CONTROL-ORIENTED MODEL REDUCTION FOR PARABOLIC CONTROL SYSTEMS

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Overview

Given Hilbert spaces

$\mathcal{X}$ – state space,

$\mathcal{U}$ – control space,

$\mathcal{Y}$ – output space,

and operators

$$\begin{aligned} \mathbf{A}: &\quad \mathrm{dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}, \\ \mathbf{B}: &\quad \mathcal{U} \to \mathcal{X}, \\ \mathbf{C}: &\quad \mathcal{X} \to \mathcal{Y}. \end{aligned}$$

Linear Distributed Parameter System (DPS)

$$\Sigma : \left\{ \begin{array}{rcl} \dot{\mathbf{x}} &=& \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} &=& \mathbf{C}\mathbf{x}, \end{array} \right. \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X},$$

i.e., abstract evolution equation together with observation equation.

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Parabolic Systems

The state $x = x(t, \xi)$ is a weak solution of a parabolic PDE with $(t, \xi) \in [0, T] \times \Omega$, $\Omega \subset \mathbb{R}^d$:

$$\partial_t x - \nabla(a(\xi).\nabla x) + b(\xi).\nabla x + c(\xi)x = B_{pc}(\xi)u(t), \quad \xi \in \Omega,\ t > 0$$

with initial and boundary conditions

$$\begin{aligned} \alpha(\xi)x + \beta(\xi)\partial_\eta x &= B_{bc}(\xi)u(t), && \xi \in \partial\Omega, && t \in [0, T] \\ x(0, \xi) &= x_0(\xi) \in \mathcal{X}, && \xi \in \Omega, && \\ y(t) &= C(\xi)x, && \xi \in \Omega, && t \in [0, T]. \end{aligned}$$

- $B_{pc} = 0 \implies$ boundary control problem
- $B_{bc} = 0 \implies$ point control problem

Assume

- $\mathbf{A}$ generates C_0-semigroup $T(t)$ on $\mathcal{X}$,
- $(\mathbf{A}, \mathbf{B})$ is exponentially stabilizable, i.e., there exists $\mathbf{F} : \mathrm{dom}(\mathbf{A}) \mapsto \mathcal{U}$ such that $\mathbf{A} + \mathbf{BF}$ generates an exponentially stable C_0-semigroup $\mathbf{S}(\mathbf{t})$;
- $(\mathbf{A}, \mathbf{C})$ is exponentially detectable, i.e., $(\mathbf{A}^*, \mathbf{C}^*)$ is exponentially stabilizable;
- $\mathbf{B}, \mathbf{C}$ are finite-rank and bounded, e.g., $\mathcal{U} = \mathbb{R}^m$, $\mathcal{Y} = \mathbb{R}^p$.

Then the system $\Sigma(A, B, C)$ has a transfer function

$$\mathbf{G} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \in L_\infty.$$

If, in addition, $\mathbf{A}$ is exponentially stable, $\mathbf{G}$ is in the Hardy space H_∞.

Weaker assumptions:
$\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is Pritchard-Salomon system, allows for certain unboundedness of $\mathbf{B}, \mathbf{C}$.

Infinite-Dimensional Systems

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(Exponentially) Stable Systems

$\mathbf{G}$ is the Laplace transform of

$$\mathbf{h}(t) := \mathbf{C}T(t)\mathbf{B}$$

and symbol of the Hankel operator $\mathbf{H} : L_2(0,\infty;\mathbb{R}^m) \mapsto L_2(0,\infty;\mathbb{R}^p)$,

$$(\mathbf{Hu})(t) := \int_0^\infty \mathbf{h}(t+\tau)u(\tau)\,d\tau.$$

$\mathbf{H}$ is compact with countable many singular values σ_j, $j = 1,\ldots,\infty$, called the Hankel singular values (HSVs) of $\mathbf{G}$. Moreover,

$$\sum_{j=1}^\infty \sigma_j < \infty.$$

HSVs are system invariants, used for approximation similar to truncated SVD.
The 2-induced operator norm is the H_∞ norm; here,

$$\|\mathbf{G}\|_{H_\infty} = \sum_{j=1}^\infty \sigma_j.$$

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Model Reduction Based on Balancing

