

# Numerical Linear Algebra for Model Reduction in Control and Simulation

Peter Benner

Professur Mathematik in Industrie und Technik  
Fakultät für Mathematik  
Technische Universität Chemnitz



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# Thanks to

Model Reduction

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References

- Enrique Quintana-Ortí, Gregorio Quintana-Ortí, Rafa Mayo, José Manuel Badía, and Sergio Barrachina (Universidad Jaume I de Castellón, Spain).
- Thilo Penzl for LyaPack.
- Ulrike Baur, Matthias Pester, Jens Saak (  ).
- Viatcheslav Sokolov ( former  ).
- Heike Faßbender (TU Braunschweig).

## Problem

Given a physical problem with dynamics described by the **states**  $x \in \mathbb{R}^n$ , where  $n$  is the dimension of the **state space**.

*Because of redundancies, complexity, etc., we want to describe the dynamics of the system using a reduced number of states.*

*This is the task of model reduction (also: dimension reduction, order reduction).*

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- A digital image with  $n_x \times n_y$  pixels can be represented as matrix  $X \in \mathbb{R}^{n_x \times n_y}$ , where  $x_{ij}$  contains color information of pixel  $(i, j)$ .
- Memory:  $4 \cdot n_x \cdot n_y$  bytes.

### Theorem: (Schmidt-Mirsky/Eckart-Young)

Best rank- $r$  approximation to  $X \in \mathbb{R}^{n_x \times n_y}$  w.r.t. spectral norm:

$$\hat{X} = \sum_{j=1}^r \sigma_j u_j v_j^T,$$

where  $X = U\Sigma V^T$  is the singular value decomposition (SVD) of  $X$ .  
The approximation error is  $\|X - \hat{X}\|_2 = \sigma_{r+1}$ .

### Idea for dimension reduction

Instead of  $X$  save  $u_1, \dots, u_r, \sigma_1 v_1, \dots, \sigma_r v_r$ .  
 $\rightsquigarrow$  memory =  $r \times (n_x + n_y)$  bytes.

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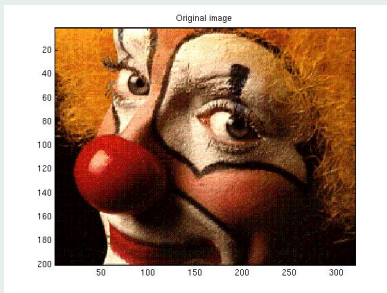
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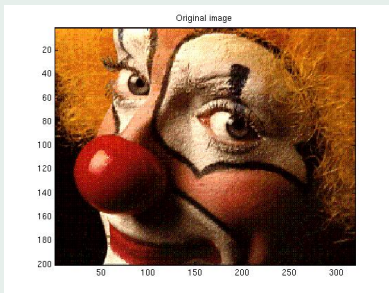
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## Example: Clown

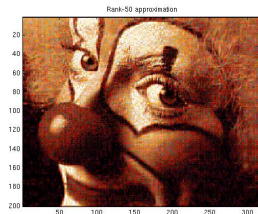


$320 \times 200$  pixel  
 $\rightsquigarrow \approx 256$  kb

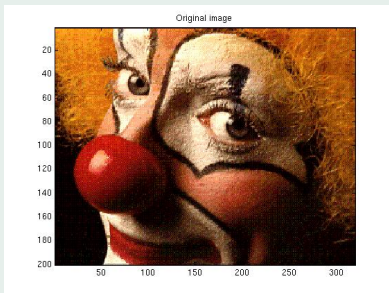
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■ rank  $r = 50$ , ≈ 104 kb

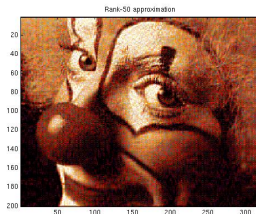


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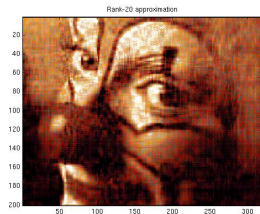


