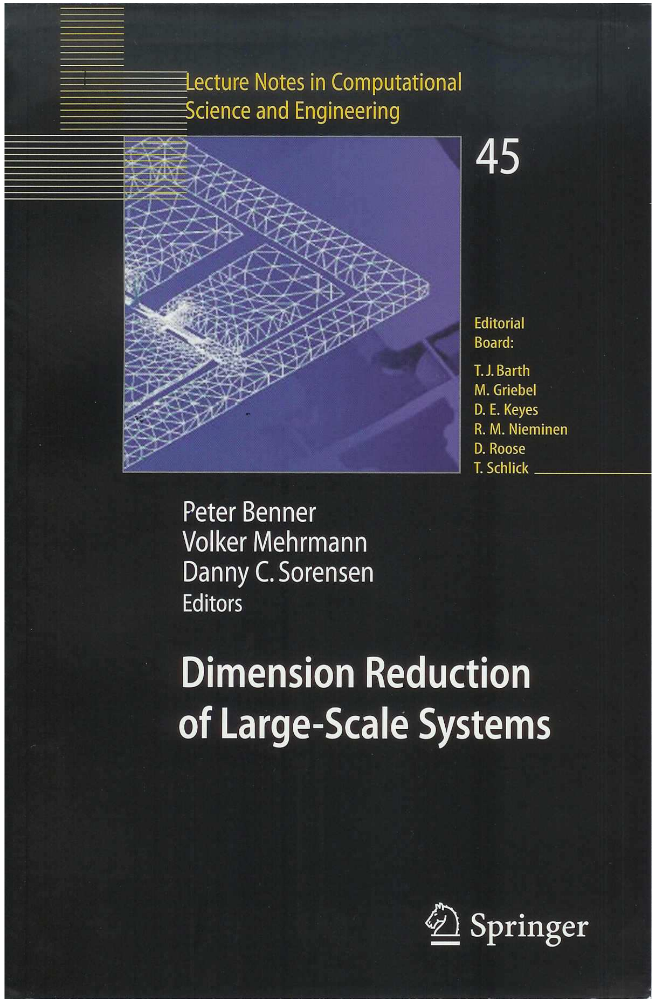




Conclusions and Open Problems

- BT (and LQG) BT perform well for model reduction of parabolic PDE control problems.
- Robust control design can be based on LQG BT.
- Need more (realistic) numerical tests.
- Other balancing schemes.
- Open Problems:
  - Optimal combination of FEM and BT error estimates/bounds.
  - Use convergence of Hankel singular values for control of mesh refinement?
  - Application to nonlinear problems: for some semilinear problems, BT approaches seem to work well.

Ad(é)



Thank you for your attention!