# Modellreduktion aus (nicht nur) systemtheoretischer Sicht

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# Overview

# Joint work with

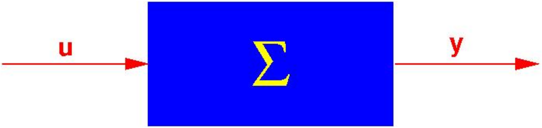
- Enrique Quintana-Ortí, Gregorio Quintana-Ortí, Rafa Mayo, José Manuel Badía, Alfredo Remón, Sergio Barrachina (Universidad Jaume I de Castellón, Spain).
- Ulrike Baur, Matthias Pester, Jens Saak ( MiT ).
- Viatcheslav Sokolov ( former MiT ).
- Heike Faßbender (TU Braunschweig).
- Infineon Technologies/Qimonda, IMTEK (U Freiburg), iwb (TU München), . . .

## Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.



# Model Reduction for Dynamical Systems

## Original System

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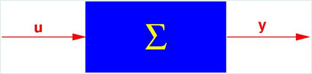


## Reduced-Order System

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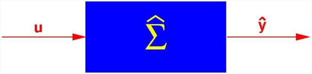


**Goal:**

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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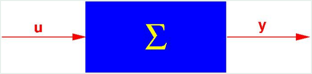


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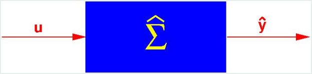


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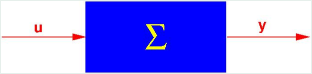


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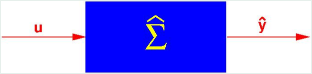


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## Linear, Time-Invariant (LTI) Systems

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## State-Space Description for I/O-Relation ($D = 0$)

$$\mathcal{S} : u \mapsto y, \quad y(t) = (h \star u)(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\, d\tau,$$

where $h(s) = \begin{cases} Ce^{A(s)}B & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$

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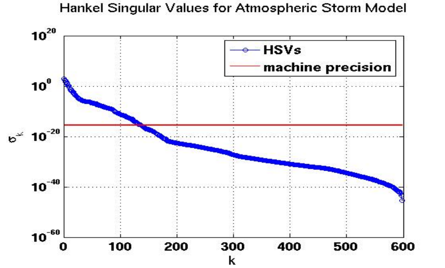

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But: computationally unfeasible for large-scale systems.

# Linear Systems in Frequency Domain

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## Laplace Transformation / Frequency Domain

Application of Laplace transformation ($x(t) \mapsto x(s)$, $\dot{x}(t) \mapsto sx(s)$) to linear system with $x(0) = 0$:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Bx(s) + Du(s),$$

yields I/O-relation in frequency domain:

$$y(s) = \Big( \underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)} \Big) u(s)$$

$G$ is the transfer function of $\Sigma$.

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Approximate the dynamical system

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by reduced-order system

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of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \le \|G - \hat{G}\|\|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G}) \le r} \|G - \hat{G}\|$.

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# Application Areas
## (Optimal) Control

### Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order $N$, where

- input $=$ output of plant,
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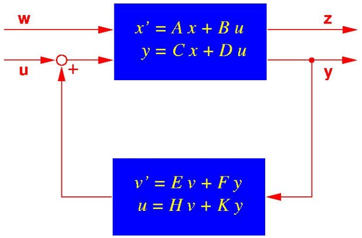
Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$

$\Rightarrow$ reduce order of original system.



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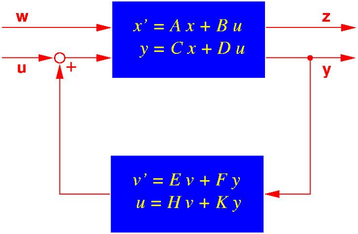

# Application Areas
## Micro Electronics

- Progressive miniaturization: **Moore's Law** states that the number of on-chip transistors doubles each 12 (now: 18) months.
- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- Increase in packing density requires modeling of interconncet to ensure that thermic/electro-magnetic effects do not disturb signal transmission.
- Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

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Model Reduction

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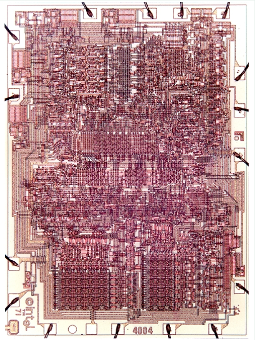
# Application Areas

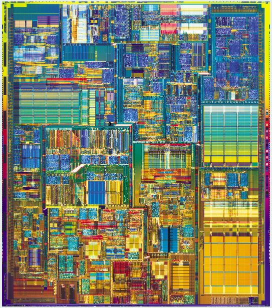
## Micro Electronics: Example for Miniaturization

### Intel 4004 (1971)




- 1 layer, $10\mu$ technology,
- 2,300 transistors,
- 64 kHz clock speed.