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- rank  $r = 50$ ,  $\approx 104$  kb



- rank  $r = 20$ ,  $\approx 42$  kb

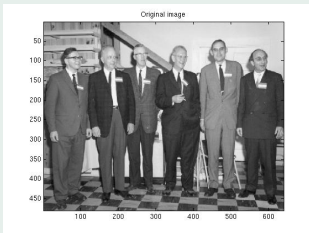


## Example: Gatlinburg

Organizing committee

Gatlinburg/Householder Meeting 1964:

*James H. Wilkinson, Wallace Givens,  
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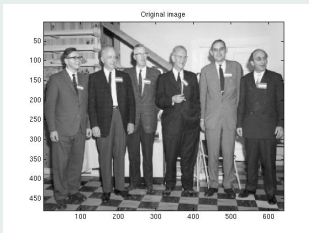


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$640 \times 480$  pixel,  $\approx 1229$  kb

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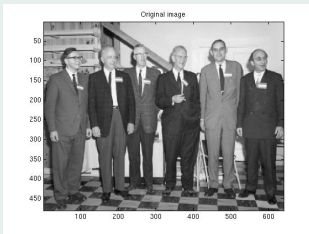


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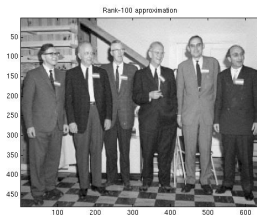
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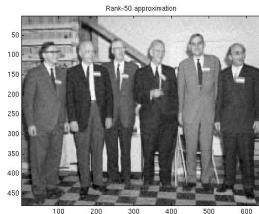
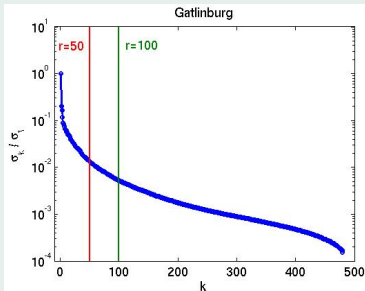
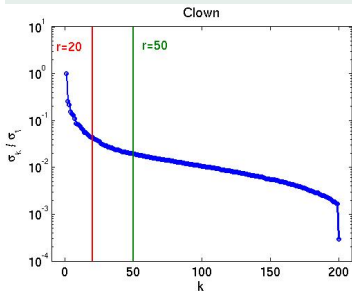


Image data compression via SVD works, if the singular values decay (exponentially).

## Singular Values of the Image Data Matrices





## Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & x(t_0) = x_0, \\ y(t) = g(t, x(t), u(t)) \end{cases}$$

with

- **states**  $x(t) \in \mathbb{R}^n$ ,
- **inputs**  $u(t) \in \mathbb{R}^m$ ,
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Goal:

 $\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$  for all admissible input signals.

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# Model Reduction for Linear Systems

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Systems Theory

**Linear Systems**

Application

Areas

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Examples

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## Linear, Time-Invariant (LTI) Systems

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## State-Space Description for I/O-Relation

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^t Ce^{A(t-\tau)} Bu(\tau) d\tau \quad \text{for all } t \in \mathbb{R}.$$

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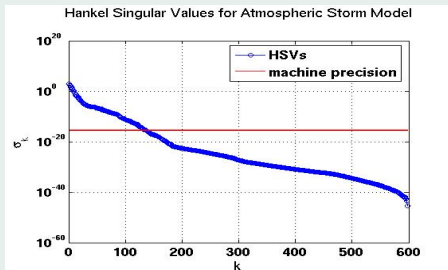
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(**Hankel norm**) well-posed.

$\Rightarrow$  solution: **Adamjan-Arov-Krein (AAK Theory, 1971/78)**.

