Accurate(?) Solution of Algebraic Bernoulli Equations

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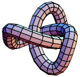

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Joint work with
Enrique Quintana-Ortí and Sergio Barrachina
(Universidad Jaume I de Castellón, Spain).

Overview

Algebraic Bernoulli Equations (ABE)

$$A^T X + XA - XGX = 0, \qquad A \in \mathbb{R}^{n\times n},\ G = G^T \in \mathbb{R}^{n\times n},$$

as special case of

$$\mathcal{L}(X) + A_0 X \left(\prod_{j=1}^{k-1} A_j X \right) A_k = 0,$$

where $\mathcal{L}(X)$ is a linear operator and $A_j \in \mathbb{R}^{n\times n}$ for $j = 0, 1, \ldots, k$.

Why Bernoulli?

Bernoulli differential equation

$$\dot{y}(t) = p(t)y(t) + q(t)y(t)^k, \quad k \neq 0, 1.$$

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...almost *Lyapunov equations:*

if X were invertible: $X^{-1}(\text{ABE})X^{-1} \Longrightarrow$

$$YA^T + AY = G, \qquad \text{where} \quad Y = X^{-1};$$

...special *algebraic Riccati equations (ARE):*

$$Q + A^T X + XA - XGX = 0 \qquad \text{with} \quad Q = 0,$$

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- ABE is homogeneous $\Longrightarrow$ $X = 0$ is a (positive semidefinite) solution;
- as special ARE, solution can be obtained from invariant subspaces of corresponding Hamiltonian matrix: if

$$\underbrace{\begin{bmatrix} A & G \\ 0 & -A^T \end{bmatrix}}_{=:H} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} W, \quad U, V, W \in \mathbb{R}^{n \times n},$$

and U is invertible, then

$$X = -VU^{-1}$$

is a solution of the ABE, where $\Lambda(A - GX) = \Lambda(W) \subset \Lambda(H)$.

If $\Lambda(W) = \Lambda(H) \cap \mathbb{C}^-$, then X is a stabilizing solution.

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Feedback stabilization problem

For $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, find $F \in \mathbb{R}^{m \times n}$ such that $\Lambda(A - BF) \subset \mathbb{C}^-$.

Corresponds to finding $u \in L_2(0, \infty; \mathbb{R}^m)$ such that the solution trajectory of

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n,$$

is asymptotically stable, i.e., $\lim_{t \to \infty} x(t; u) = 0$.

If F solves the feedback stabilization problem, then $u(t) = -Fx(t)$.

Solution by ABE

If X is a stabilizing solution of the ABE with $G = BB^T$, then

$$F := B^T X$$

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Problem

Approximate the dynamical system

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}, \end{cases}$$

by reduced-order system

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \qquad \hat{A} \in \mathbb{R}^{r\times r}, \quad \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \qquad \hat{C} \in \mathbb{R}^{p\times r}, \quad \hat{D} \in \mathbb{R}^{p\times m}, \end{cases}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|.$$

Model Reduction

Balanced Truncation (BT)

For A stable, BT is based on balancing thr system Gramians, defined by

$$P = \int_0^\infty e^{At} BB^T e^{A^T t}\, dt, \quad Q = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt.$$

For unstable A, integrals diverge!

Frequency-domain definition of Gramians

$$\begin{aligned} P &:= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega - A)^{-1} BB^T (j\omega - A)^{-H}\, d\omega, \\ Q &:= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega - A)^{-H} C^T C (j\omega - A)^{-1}\, d\omega. \end{aligned}$$

Properties [Zhou/Salomon/Wu 1999]

- Well-defined if $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$; for stable A, definitions coincide.
- (A, B) controllable $\Leftrightarrow P > 0$; (A, C) observable $\Leftrightarrow Q > 0$.
- BT can be based on P, Q, BT error bound holds!

