BALANCING-RELATED MODEL REDUCTION METHODS FOR LARGE-SCALE SYSTEMS

Peter Benner

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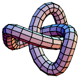
Fakultät für Mathematik
Technische Universität Chemnitz



31st Conference of the
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Woudschoten, October 12, 2006

Overview

Thanks to

- Enrique Quintana-Ortí, Gregorio Quintana-Ortí, Rafa Mayo, José Manuel Badía, and Sergio Barrachina (Universidad Jaume I de Castellón, Spain).
- Thilo Penzl for LyaPack.
- Members of the working group MiIT:
  Ulrike Baur, Sabine Görner, Matthias Pester, Jens Saak, René Schneider, and Viatcheslav Sokolov.

Problem

Given a physical problem with dynamics described by the states $x \in \mathbb{R}^n$, where n is the dimension of the state space.

Because of redundancies, complexity, etc., we want to describe the dynamics of the system using a reduced number of states.

This is the task of model reduction (also: dimension reduction, order reduction).

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Motivation: Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory: $4 \cdot n_x \cdot n_y$ bytes.

Theorem: (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum\nolimits_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U\Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\|X - \widehat{X}\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r, \sigma_1 v_1, \ldots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

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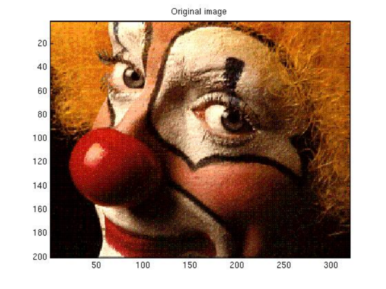
Example: Image Compression by Truncated SVD

Example: Clown



320×200 pixel
$\rightsquigarrow \approx 256$ kb

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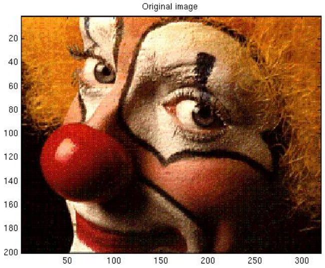

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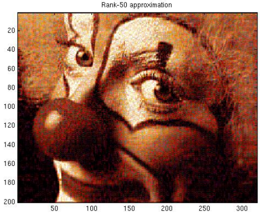

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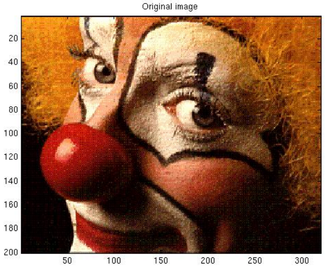

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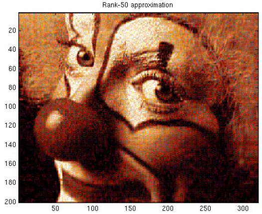
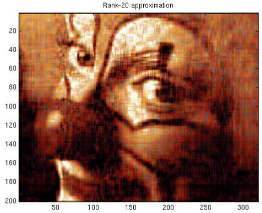


- rank $r = 20$, ≈ 42 kb



BALANCING-RELATED MODEL REDUCTION

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Introduction
Model Reduction
Systems Theory
Linear Systems
Application Areas
Model Reduction
Examples
Current and Future Work
References

Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:

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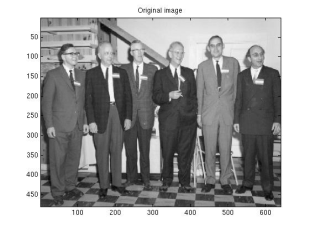

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Dimension Reduction via SVD

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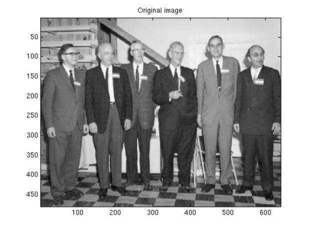