### Intel Pentium IV (2001)



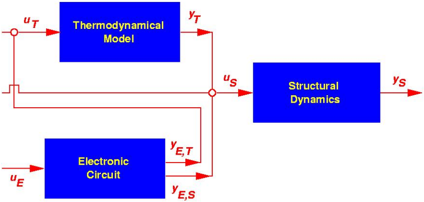
- 7 layers, $0.18\mu$ technology,
- 42,000,000 transistors,
- 2 GHz clock speed,
- 2km of interconnect.

# Application Areas
## MEMS/Microsystems

Typical problem in MEMS simulation:
coupling of different models (thermic, structural, electric, electro-magnetic) during simulation.

Problems and Challenges:

- Reduce simulation times by replacing sub-systems with their reduced-order models.
- Stability properties of coupled system may deteriorate through model reduction even when stable sub-systems are replaced by stable reduced-order models.
- Multi-scale phenomena.



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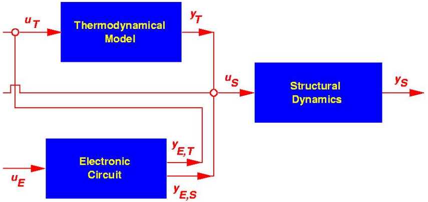

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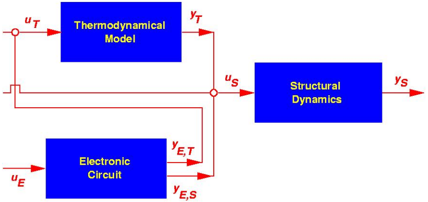

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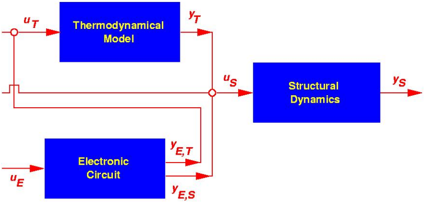

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of $G$ in $\mathbb{C}^-$),
  - minimum phase (zeroes of $G$ in $\mathbb{C}^-$),
  - passivity ("system does not generate energy").

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  - stability (poles of $G$ in $\mathbb{C}^-$),
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  - passivity ("system does not generate energy").

- Automatic generation of compact models.
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$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

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2. Padé-Approximation and Krylov Subspace Methods
3. Balanced Truncation
4. many more...

# Model Reduction



1. Modal Truncation
2. Padé-Approximation and Krylov Subspace Methods
3. Balanced Truncation
4. many more...

Joint feature of many methods: Galerkin or Petrov-Galerkin-type projection of state-space onto low-dimensional subspace $\mathcal{V}$ along $\mathcal{W}$: assume $x \approx VW^T x =: \tilde{x}$, where

$$\mathrm{range}\,(V) = \mathcal{V}, \quad \mathrm{range}\,(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ and

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|.$$

# Modal Truncation

## Idea:

Project state-space onto $A$-invariant subspace $\mathcal{V}$, where

$$\mathcal{V} = \operatorname{span}(v_1, \ldots, v_r),$$

$v_k$ = eigenvectors corresp. to "dominant" modes $\equiv$ eigenvalues of $A$.

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## Properties:

- Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.
- Error bound:

$$\|G - \hat{G}\|_\infty \leq \mathrm{cond}_2(T)\, \|C_2\|_2 \|B_2\|_2 \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\mathrm{Re}(\lambda)|},$$

where $T^{-1}AT = \mathrm{diag}(A_1, A_2)$.

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## Difficulties:

- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
  (Litz 1979: use Jordan canoncial form; otherwise merely heuristic criteria.)
- Error bound not computable for really large-scale probems.
- New direction: AMLS (automated multilevel substructuring) [Benninghof/Lehoucq '04, Elssel/Voss '05, Blömeling '06].