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Computation of Gramians for Unstable Systems

If (A, B) stabilizable, (A, C) detectable, and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then P, Q are solutions of the Lyapunov equations

$$\begin{aligned}(A - BB^T X)P + P(A - BB^T X)^T + BB^T &= 0,\\ (A - YC^T C)^T Q + Q(A - YC^T C) + C^T C &= 0,\end{aligned}$$

where X and Y are the stabilizing solutions of the dual ABEs

$$\begin{aligned}A^T X + XA - XBB^T X &= 0,\\ AY + YA^T - YC^T CY &= 0.\end{aligned}$$

Further Applications

- Coprime factorization problems for rational transfer functions,
- computing (sub-)optimal H_∞ controllers using a mixed sensitivity approach ($\mathcal{S}$-over-$\mathcal{KS}$ design).

Theorem

Consider the ABE

$$A^T X + XA - XBB^T X = 0 \tag{1}$$

with (A, B) controllable. Then

- there exist symmetric solutions $X_+ \geq 0$, $X_- \leq 0$, with $X_- \leq X \leq X_+$ for all solutions X of the ABE;
- X_- is the unique solution satisfying $\Lambda(A - BB^T X_-) \subset \mathbb{C}^+ \cup \imath\mathbb{R}$;
- X_+ is the unique solution satisfying $\Lambda(A - BB^T X_+) \subset \mathbb{C}^- \cup \imath\mathbb{R}$.
- If $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then X_- is the unique anti-stabilizing solution and X_+ is the unique stabilizing solution of the ABE.

Corollary

If (A, B) is stabilizable and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then the ABE (1) has a unique stabilizing solution X_+ and $X_+ \geq 0$.

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Consider again the ABE (1), i.e.,

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with (A, B) stabilizable, $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, and its unique stabilizing solution X_+.
Then

$$\operatorname{rank}(X_+) = k,$$

where k is the number of eigenvalues of A in $\mathbb{C}^+$.

Corollary

If (A, B) is stabilizable and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then the ABE (1) has a unique stabilizing solution

$$X_+ = Y_+ Y_+^T, \quad \text{where} \quad Y_+ \in \mathbb{R}^{n \times k}.$$

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Numerical Solution of ABEs

$A^T X + XA - XGX = 0$

Recall:

- X can be obtained from invariant subspaces of
$$H = \begin{bmatrix} A & G \\ 0 & -A^T \end{bmatrix}.$$
- ABE is homogeneous ARE $\Longrightarrow$ ABE can be solved by any method for AREs, e.g., invariant subspace methods.

In particular, if $\mathcal{P}$ is a projector onto anti-stable, H-invariant subspace, then $\begin{bmatrix} I_n \\ -X_+ \end{bmatrix} \in \ker \mathcal{P}$

$\Longrightarrow$ solve consistent least-squares problem

$$0 = \mathcal{P} \begin{bmatrix} I_n \\ -X_+ \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I_n \\ -X_+ \end{bmatrix} \Leftrightarrow \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} X_+ = \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix}$$

$\rightsquigarrow$ compute projector from $\mathcal{P} = \frac{1}{2}(I_n + \operatorname{sign}(H))$.

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The Matrix Sign Function

Definition

For $Z \in \mathbb{R}^{n \times n}$ with $\Lambda(Z) \cap \imath\mathbb{R} = \emptyset$ and Jordan canonical form

$$Z = S^{-1} \begin{bmatrix} J^+ & 0 \\ 0 & J^- \end{bmatrix} S$$

the matrix sign function is

$$\operatorname{sign}(Z) := S \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} S^{-1}.$$

The Matrix Sign Function

Computation of $\operatorname{sign}(Z)$

$\operatorname{sign}(Z)$ is root of $I_n \Longrightarrow$ use Newton's method to compute it:

$$Z_0 \leftarrow Z, \qquad Z_{j+1} \leftarrow \frac{1}{2}\left(c_j Z_j + \frac{1}{c_j} Z_j^{-1}\right), \qquad j = 1, 2, \ldots$$

$\Longrightarrow \quad \operatorname{sign}(Z) = \lim_{j\to\infty} Z_j.$

($c_j > 0$ is scaling parameter for convergence acceleration and rounding error minimization.)