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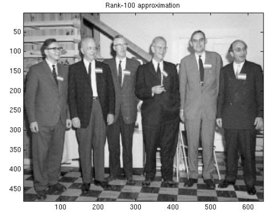

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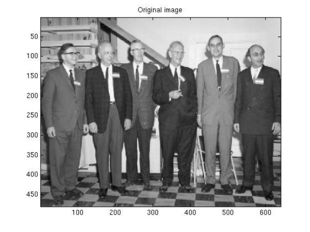

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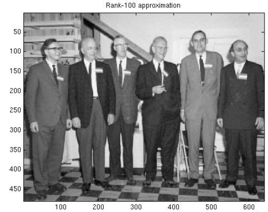
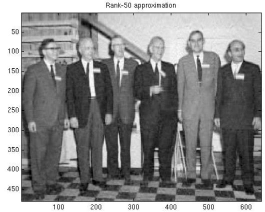


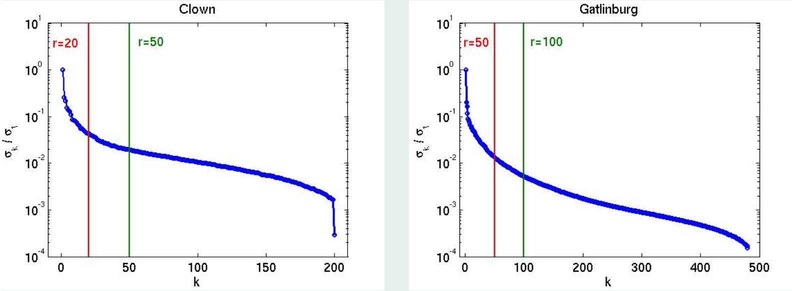
- rank $r = 50$, ≈ 224 kb



Background

Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices



Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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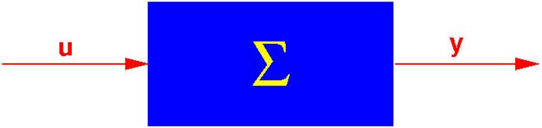

Model Reduction for Dynamical Systems

Original System

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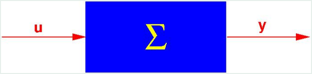

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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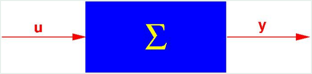

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$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

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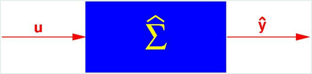

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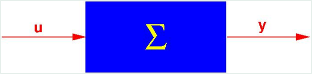

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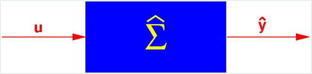

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Linear, Time-Invariant (LTI) Systems

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\begin{aligned}
f(t,x,u) &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m},\\
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Model Reduction for Linear Systems

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State-Space Description for I/O-Relation

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\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{\infty} Ce^{A(t-\tau)}B \, d\tau \quad \text{for all } t \in \mathbb{R}.
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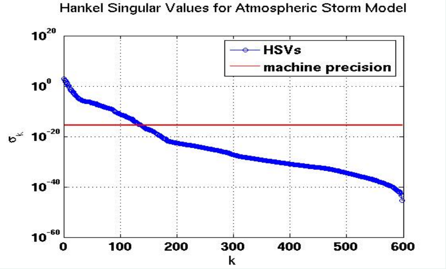

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$\Rightarrow$ solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

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But: computationally unfeasible for large-scale systems.

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Laplace Transformation / Frequency Domain

Application of Laplace transformation ($x(t) \mapsto x(s)$, $\dot{x}(t) \mapsto sx(s)$) to linear system with $x(0) = 0$:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Bx(s) + Du(s),$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)}\Big)u(s)$$

G is the transfer function of Σ.

Linear Systems in Frequency Domain

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Problem

Approximate the dynamical system

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by reduced-order system

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of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\|\|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G})\leq r} \|G - \hat{G}\|$.