Motivation

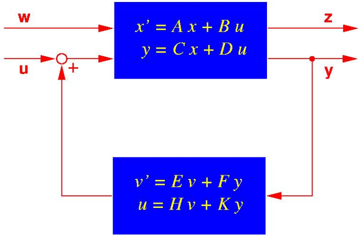
Designing a controller for parabolic control systems requires semi-discretization in space, control design for n-dim. system.

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$



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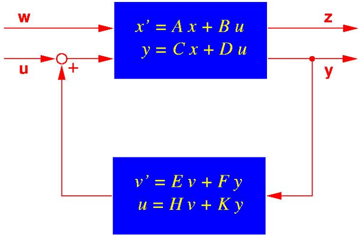

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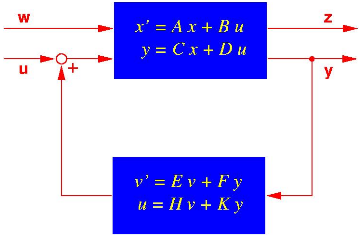

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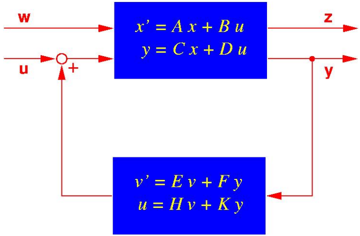

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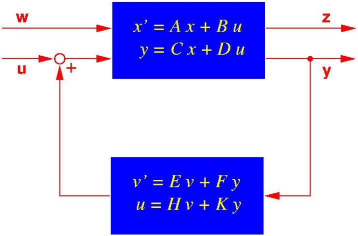

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Definition: [Curtain/Glover/(Partington) 1986,1988]

For $\mathbf{G} \in H_\infty$, $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a balanced realization of $\mathbf{G}$ if the controllability and observability Gramians, given by the unique self-adjoint positive semidefinite solutions of the Lyapunov equations

$$\begin{aligned} \mathbf{APz} + \mathbf{PA}^*\mathbf{z} + \mathbf{BB}^*\mathbf{z} &= 0 \quad \forall\ \mathbf{z} \in \mathrm{dom}(\mathbf{A}^*) \\ \mathbf{A}^*\mathbf{Qz} + \mathbf{QAz} + \mathbf{C}^*\mathbf{Cz} &= 0 \quad \forall\ \mathbf{z} \in \mathrm{dom}(\mathbf{A}) \end{aligned}$$

satisfy $\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\sigma_j) =: \mathbf{\Sigma}$.

Abstract balanced truncation [GLOVER/CURTAIN/PARTINGTON 1988]

Given balanced realization with

$$\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\sigma_j) = \mathbf{\Sigma},$$

choose r with $\sigma_r > \sigma_{r+1}$ and partition $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ according to

$$\mathbf{P}_r = \mathbf{Q}_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r),$$

so that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_r & * \\ * & * \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_r \\ * \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_r & * \end{bmatrix},$$

then the reduced-order model is the stable system $\Sigma_r(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$ with transfer function $\mathbf{G}_r$ satisfying

$$\|\mathbf{G} - \mathbf{G}_r\|_{H_\infty} \leq 2 \sum_{j=r+1}^{\infty} \sigma_j.$$

Balanced truncation only applicable for *stable* systems.
Now: unstable systems

Definition: [Curtain 2003].