Solving ABEs with the Matrix Sign Function Method

1. Apply sign function iteration $Z \leftarrow \frac{1}{2}(Z + Z^{-1})$ to H:

$$H + H^{-1} = \begin{bmatrix} A & G \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} A^{-1} & A^{-1}GA^{-T} \\ 0 & -A^{-T} \end{bmatrix}$$

$\Longrightarrow$ Sign function iteration for ABE:

$$\begin{aligned} A_0 &\leftarrow A, \quad A_{j+1} \leftarrow \tfrac{1}{2}\left(A_j + A_j^{-1}\right), \\ G_0 &\leftarrow G, \quad G_{j+1} \leftarrow \tfrac{1}{2}\left(G_j + A_j^{-1} G_j A_j^{-T}\right), \end{aligned} \qquad j = 0, 1, 2, \ldots.$$

Define $A_\infty := \lim_{j\to\infty} A_j$, $G_\infty := \lim_{j\to\infty} G_j$.

2. Solve linear least-squares problems

$$\begin{bmatrix} G_\infty \\ I_n - A_\infty^T \end{bmatrix} X_+ = \begin{bmatrix} A_\infty + I_n \\ 0_n \end{bmatrix}.$$

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Computing Solution Factors

Consider $A^T X + XA - XBB^T X = 0$ and recall that

$$X_+ = Y_+ Y_+^T, \quad Y_+ \in \mathbb{R}^{n \times k}.$$

Sign function iteration can be re-written as for Lyapunov equations:

$$\begin{aligned} B_{j+1} B_{j+1}^T = G_{j+1} &= \frac{1}{2}\left(G_j + A_j^{-1} G_j A_j^{-T}\right) \\ &= \frac{1}{2}\left(B_j B_j^T + A_j^{-1} B_j B_j^T A_j^{-T}\right) \\ &= \frac{1}{2}\left[\begin{array}{cc} B_j & A_j^{-1} B_j \end{array}\right] \left[\begin{array}{cc} B_j & A_j^{-1} B_j \end{array}\right]^T \end{aligned}$$

and use RRQR of $\left[\begin{array}{cc} B_j & A_j^{-1} B_j \end{array}\right]$ for column compression.

But: still need $G_\infty = \lim_{j \to \infty} B_j B_j^T$ for least-squares solution and Y_+ not directly obtained from least-squares problem!

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Computing Solution Factors

Theorem [B. 2006]

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$$A^T X + XA - XBB^T X = 0.$$

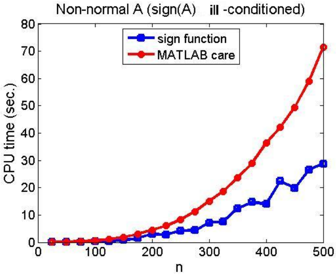
If $B_\infty = \lim_{k\to\infty} B_k$ is obtained from the factorized form of the sign function iteration for the ABE, then a full-rank factor $Y_+ \in \mathbb{R}^{n\times k}$ of X_+ is given by

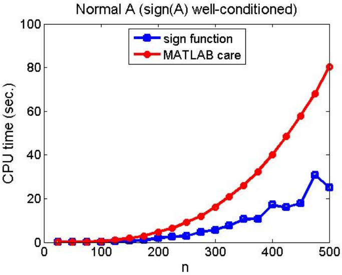
$$Y_+ = \sqrt{2} Q_Y R^{-1}, \qquad (2)$$

where the columns of Q_Y form a basis of $\ker\left(I - \operatorname{sign}(A)^T\right)$ and R is the upper triangular factor in the "skinny" QR factorization of $B_\infty^T Q_Y$.

Numerical Examples

Random ABE with $n = 25 : 25 : 500$, $m = n/5$.