# Padé Approximation



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- Consider

$$E\dot{x} = Ax + Bu, \quad y = Cx$$

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$$G(s) \quad = \quad m_0 + m_1(s - s_0) + m_2(s - s_0)^2 + \ldots$$
- As reduced-order model use $r$th Padé approximant $\hat{G}$ to $G$:
$$G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^{2r}),$$
i.e., $m_j = \widehat{m}_j$ for $j = 0, \ldots, 2r - 1$
$\rightsquigarrow$ moment matching if $s_0 < \infty$,
$\rightsquigarrow$ partial realization if $s_0 = \infty$.

# Padé Approximation

## Padé-via-Lanczos Method (PVL)

- Moments need not be computed explicitly; moment matching is equivalent to projecting state-space onto

$$\mathcal{V} = \mathrm{span}(\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{r-1}B) = \mathcal{K}(\tilde{A}, \tilde{B}, r)$$

(where $\tilde{A} = (s_0 E - A)^{-1}E,\ \tilde{B} = (s_0 E - A)^{-1}B$) along

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- Computation via unsymmetric Lanczos method, yields system matrices of reduced-order model as by-product.
- PVL applies w/o changes for singular $E$ if $s_0 \notin \Lambda(A, E)$:
  - for $s_0 \neq \infty$: Gallivan/Grimme/Van Dooren 1994, Freund/Feldmann 1996, Grimme 1997
  - for $s_0 = \infty$: B./Sokolov 2005

## Padé-via-Lanczos Method (PVL)

**Partial realization for descriptor systems:** [B./Sokolov, SCL, 2006]

For nonsingular $E$

$$\text{moments} = \text{Markov parameters} = C(E^{-1}A)^j E^{-1}B, \quad j = 0, 1, \ldots$$

Question: for $E$ singular, what is the correct generalized inverse here?

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Answer: {2}-inverse $E^{\{2\}} = Q \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} P$, where

$$sPEQ - PAQ = s \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

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- $P, Q$ can be computed w/o WCF in many applications.
- Numerically, use Lanczos applied to $\{E^{\{2\}}A, E^{\{2\}}B, C\}$.

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New direction: moment matching yields rational interpolation of $G^{(j)}(s)$ for $j = 0, \ldots, 2r - 1$ at $s = s_0$.
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Instead: use rational (Hermite) interpolation at $s_j$, $j = 0, \ldots, r$.
Question: where to put $s_j$? ($\rightsquigarrow$ next talk!)

# Balanced Truncation



## Idea:

- A system $\Sigma$, realized by $(A, B, C, D)$, is called balanced, if solutions $P, Q$ of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of $\Sigma$.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (A_{11}, B_1, C_1, D)$.

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## Motivation:

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

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$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\, dt}{\int_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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$\Longrightarrow$ Truncate states corresponding to "small" HSVs

$\Longrightarrow$ complete analogy to best approximation via SVD!

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3. Set
$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$
4. Reduced model is $(W^T AV, W^T B, CV, D)$.

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- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
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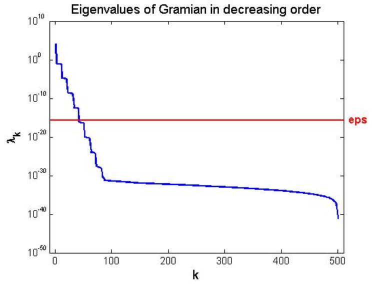
New algorithmic ideas from numerical linear algebra:

- Instead of Gramians $P, Q$ compute $S, R \in \mathbb{R}^{n \times k}$, $k \ll n$, such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

- Compute $S, R$ with problem-specific Lyapunov solvers of "low" complexity directly.



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New algorithmic ideas from numerical linear algebra:

**Parallelization:**

- Efficient parallel algorithms based on matrix sign function.
- Complexity $\mathcal{O}(n^3/q)$ on $q$-processor machine.
- Software library PLiCMR with WebComputing interface.

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**Formatted Arithmetic:**

For special problems from PDE control use implementation based on hierarchical matrices and matrix sign function method (Baur/B.), complexity $\mathcal{O}(n \log^2(n) r^2)$.