For $\mathbf{G} \in L_\infty$, $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is an LQG-balanced realization of $\mathbf{G}$ if the unique self-adjoint, positive semidefinite, stabilizing solutions of the operator Riccati equations

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are bounded and satisfy $\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\gamma_j) =: \mathbf{\Gamma}$.
($\mathbf{P}$ stabilizing $\Leftrightarrow$ $\mathbf{A} - \mathbf{PC}^*\mathbf{C}$ generates exponentially stable C_0-semigroup.)

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Computation of Reduced-Order Systems

Spatial discretization (FEM, FDM) $\rightsquigarrow$ finite-dimensional system on $\mathcal{X}_n \subset \mathcal{X}$ with $\dim \mathcal{X}_n = n$:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, with corresponding

- algebraic Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

- algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= \mathcal{R}_f(P) := AP + PA^T - PC^TCP + BB^T, \\ 0 &= \mathcal{R}_c(Q) := A^TQ + QA - QBB^TQ + C^TC. \end{aligned}$$

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Convergence of Gramians

Theorem [Curtain 2003]

Under given assumptions for $\mathbf{\Sigma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, the solutions of the algebraic Lyapunov equations on $\mathcal{X}_n$ converge in the nuclear norm to the solutions of the corresponding operator equations and the transfer functions converge in the gap topology if the n-dimensional approximations satisfy the assumptions:

- $\exists$ orthogonal projector $\Pi_n : \mathcal{X} \mapsto \mathcal{X}_n$ such that

$$\Pi_n \mathbf{z} \to \mathbf{z} \ (n \to \infty) \quad \forall \mathbf{z} \in \mathcal{X}, \quad B = \Pi_n \mathbf{B}, \qquad C = \mathbf{C}|_{\mathcal{X}_n}.$$

- For all $\mathbf{z} \in \mathcal{X}$ and $n \to \infty$,

$$e^{At}\Pi_n \mathbf{z} \to T(t)\mathbf{z}, \qquad (e^{At})^* \Pi_n \mathbf{z} \to T(t)^* \mathbf{z},$$

  uniformly in t on bounded intervals.
- A is uniformly exponentially stable.

Convergence of Gramians

Theorem [Curtain 2003]

Under given assumptions for $\mathbf{\Sigma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, the stabilizing solutions of the algebraic Riccati equations on $\mathcal{X}_n$ converge in the nuclear norm to the solutions of the corresponding operator equations and the transfer functions converge in the gap topology if the n-dimensional approximations satisfy the assumptions:

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  uniformly in t on bounded intervals.
- (A, B, C) is uniformly exponentially stabilizable and detectable.

Error Bounds

For control applications, want to estimate/bound

$$\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^m)} \quad \text{or} \quad \|\mathbf{y}(t) - y_r(t)\|_2.$$

Error bound includes approximation errors caused by

- Galerkin projection/spatial FEM discretization,
- model reduction.

Ultimate goal

Balance the discretization and model reduction errors vs. each other in fully adaptive discretization scheme.

Output Error Bound

Assume $\mathbf{C} \in \mathcal{L}(\mathcal{X}, \mathbb{R}^p)$ bounded, $C = \mathbf{C}|_{\mathcal{X}_n}$, $\mathcal{X}_n \subset \mathcal{X}$. Then:

$$
\begin{aligned}
\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^p)} &\leq \|\mathbf{y} - y\|_{L_2(0,T;\mathbb{R}^p)} + \|y - y_r\|_{L_2(0,T;\mathbb{R}^p)} \\
&= \|\mathbf{C}\mathbf{x} - Cx\|_{L_2(0,T;\mathbb{R}^p)} + \|y - y_r\|_{L_2(0,T;\mathbb{R}^p)} \\
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Balanced truncation:

$$\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^p)} \leq c\|\mathbf{x} - x\|_{L_2(0,T;\mathcal{X})} + 2\|u\|_{L_2(0,T;\mathbb{R}^p)} \sum_{j=r+1}^{n} \sigma_j .$$