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**But: computationally unfeasible for large-scale systems.**

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## Laplace Transformation / Frequency Domain

Application of Laplace transformation ( $x(t) \mapsto x(s)$ ,  $\dot{x}(t) \mapsto sx(s)$ ) to linear system with  $x(0) = 0$ :

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Bx(s) + Du(s),$$

yields I/O-relation in frequency domain:

$$y(s) = \underbrace{\left( C(sI_n - A)^{-1}B + D \right)}_{=: G(s)} u(s)$$

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by reduced-order system

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of order  $r \ll n$ , such that

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$\implies$  Approximation problem:  $\min_{\text{order}(\hat{G}) \leq r} \|G - \hat{G}\|$ .



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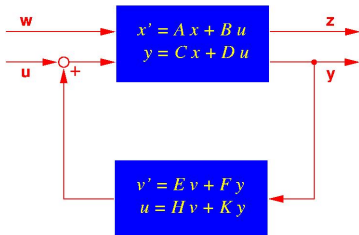
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- input = output of plant,
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Modern (LQG-/ $\mathcal{H}_2$ -/ $\mathcal{H}_\infty$ -) control design:  $N \geq n$

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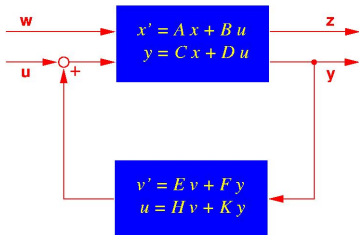
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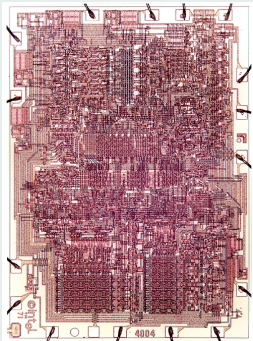
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- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- Increase in packing density requires modeling of interconnect to ensure that thermic/electro-magnetic effects do not disturb signal transmission.
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- Progressive miniaturization: **Moore's Law** states that the number of on-chip transistors doubles each 12 (now: 18) months.
- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
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- Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when
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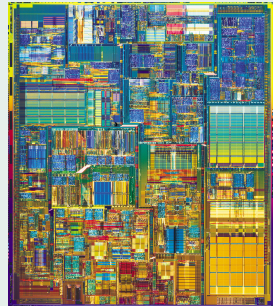
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### Intel 4004 (1971)



- 1 layer,  $10\mu$  technology,
- 2,300 transistors,
- 64 kHz clock speed.

### Intel Pentium IV (2001)



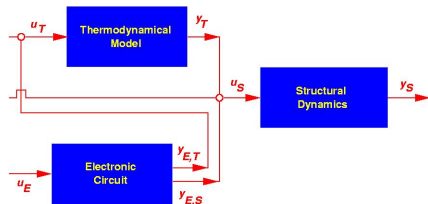
- 7 layers,  $0.18\mu$  technology,
- 42,000,000 transistors,
- 2 GHz clock speed,
- **2km of interconnect.**



Typical problem in MEMS simulation:  
coupling of different models (thermic, structural, electric, electro-magnetic) during simulation.

Problems and Challenges:

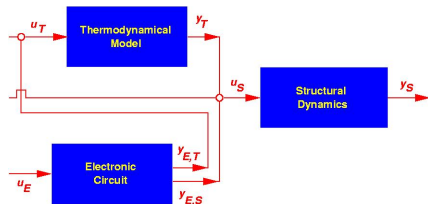
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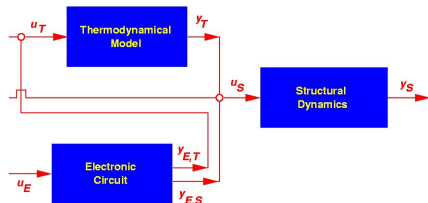
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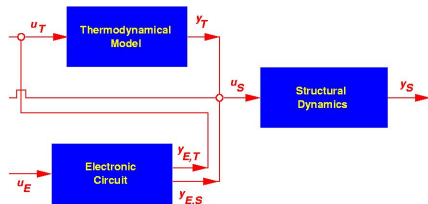
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- **Automatic generation of compact models.**
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

⇒ Need computable error bound/estimate!

- Preserve physical properties:

→ *non-linear* systems of ODEs  
→ *non-linear* systems of DAEs  
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- 2 Padé-Approximation and Krylov Subspace Methods
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- 4 many more. . .