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New algorithmic ideas from numerical linear algebra:

**Sparse Balanced Truncation:**

- Sparse implementation using sparse Lyapunov solver (ADI+MUMPS/SuperLU).
- Complexity $\mathcal{O}(n(k^2 + r^2))$.
- Software:
  - + MATLAB toolbox LYAPACK (PENZL 1999),
  - + Software library SPARED with WebComputing interface.
    (BADÍA/B./QUINTANA-ORTÍ/QUINTANA-ORTÍ since 2003)

# Balanced Truncation for Unstable Systems



For $A$ stable, Gramians are defined by

$$P = \int_0^\infty e^{At} B B^T e^{A^T t}\, dt, \quad Q = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt.$$

For unstable $A$, integrals diverge!

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## Frequency-domain definition of Gramians

$$\begin{aligned} P &:= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega - A)^{-1} BB^T (j\omega - A)^{-H}\, d\omega, \\ Q &:= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega - A)^{-H} C^T C (j\omega - A)^{-1}\, d\omega. \end{aligned}$$

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- Well-defined if $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$!
- For stable $A$, definitions coincide.
- Balancing/balanced truncation can be based on $P$, $Q$. Moreover, BT error bound holds!

# Balanced Truncation for Unstable Systems

## Computation of Unstable Gramians

If $(A, B)$ stabilizable, $(A, C)$ detectable, and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then $P, Q$ are solutions of the Lyapunov equations

$$
\begin{aligned}
(A - BB^T X)P + P(A - BB^T X)^T + BB^T &= 0, \\
(A - YC^T C)^T Q + Q(A - YC^T C) + C^T C &= 0,
\end{aligned}
$$

where $X$ and $Y$ are the stabilizing solutions of the dual algebraic Bernoulli equations

$$
\begin{aligned}
A^T X + XA - XBB^T X &= 0, \\
AY + YA^T - YC^T CY &= 0.
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## Theorem [B. 2006]

Let $(A, B)$ be stabilizable, $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, and $X_+$ be the unique stabilizing solution of the ABE

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a) $\operatorname{rank}(X_+) = k$, where $k$ is the number of eigenvalues of $A$ in $\mathbb{C}^+$.

b) A full-rank factor $Y_+ \in \mathbb{R}^{n\times k}$ of $X_+$ is given by

$$Y_+ = \sqrt{2} Q_Y R^{-1},$$

where colspan$(Q_Y)$ is basis of anti-stable $A$-invariant subspace, $R$ is defined via $\operatorname{sign}\left(\begin{bmatrix} A^T & BB^T \\ 0 & -A \end{bmatrix}\right)$.

## Theorem [B. 2006]

Let $(A, B)$ be stabilizable, $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, and $X_+$ be the unique stabilizing solution of the ABE

$$A^T X + XA - XBB^T X = 0.$$

Then

a) $\operatorname{rank}(X_+) = k$, where $k$ is the number of eigenvalues of $A$ in $\mathbb{C}^+$.

b) A full-rank factor $Y_+ \in \mathbb{R}^{n\times k}$ of $X_+$ is given by

$$Y_+ = \sqrt{2} Q_Y R^{-1},$$

where colspan$(Q_Y)$ is basis of anti-stable $A$-invariant subspace, $R$ is defined via $\operatorname{sign}\left(\begin{bmatrix} A^T & BB^T \\ 0 & -A \end{bmatrix}\right)$.

Efficient solution of ABEs: sign-function based computation of $Y_+$ [Barrachina/B./Quintana-Ortí].

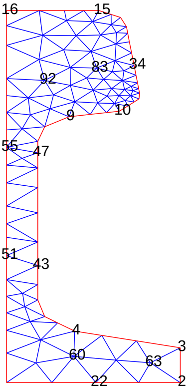
Current work: solvers for large-scale ABEs with (data-)sparse $A$.

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, & \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa(u_k - x), & \xi \in \Gamma_k,\ 1 \leq k \leq 7, \\
\frac{\partial}{\partial n} x &= 0, & \xi \in \Gamma_7.
\end{aligned}
$$

$\implies m = 7, p = 6.$

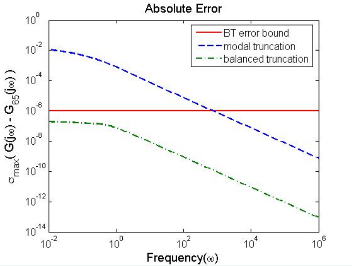
- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841$.




Source: Physical model: courtesy of Mannesmann/Demag.

Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

## $n = 1357$, Absolute Error




- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

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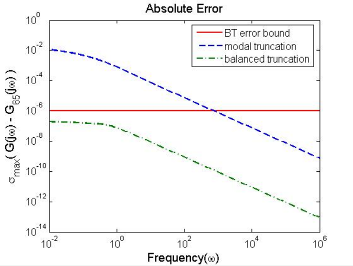


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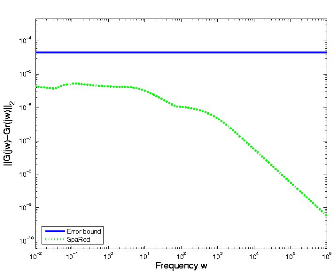
### $n = 79841$, Absolute error

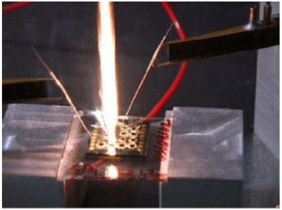


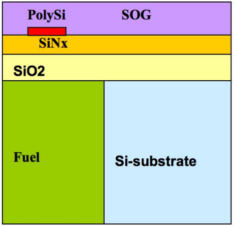

- BT model computed using SpaRed,
- computation time: 8 min.

# Examples

## MEMS: Microthruster

- Co-integration of solid fuel with silicon micromachined system.
- Goal: Ignition of solid fuel cells by electric impulse.
- Application: nano satellites.
- Thermo-dynamical model, ignition via heating an electric resistance by applying voltage source.
- Design problem: reach ignition temperature of fuel cell w/o firing neighbouring cells.
- Spatial FEM discretizatoin of thermo-dynamical model $\rightsquigarrow$ linear system, $m = 1$, $p = 7$.






Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Courtesy of C. Rossi, LAAS-CNRS/EU project "Micropyros".

# Examples

## MEMS: Microthruster

- axial-symmetric 2D model
- FEM discretisation using linear (quadratic) elements $\rightsquigarrow$ $n = 4,257$ $(11,445)$ $m = 1$, $p = 7$.
- Reduced model computed using SpaRed. modal truncation using ARPACK, and Z. Bai's PVL implementation.

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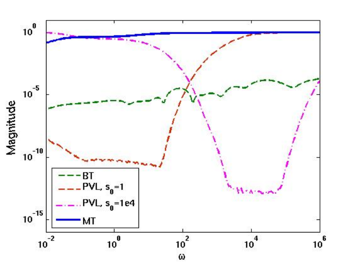
### Relative error $n = 4,257$



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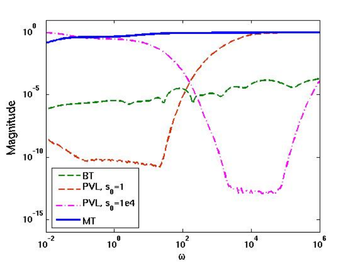
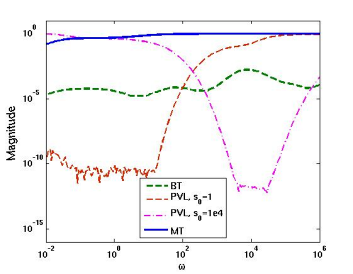



### Relative error $n = 11,445$



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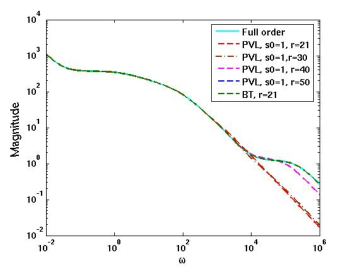
### Frequency Response BT/PVL



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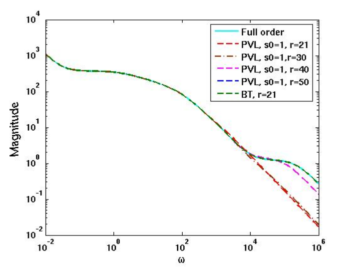
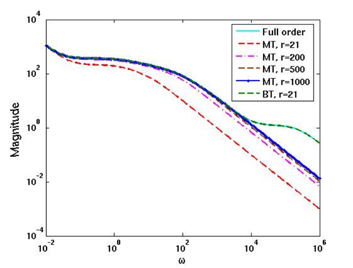


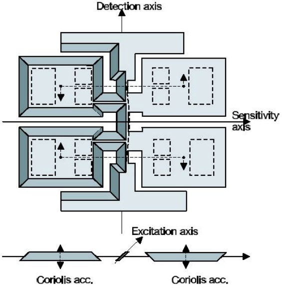

### Frequency Response BT/MT



# Examples

## MEMS: Microgyroscope (Butterfly Gyro)

- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.




- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

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Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Courtesy of D. Billger (Imego Institute, Göteborg), Saab Bofors Dynamics AB.

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722,\ m = 1,\ p = 12.$
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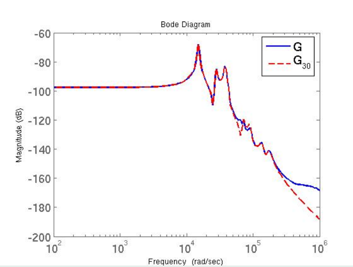
### Frequency Repsonse Analysis



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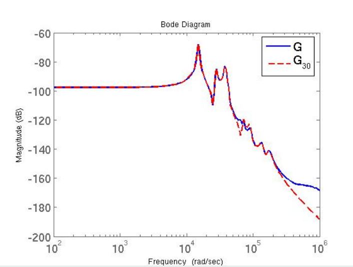
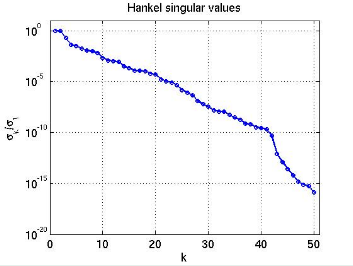



### Hankel Singular Values



# Interconnect

- RLC circuit, characteristic curve has falling edge at $\omega = 100\,\text{Hz}$.
- $n = 1999$, $m = p = 2$, reduced model using PLICMR: $r = 20$.

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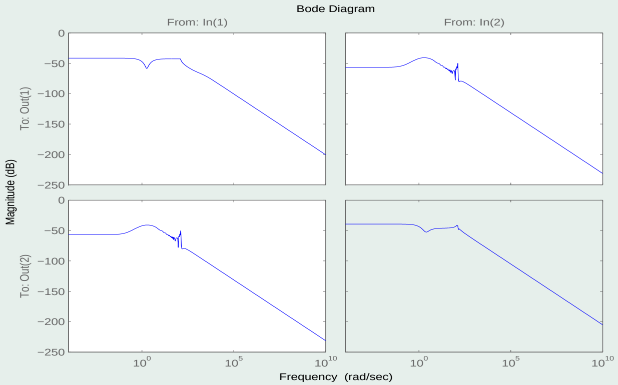
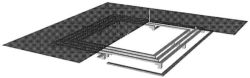
## Accuracy of Reduced-Order Model



- Passive device used for RF filters etc.
- $n = 500$, $m = 1$, $p = 1$.
- Numerical rank of Gramians is 34.
- $r = 11$ model computed by PLiCMR.



Courtesy of MIT/Jing-Rebecca Li

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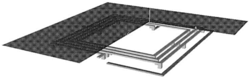
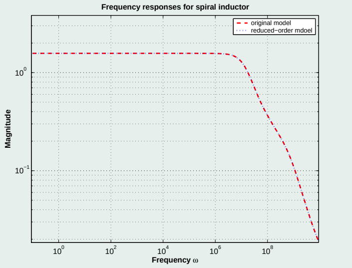


### Frequency Repsonse Analysis




Courtesy of MIT/Jing-Rebecca Li

# Examples
## Micro Electronics: Spiral Inductor

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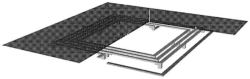

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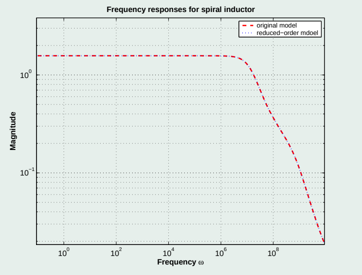
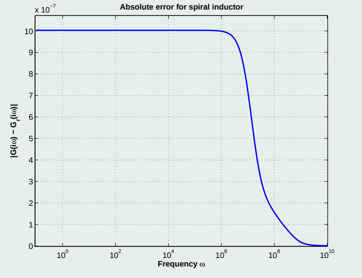



### Absolute Error




Courtesy of MIT/Jing-Rebecca Li

BT is often criticized for its bias towards the input-output behavior of the system. But states can also be reconstructed using

$$x(t) \approx V x_r(t).$$

**Example:** 2D heat equation with localized heat source, $64 \times 64$ grid, $r = 6$ model by BT, simulation for $u(t) = 10\cos(t)$.