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Output Error Bound

Assume $\mathbf{C} \in \mathcal{L}(\mathcal{X}, \mathbb{R}^p)$ bounded, $C = \mathbf{C}|_{\mathcal{X}_n}$, $\mathcal{X}_n \subset \mathcal{X}$. Then:

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General form for $A, G = G^T, W = W^T \in \mathbb{R}^{n \times n}$ given and $P \in \mathbb{R}^{n \times n}$ unknown:

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\begin{aligned}
0 &= \mathcal{L}(Q) := A^T Q + QA + W, \\
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In large scale applications from semi-discretized control problems for PDEs,

- $n = 10^3 - 10^6$ ($\Longrightarrow 10^6 - 10^{12}$ unknowns!),
- A has sparse representation ($A = -M^{-1}K$ for FEM),
- G, W low-rank with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n \times m}$, $m \ll n$, $\quad C \in \mathbb{R}^{p \times n}$, $p \ll n$.
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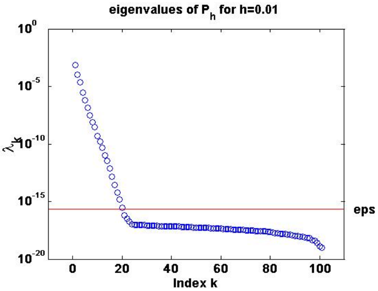
Low-Rank Approximation

ARE $0 = A^TQ + QA - QBB^TQ + CC^T$

Consider spectrum of ARE solution (analogous for Lyapunov equations).

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \implies n = 101$.



Idea: $Q = Q^T \geq 0 \implies$

$$Q = ZZ^T = \sum_{k=1}^{n} \lambda_k z_k z_k^T \approx Z^{(r)}(Z^{(r)})^T = \sum_{k=1}^{r} \lambda_k z_k z_k^T .$$

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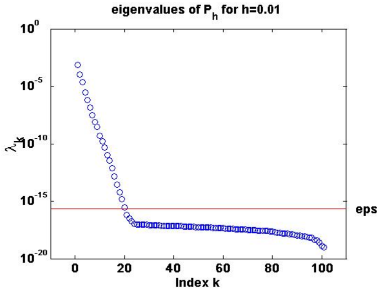

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PDE Model Reduction
Peter Benner
DPS
Model Reduction Based on Balancing
Large Matrix Equations
ADI for Lyapunov
Newton's Method for AREs
LQR Problem
Numerical Results
Conclusions and Open Problems

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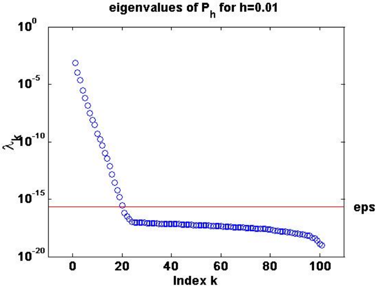

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ADI Method for Lyapunov Equations

- For $A \in \mathbb{R}^{n\times n}$ stable, $B \in \mathbb{R}^{n\times m}$ ($w \ll n$), consider Lyapunov equation

$$AX + XA^T = -BB^T.$$

- ADI Iteration: [Wachspress 1988]

$$\begin{aligned}(A + p_k I)X_{(j-1)/2} &= -BB^T - X_{k-1}(A^T - p_k I)\\ (A + \overline{p_k} I){X_k}^T &= -BB^T - X_{(j-1)/2}(A^T - \overline{p_k} I)\end{aligned}$$

with parameters $p_k \in \mathbb{C}^-$ and $p_{k+1} = \overline{p_k}$ if $p_k \notin \mathbb{R}$.

- For $X_0 = 0$ and proper choice of p_k: $\lim_{k\to\infty} X_k = X$ superlinear.
- Re-formulation using $X_k = Y_k Y_k^T$ yields iteration for $Y_k \ldots$

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Factored ADI Iteration

Lyapunov equation $0 = AX + XA^T = -BB^T$.