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$$\begin{array}{rcl} \dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u, \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u, \end{array} \qquad \begin{array}{ll} \hat{A} \in \mathbb{R}^{r \times r}, & \hat{B} \in \mathbb{R}^{r \times m}, \\ \hat{C} \in \mathbb{R}^{p \times r}, & \hat{D} \in \mathbb{R}^{p \times m}. \end{array}$$

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$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\|\|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}(\hat{G}) \leq r} \|G - \hat{G}\|$.

Application Areas

(Optimal) Control

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
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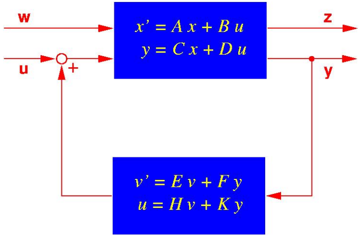
Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$

$\Rightarrow$ reduce order of original system.



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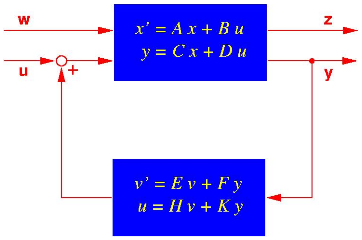

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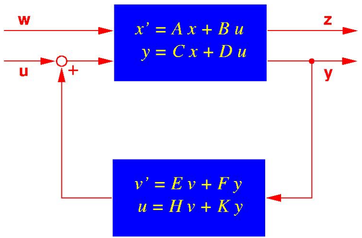

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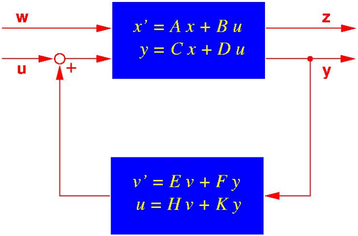

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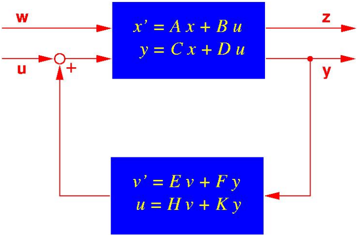

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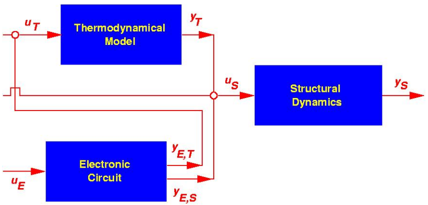

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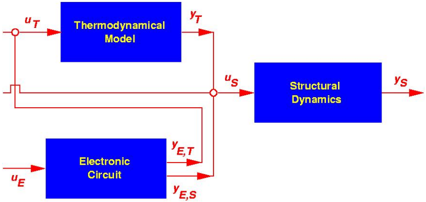

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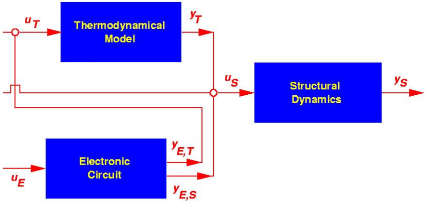

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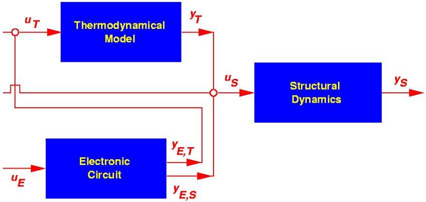

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2. Padé-Approximation and Krylov Subspace Methods
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4. many more...

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Joint feature of many methods: Galerkin or Petrov-Galerkin-type projection of state-space onto low-dimensional subspace $\mathcal{V}$ along $\mathcal{W}$: assume $x \approx VW^T x =: \tilde{x}$, where

$$\mathrm{range}\,(V) = \mathcal{V}, \quad \mathrm{range}\,(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ and

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|.$$

Modal Truncation

Idea:

Project state-space onto A-invariant subspace $\mathcal{V}$, where

$$\mathcal{V} = \operatorname{span}(v_1, \ldots, v_r),$$

v_k = eigenvectors corresp. to "dominant" modes $\equiv$ eigenvalues of A.