# Examples

## Reconstruction of the State

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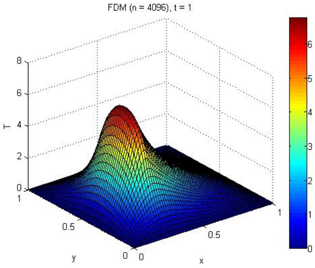

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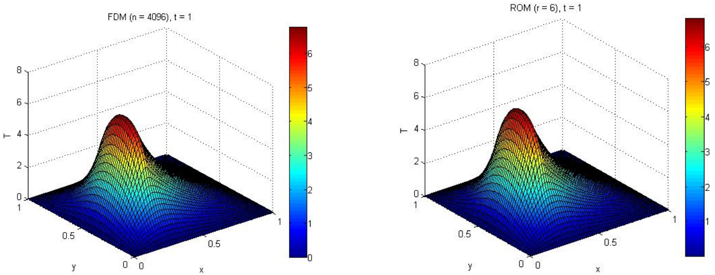
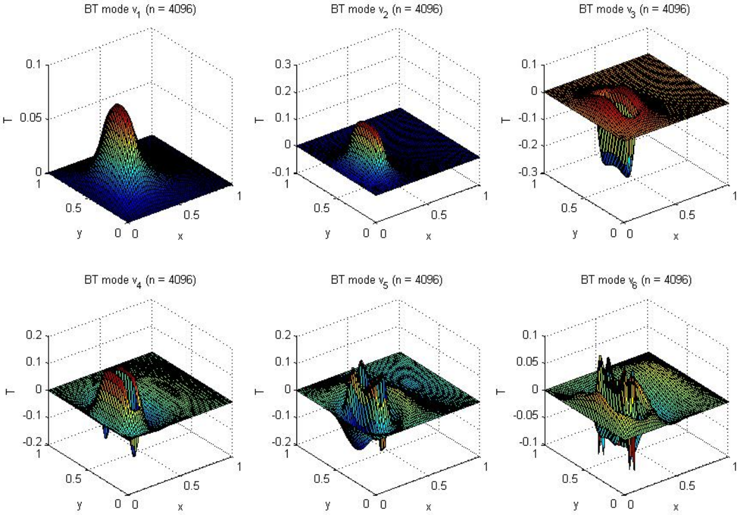


# Examples

## BT modes are intelligent ansatz functions for Galerkin projection



# Current and Future Work

## Parametric Models

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x + D(p)u,$$

where $p \in \mathbb{R}^s$ are free parameters which should be preserved in the reduced-order model.

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- Frequently: $B, C, D$ parameter independent,

$$A(p) = A_0 + p_1 A_1 + \ldots + p_s A_s.$$

  $\Rightarrow$ (Modified) linear model reduction methods applicable.
- Multipoint expansion combined with Padé-type approx. possible.
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$$x \approx V\hat{x}, \quad \dot{\hat{x}} = W^T f(V\hat{x}, u)$$

is in general not model reduction!

# Current and Future Work

## Nonlinear Systems

- Linear projection

$$x \approx V\hat{x}, \quad \dot{\hat{x}} = W^T f(V\hat{x}, u)$$

  is in general not model reduction!
- Need specific methods
  - POD + balanced truncation $\rightsquigarrow$ empirical Gramians (Lall/Marsden/Glavaski 1999/2002),
  - Approximate inertial manifold method ($\sim$ static condensation for nonlinear systems).

# Current and Future Work

## Nonlinear Systems

- Linear projection

$$x \approx V\hat{x}, \quad \dot{\hat{x}} = W^T f(V\hat{x}, u)$$

  is in general not model reduction!

- Exploit structure of nonlinearities, e.g., in optimal control of linear PDEs with nonlinear BCs $\rightsquigarrow$
  - bilinear control systems $\dot{x} = Ax + \sum_j N_j x u_j + Bu$,
  - formal linear systems (cf. Föllinger 1982)

$$\dot{x} = Ax + N\,g(Hx) + Bu = Ax + \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} u \\ g(z) \end{bmatrix},$$

  where $z := Hx \in \mathbb{R}^\ell$, $\ell \ll n$.

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Thanks for your attention!