Setting $X_k = Y_k Y_k^T$, some algebraic manipulations $\Longrightarrow$

Algorithm [Penzl 1997, Li/White 2002, B./Li/Penzl 1999/2006]

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A + p_1 I)^{-1}B, \qquad Y_1 \leftarrow V_1$$

FOR $j = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1}V_{k-1}\right),$$

$$Y_k \leftarrow \begin{bmatrix} Y_{k-1} & V_k \end{bmatrix}$$

At convergence, $Y_{k_{\max}} Y_{k_{\max}}^T \approx X$, where

$$Y_{k_{\max}} = \begin{bmatrix} V_1 & \ldots & V_{k_{\max}} \end{bmatrix}, \quad V_k = \boxed{} \in \mathbb{C}^{n \times m}$$

Note: Implementation in real arithmetic possible by combining two steps.

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[Kleinman '68, Mehrmann '91, Lancaster/Rodman '95, B./Byers '94/'98, B. '97, Guo/Laub '99]

- Consider $0 = \mathcal{R}(Q) = C^T C + A^T Q + QA - QBB^T Q$.
- Frechét derivative of $\mathcal{R}(Q)$ at Q:

$$\mathcal{R}'_Q : Z \to (A - BB^T Q)^T Z + Z(A - BB^T Q).$$

- Newton-Kantorovich method:

$$Q_{j+1} = Q_j - \left(\mathcal{R}'_{Q_j}\right)^{-1} \mathcal{R}(Q_j), \quad j = 0, 1, 2, \ldots$$

Newton's method (with line search) for AREs

FOR $j = 0, 1, \ldots$

1. $A_j \leftarrow A - BB^T Q_j =: A - BK_j$.
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Properties and Implementation

- Convergence for K_0 stabilizing:
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  - $\lim_{j\to\infty} Q_j = Q_* \geq 0$ (locally quadratic).
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Properties and Implementation

- Convergence for K_0 stabilizing:
  - $A_j = A - BK_j = A - BB^T Q_j$ is stable $\forall\ j \geq 0$.
  - $\lim_{j\to\infty} \|\mathcal{R}(Q_j)\|_F = 0$ (monotonically).
  - $\lim_{j\to\infty} Q_j = Q_* \geq 0$ (locally quadratic).
- Need large-scale Lyapunov solver; here, ADI iteration: linear systems with dense, but "sparse+low rank" coefficient matrix A_j:

$$A_j = A - B \cdot K_j$$

$$= \boxed{\text{sparse}} - \boxed{m} \cdot \boxed{}$$

- $m \ll n \implies$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j)^{-1} = (I_n + A^{-1}B(I_m - K_jA^{-1}B)^{-1}K_j)A^{-1}.$$

- BUT: $Q = Q^T \in \mathbb{R}^{n\times n} \implies n(n+1)/2$ unknowns!

Re-write Newton's method for AREs

$$A_j^T N_j + N_j A_j = -\mathcal{R}(Q_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(Q_j + N_j)}_{=Q_{j+1}} + \underbrace{(Q_j + N_j)}_{=Q_{j+1}} A_j = \underbrace{-C^T C - Q_j B B^T Q_j}_{=:-W_j W_j^T}$$

Set $Q_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

Factored Newton Iteration [B./Li/Penzl 1999/2006]

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and use 'sparse + low-rank' structure of A_j.

Low-Rank Newton-ADI for AREs

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LQR Problem

Linear-Quadratic Regulator Problem

Linear-quadratic optimization problem w/o control/state constraints:

$$\min_{\mathbf{u}\in L_2} \int_0^\infty \langle \mathbf{C}\mathbf{x}(t), \mathbf{C}\mathbf{x}(t)\rangle_{\mathcal{Y}} + \langle \mathbf{u}(t), \mathbf{u}(t)\rangle_{\mathcal{U}}\, dt$$

subject to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \ \mathbf{x}(0) = \mathbf{x}_0.$

Solution: feedback control law ($\rightsquigarrow$ static feedback controller)

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t) := \mathbf{B}^*\mathbf{Q}\mathbf{x}(t)$$

(with $\mathbf{Q}$ as in LQG operator Riccati equation).
Finite-dimensional approximation is

$$u(t) = K_* x(t) := B^T Q_* x(t),$$

where Q_* is the stabilizing solution of the corresponding ARE.