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Properties:

- Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.
- Error bound:

$$\|G - \hat{G}\|_\infty \leq \mathrm{cond}_2(T)\, \|C_2\|_2 \|B_2\|_2 \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\mathrm{Re}(\lambda)|},$$

where $T^{-1}AT = \mathrm{diag}(A_1, A_2)$.

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Difficulties:

- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
  (LITZ 1979: use Jordan canoncial form; otherwise merely heuristic criteria, e.g., VARGA '95.)
- Error bound not computable for really large-scale problems.

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Idea:

- Consider

$$E\dot{x} = Ax + Bu, \quad y = Cx$$

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$$G(s) \quad = \quad m_0 + m_1(s - s_0) + m_2(s - s_0)^2 + \ldots$$
- As reduced-order model use rth Padé approximate $\hat{G}$ to G:
$$G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^{2r}),$$
i.e., $m_j = \widehat{m}_j$ for $j = 0, \ldots, 2r - 1$

  $\rightsquigarrow$ moment matching if $s_0 < \infty$,

  $\rightsquigarrow$ partial realization if $s_0 = \infty$.

Padé Approximation

Padé-via-Lanczos Method (PVL)

- Moments need not be computed explicitly; moment matching is equivalent to projecting state-space onto

$$\mathcal{V} = \mathrm{span}(\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{r-1}B) = \mathcal{K}(\tilde{A}, \tilde{B}, r)$$

(where $\tilde{A} = (s_0E - A)^{-1}E$, $\tilde{B} = (s_0E - A)^{-1}B$) along

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- PVL applies w/o changes for singular E if $s_0 \notin \Lambda(A, E)$:
  - for $s_0 \neq \infty$: Gallivan/Grimme/Van Dooren 1994, Freund/Feldmann 1996, Grimme 1997
  - for $s_0 = \infty$: B./Sokolov 2005

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- Good approximation quality only locally.
- Preservation of physical properties only in special cases; usually requires post processing which (partially) destroys moment matching properties.

Balanced Truncation

Idea:

- A system Σ, realized by (A, B, C, D), is called balanced, if solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (A_{11}, B_1, C_1, D)$.

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Motivation:

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

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$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int\limits_0^\infty y(t)^T y(t)\,dt}{\int\limits_{-\infty}^0 u(t)^T u(t)\,dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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$\Longrightarrow$ Truncate states corresponding to "small" HSVs

$\Longrightarrow$ complete analogy to best approximation via SVD!

Implementation: SR Method

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3. Set
$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$
4. Reduced model is $(W^T AV, W^T B, CV, D)$.

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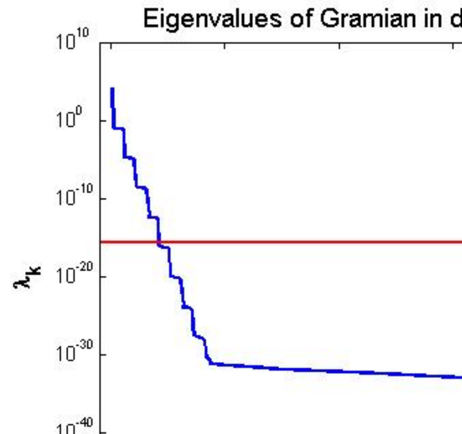
New algorithmic ideas from numerical linear algebra:

- Instead of Gramians P, Q compute $S, R \in \mathbb{R}^{n \times k}$, $k \ll n$, such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

- Compute S, R with problem-specific Lyapunov solvers of "low" complexity directly.