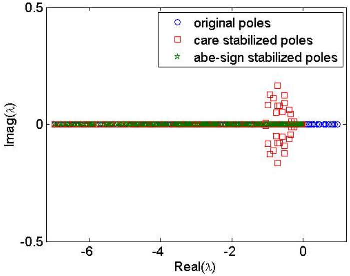
Numerical Examples

Numerical Accuracy: $\frac{\|A^T X_+ + X_+ A - X_+ BB^T X_+\|_1}{\|X_+\|_1}$.

Example	n	care	#Iter.	sign	sign_fac
CAREX 1.1	2	8.08e−22	2	8.08e−22	8.08e−22
CAREX 1.2	2	1.18e−15	3	1.26e−14	1.14e−14
CAREX 2.1	2	0.00e+00	3	0.00e+00	9.77e−16
CAREX 2.3	2	6.68e−46	4	1.00e+06	0.00e+00
CAREX 2.4	2	0.00e+00	3	1.61e−09	8.24e−12
CAREX 2.5	2	1.78e−15	3	1.78e−15	2.49e−15
CAREX 2.6	3	1.98e−09	5	2.82e−09	3.22e−09
CAREX 2.7	4	1.85e−10	5	7.92e−12	5.30e−10
CAREX 2.8	4	3.40e−12	3	5.55e−11	5.55e−11
CAREX 3.1	39	7.61e−16	4	9.71e−11	6.14e−16
CAREX 3.2	64	6.75e−15	24	1.73e−09	1.72e−14
CAREX 4.1a	21	1.97e−20	6	1.00e+00	1.00e+00
CAREX 4.1b	21	3.30e+00	6	8.66e−01	6.54e−08
CAREX 4.3	60	6.78e−15	21	2.85e−12	8.16e−15
RLC	199	2.72e−16	30	6.08e−11	1.41e−15

Numerical Examples

Here, mimic finite-differences discretization of control problem for linear reaction-diffusion problem on unit square:
$A = -\Delta_h + I,\ B = I_{n,m},$
where $h = 1/21,\ m = 20.$



Results for different solvers:

	$\frac{\|B(X_+)\|_1}{\|X_+\|_1}$	CPU time
`care`:	$3.5 \cdot 10^{-5}$	13.8 sec
`sign`:	$4.8 \cdot 10^{-5}$	8.0 sec
`sign_fac`:	$5.2 \cdot 10^{-10}$	7.4 sec

Summary and Outlook

- Sign function-based ABE solver
  - is sometimes more accurate than Schur decomposition-based approach,
  - is often faster than Schur decomposition-based approach,
  - can compute solution factor without forming X_+.

- **Work in progress:**
  - $\mathcal{H}$-matrix based implementation, using normal equations for solving the least-squares problem [BAUR 2007].
  - ADI-like or doubling-type iteration for large and sparse ABEs.
  - Hammarling-style algorithm for computing Y_+?

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References

1. Sergio Barrachina, Peter Benner, Enrique S. Quintana-Ortí, and Gregorio Quintana-Ortí: *Parallel Algorithms for Balanced Truncation of Large-Scale Unstable Systems*, in PROCEEDINGS OF 44TH IEEE CONFERENCE ON DECISION AND EUROPEAN CONTROL CONFERENCE ECC 2005, pp. 2248-2253, 2005.
2. Sergio Barrachina, Peter Benner, and Enrique S. Quintana-Ortí: *Parallel Solution of Large-Scale Algebraic Bernoulli Equations with the Matrix Sign Function Method*, INTERNATIONAL JOURNAL OF COMPUTATIONAL SCIENCE AND ENGINEERING, to appear.
3. Peter Benner, Enrique S. Quintana-Ortí, and Gregorio Quintana-Ortí: *Solving Large-Scale Generalized Algebraic Bernoulli Equations via the Matrix Sign Function*, Preprint, TU Chemnitz, March 2006.

Thanks for your attention!