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Linear-quadratic optimization problem w/o control/state constraints:

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subject to $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \;\; \mathbf{x}(0) = \mathbf{x}_0.$

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K_* can be computed by direct feedback iteration:

- jth Newton iteration:

$$K_j = B^T Z_j Z_j^T = \sum_{k=1}^{k_{\max}} (B^T V_{j,k}) V_{j,k}^T \quad \xrightarrow{j\to\infty} \quad K_* = B^T Z_* Z_*^T$$

- K_j can be updated in ADI iteration, no need to even form Z_j, need only fixed workspace for $K_j \in \mathbb{R}^{m \times n}$!

LQR solution for the reduced-order model yields

$$u_r(t) = K_{r,*}x_r(t) := B_r Q_{r,*}x_r(t).$$

Theorem

Let K_* be the feedback matrix computed from finite-dimensional approximation to LQR problem, $K_{r,*}$ the feedback matrix obtained from the LQR problem for the LQG reduced-order model obtained using the projector VW^T, then

$$K_{r,*} = K_* V^T.$$

Consequence: the reduced-order optimal control can be computed as by-product in the model reduction process!
Similar result for LQG controller.

Optimal Control from Reduced-Order Model

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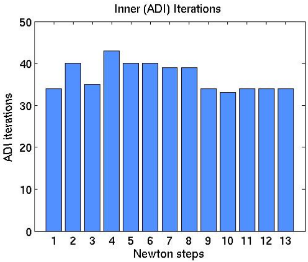
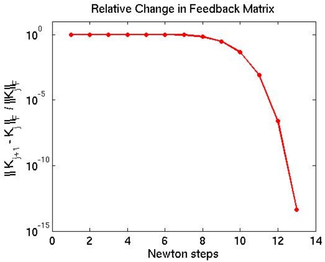
Similar result for LQG controller.

Numerical Results

Performance of Matrix Equation Solvers

- Linear 2D heat equation with homogeneous Dirichlet boundary and point control/observation.
- FD discretization on uniform 150×150 grid.
- $n = 22.500$, $m = p = 1$, 10 shifts for ADI iterations.
- Convergence of large-scale matrix equation solvers:





Performance of Newton's method for accuracy $\sim 1/n$

grid	unknowns	$\frac{\|\mathcal{R}(P)\|_F}{\|P\|_F}$	it. (ADI it.)	CPU (sec.)
8×8	2,080	4.7e-7	2 (8)	0.47
16×16	32,896	1.6e-6	2 (10)	0.49
32×32	524,800	1.8e-5	2 (11)	0.91
64×64	8,390,656	1.8e-5	3 (14)	7.98
128×128	134,225,920	3.7e-6	3 (19)	79.46

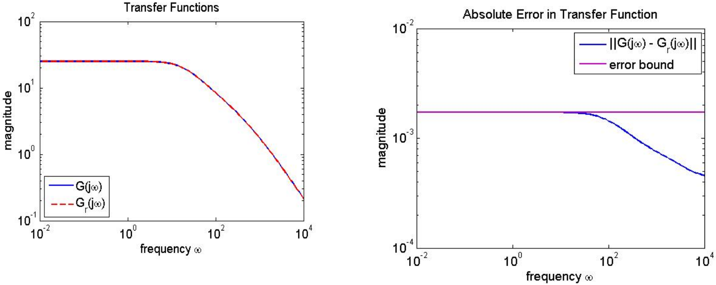
Here,

- Convection-diffusion equation,
- $m = 1$ input and $p = 2$ outputs,
- $Q = Q^T \in \mathbb{R}^{n \times n} \Rightarrow \frac{n(n+1)}{2}$ unknowns.