Balanced Truncation

Properties:

General misconception: complexity $\mathcal{O}(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

New algorithmic ideas from numerical linear algebra:

Parallelization:

- Efficient parallel algorithms based on matrix sign function.
- Complexity $\mathcal{O}(n^3/q)$ on q-processor machine.
- Software library PLiCMR with WebComputing interface.

(B./Quintana-Ortí/Quintana-Ortí since 1999)

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Formatted Arithmetic:

For special problems from PDE control use implementation based on hierarchical matrices and matrix sign function method (Baur/B.), complexity $\mathcal{O}(n \log^2(n) r^2)$.

Balanced Truncation

Properties:

General misconception: complexity $\mathcal{O}(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

New algorithmic ideas from numerical linear algebra:

Sparse Balanced Truncation:

- Sparse implementation using sparse Lyapunov solver (ADI+MUMPS/SuperLU).
- Complexity $\mathcal{O}(n(k^2 + r^2))$.
- Software:
  - MATLAB toolbox LYAPACK (PENZL 1999),
  - Software library SPARED with WebComputing interface.
    (BADÍA/B./QUINTANA-ORTÍ/QUINTANA-ORTÍ since 2003)

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) MULLIS/ROBERTS '76, MOORE '81

- P = controllability Gramian of system given by (A, B, C, D).
- Q = observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

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LQG Balanced Truncation (LQGBT) — Jonckheere/Silverman '83

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

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Balanced Stochastic Truncation (BST) Desai/Pal '84, Green '88

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

Balancing-Related Model Reduction

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Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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Positive-Real Balanced Truncation (PRBT) — Green '88

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^T Q + Q\bar{A} + QB\bar{R}^{-1}B^T Q + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

Balancing-Related Model Reduction

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Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

Balancing-Related Model Reduction
Properties

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).
- Computable error bounds, e.g.,

$$\text{BT:} \quad \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT},$$

$$\text{LQGBT:} \quad \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}$$

$$\text{BST:} \quad \|G - G_r\|_\infty \leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty,$$

- Can be combined with singular perturbation approximation for steady-state performance.
- Computations can be modularized.

Examples

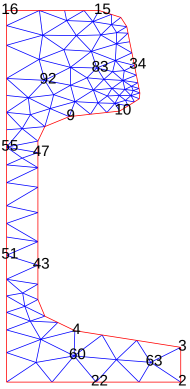
Optimal Control: Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

$$\begin{aligned} c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, && \xi \in \Omega \\ \lambda \frac{\partial}{\partial n} x &= \kappa(u_k - x), && \xi \in \Gamma_k,\ 1 \leq k \leq 7, \\ \frac{\partial}{\partial n} x &= 0, && \xi \in \Gamma_7. \end{aligned}$$

$\Longrightarrow m = 7, p = 6.$

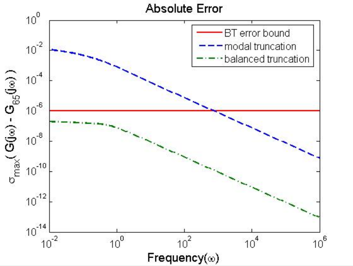
- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841$.



Source: Physical model: courtesy of Mannesmann/Demag.

Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

$n = 1357$, Absolute Error



- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

Examples

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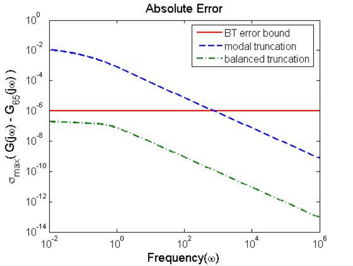

- BT model computed with sign function method,
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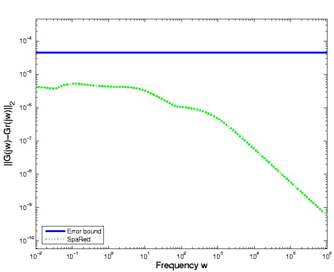
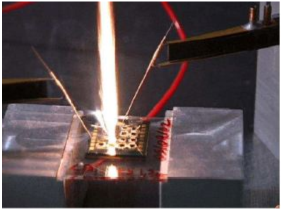
$n = 79841$, Absolute error

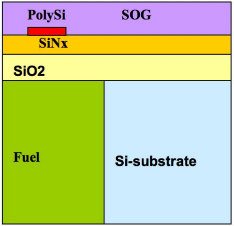


- BT model computed using SpaRed,
- computation time: 8 min.