Numerical Results

Model Reduction Performance

- Numerical ranks of Gramians are 31 and 26, respectively.
- Computed reduced-order model (BT): $r = 6$ ($\sigma_7 = 5.8 \cdot 10^{-4}$),
- BT error bound $\delta = 1.7 \cdot 10^{-3}$.



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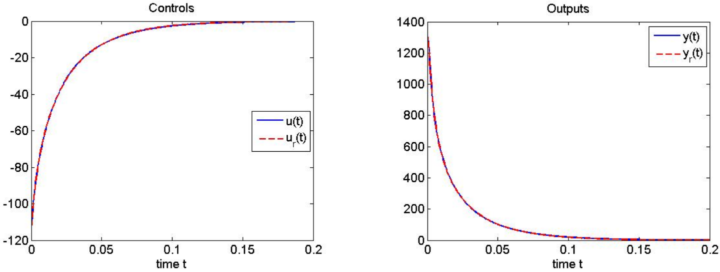
- Computed controls and outputs (implicit Euler):



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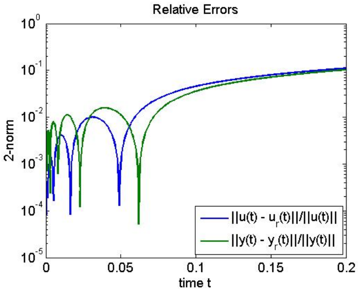
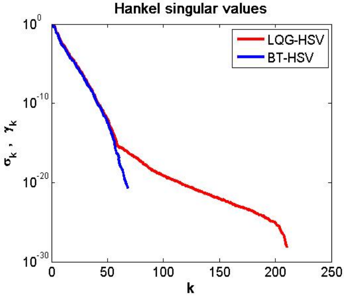


- Boundary control problem for 2D heat flow in copper on rectangular domain; control acts on two sides via Robins BC.
- FDM $\rightsquigarrow$ $n = 4496$, $m = 2$; 4 sensor locations $\rightsquigarrow$ $p = 4$.
- Numerical ranks of BT Gramians are 68 and 124, respectively, for LQG BT both have rank 210.
- Computed reduced-order model: $r = 10$.



Source: *COMPl$_e$ib* v1.1, `www.compleib.de`.

Numerical Results

Model Reduction Performance: BT vs. LQG BT

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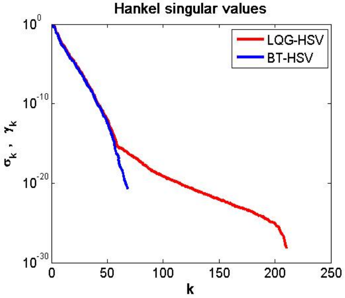

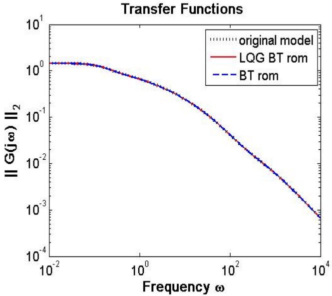

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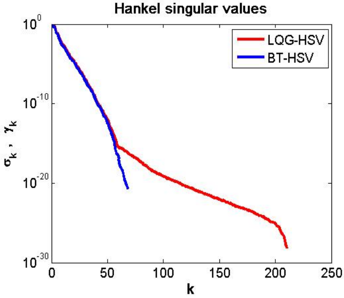

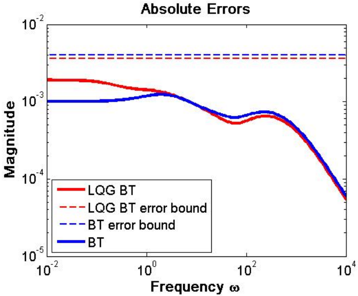

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BT is often criticized for its bias towards the input-output behavior of the system. But states can also be reconstructed using

$$x(t) \approx V x_r(t).$$

Example: 2D heat equation with localized heat source, 64×64 grid, $r = 6$ model by BT, simulation for $u(t) = 10\cos(t)$.

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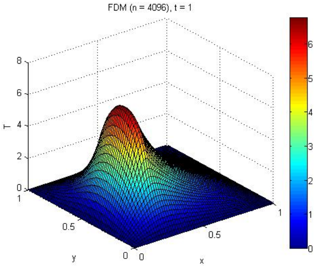

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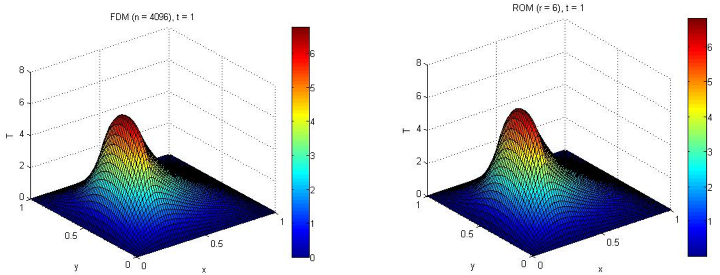
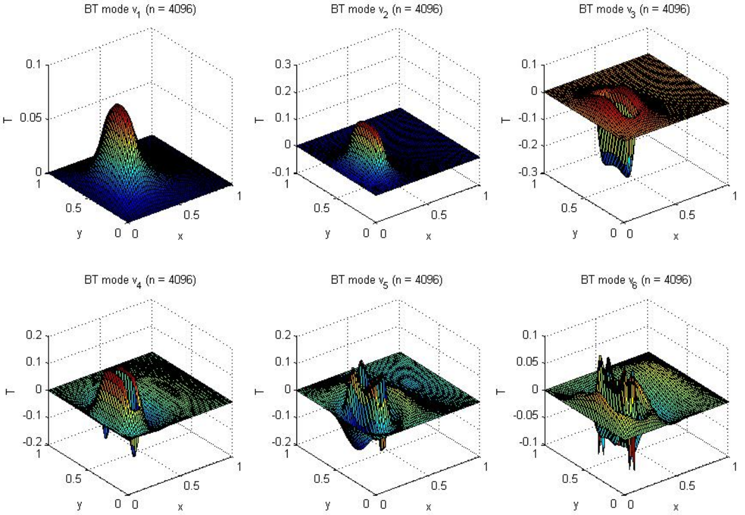


Numerical Results

BT modes are shape functions for Galerkin projection



Conclusions and Open Problems

- BT (and LQG) BT perform well for model reduction of (as of yet, simple) parabolic PDE control problems.
- Robust control design can be based on LQG BT (see CURTAIN 2004).
- Need more numerical tests.
- Find implementations for other balancing schemes (H_∞-/bounded real BT,...).
- Open Problems:
  - Optimal combination of FEM and BT error estimates/bounds — use convergence of Hankel singular values for control of mesh refinement?
  - BT modes are intelligent ansatz functions for (Petrov-)Galerkin projection—how to exploit?
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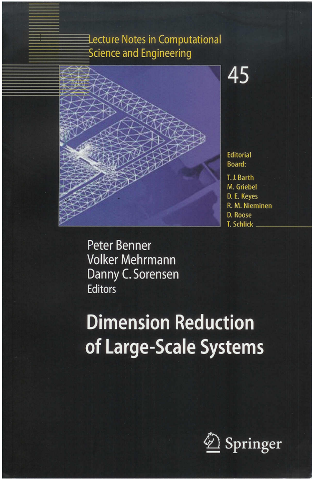
Ad(é)



Thank you for your attention!