Examples
MEMS: Microthruster

- Co-integration of solid fuel with silicon micromachined system.
- Goal: Ignition of solid fuel cells by electric impulse.
- Application: nano satellites.
- Thermo-dynamical model, ignition via heating an electric resistance by applying voltage source.
- Design problem: reach ignition temperature of fuel cell w/o firing neighbouring cells.
- Spatial FEM discretization of thermo-dynamical model $\rightsquigarrow$ linear system, $m = 1$, $p = 7$.





Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Courtesy of C. Rossi, LAAS-CNRS/EU project "Micropyros".

- axial-symmetric 2D model
- FEM discretisation using linear (quadratic) elements $\rightsquigarrow$ $n = 4,257$ $(11,445)$ $m = 1$, $p = 7$.
- Reduced model computed using SpaRed. modal truncation using ARPACK, and Z. Bai's PVL implementation.

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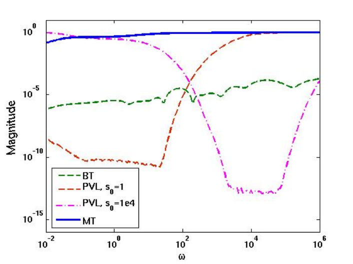
Relative error $n = 4,257$



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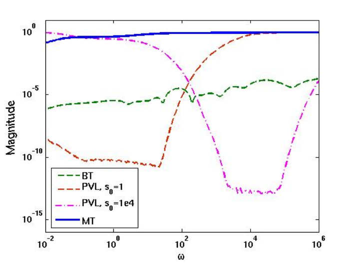
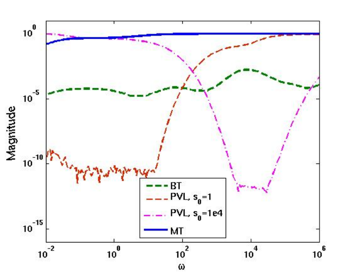


Relative error $n = 11,445$



BALANCING-RELATED MODEL REDUCTION
Peter Benner
Introduction
Model Reduction
Examples
Optimal Cooling
Microthruster
Butterfly Gyro
Current and Future Work
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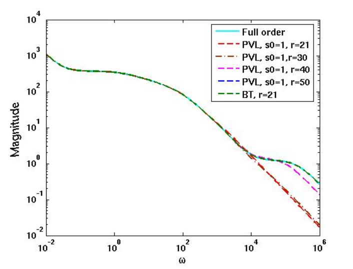
Frequency Response BT/PVL



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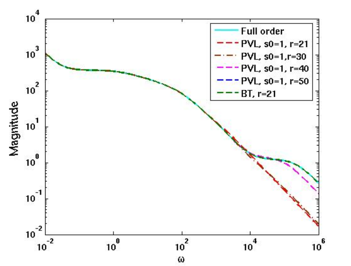
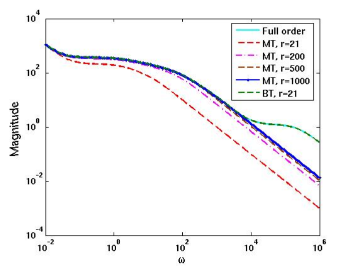
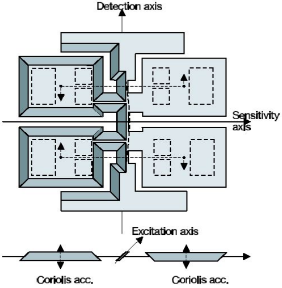


Frequency Response BT/MT



- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.



Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark
Courtesy of D. Billger (Imego Institute, Göteborg), Saab Bofors Dynamics AB.

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722$, $m = 1$, $p = 12$.
- Reduced model computed using SPARED, $r = 30$.

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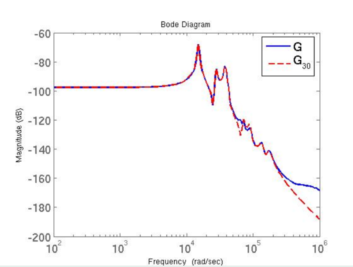
Frequency Repsonse Analysis



Examples
MEMS: Butterfly Gyro

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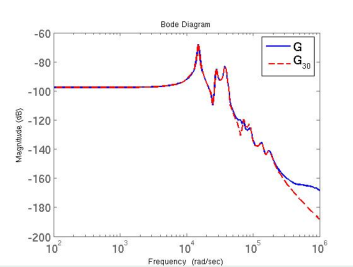
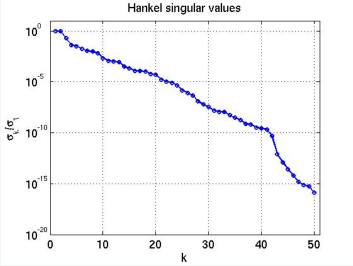


Hankel Singular Values



Current and Future Work

- **Parametric Models**

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x + D(p)u,$$

where $p \in \mathbb{R}^s$ is free parameter vector; parameters should be preserved in the reduced-order model.

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- Nonlinear Systems
  Exploit structure of nonlinearities, e.g., in optimal control of linear PDEs with nonlinear BCs $\rightsquigarrow$

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  - bilinear control systems $\dot{x} = Ax + \sum_j N_j x u_j + Bu$,

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  Exploit structure of nonlinearities, e.g., in optimal control of linear PDEs with nonlinear BCs $\rightsquigarrow$
  - bilinear control systems $\dot{x} = Ax + \sum_j N_j x u_j + Bu$,
  - formal linear systems (cf. FÖLLINGER 1982)

$$\dot{x} = Ax + N\,g(Hx) + Bu = Ax + \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} u \\ g(z) \end{bmatrix},$$

  where $z := Hx \in \mathbb{R}^\ell$, $\ell \ll n$.

References

1. G. Obinata and B.D.O. Anderson.
Model Reduction for Control System Design.
Springer-Verlag, London, UK, 2001.

2. Z. Bai.
Krylov subspace techniques for reduced-order modeling of large-scale dynamical systems.
APPL. NUMER. MATH, 43(1–2):9–44, 2002.

3. R. Freund.
Model reduction methods based on Krylov subspaces.
ACTA NUMERICA, 12:267–319, 2003.

4. P. Benner, E.S. Quintana-Ortí, and G. Quintana-Ortí.
State-space truncation methods for parallel model reduction of large-scale systems.
PARALLEL COMPUT., 29:1701–1722, 2003.

5. P. Benner, V. Mehrmann, and D. Sorensen (editors).
Dimension Reduction of Large-Scale Systems.
LECTURE NOTES IN COMPUTATIONAL SCIENCE AND ENGINEERING, Vol. 45,
Springer-Verlag, Berlin/Heidelberg, Germany, 2005.

6. A.C. Antoulas.
Lectures on the Approximation of Large-Scale Dynamical Systems.
SIAM Publications, Philadelphia, PA, 2005.

7. P. Benner, R. Freund, D. Sorensen, and A. Varga (editors).
Special issue on *Order Reduction of Large-Scale Systems*.
LINEAR ALGEBRA APPL., June 2006.

Thanks for your attention!