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Joint feature of many methods: **Galerkin** or **Petrov-Galerkin-type projection** of state-space onto low-dimensional subspace  $\mathcal{V}$  along  $\mathcal{W}$ : assume  $x \approx VW^T x =: \tilde{x}$ , where

$$\text{range}(V) = \mathcal{V}, \quad \text{range}(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with  $\hat{x} = W^T x$ , we obtain  $x \approx V\hat{x}$  and

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|.$$



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Idea:

Project state-space onto  $A$ -invariant subspace  $\mathcal{V}$ , where

$$\mathcal{V} = \text{span}(v_1, \dots, v_r),$$

$v_k$  = eigenvectors corresp. to “dominant” **modes**  $\equiv$  **eigenvalues** of  $A$ .

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## Properties:

- Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.
- Error bound:

$$\|G - \hat{G}\|_{\infty} \leq \text{cond}_2(T) \|C_2\|_2 \|B_2\|_2 \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\text{Re}(\lambda)|},$$

where  $T^{-1}AT = \text{diag}(A_1, A_2)$ .

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## Difficulties:

- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.  
(LITZ 1979: use Jordan canonical form; otherwise merely heuristic criteria.)
- Error bound not computable for really large-scale problems.



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- As reduced-order model use **rth Padé approximant**  $\hat{G}$  to  $G$ :

$$G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^{2r}),$$

i.e.,  $m_j = \hat{m}_j$  for  $j = 0, \dots, 2r - 1$

$\rightsquigarrow$  **moment matching** if  $s_0 < \infty$ ,

$\rightsquigarrow$  **partial realization** if  $s_0 = \infty$ .

## Padé-via-Lanczos Method (PVL)

- Moments need not be computed explicitly; moment matching is equivalent to **projecting** state-space **onto**

$$\mathcal{V} = \text{span}(\tilde{B}, \tilde{A}\tilde{B}, \dots, \tilde{A}^{r-1}\tilde{B}) = \mathcal{K}(\tilde{A}, \tilde{B}, r)$$

(where  $\tilde{A} = (s_0 E - A)^{-1} E$ ,  $\tilde{B} = (s_0 E - A)^{-1} B$ ) along

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- Computation via unsymmetric Lanczos method, yields system matrices of reduced-order model as by-product.
- PVL applies w/o changes for singular  $E$  if  $s_0 \notin \Lambda(A, E)$ :
  - for  $s_0 \neq \infty$ : GALLIVAN/GRIMME/VAN DOOREN 1994,  
FREUND/FELDMANN 1996, GRIMME 1997
  - for  $s_0 = \infty$ : B./SOKOLOV 2005



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- Good approximation quality only locally.
- Preservation of physical properties only in very special cases; usually requires post processing which (partially) destroys moment matching properties.

## Idea:

- A system  $\Sigma$ , realized by  $(A, B, C, D)$ , is called **balanced**, if solutions  $P, Q$  of the **Lyapunov equations**

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0,$$

satisfy:  $P = Q = \text{diag}(\sigma_1, \dots, \sigma_n)$  with  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$ .

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- Compute balanced realization of the system via **state-space transformation**

$$\begin{aligned} T : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

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- Truncation  $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (A_{11}, B_1, C_1, D)$ .

## Motivation:

HSV are **system invariants**: they are preserved under  $\mathcal{T}$  and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

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In balanced coordinates ... **energy transfer from  $u_-$  to  $y_+$** :

$$E := \sup_{\substack{u \in L_2(-\infty, 0] \\ x(0) = x_0}} \frac{\int_0^{\infty} y(t)^T y(t) dt}{\int_{-\infty}^0 u(t)^T u(t) dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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⇒ **Truncate states corresponding to “small” HSVs**

⇒ **complete analogy to best approximation via SVD!**





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- 3 Set

$$W = R^T V_1 \Sigma_1^{-1/2}, \quad V = S^T U_1 \Sigma_1^{-1/2}.$$

- 4 Reduced model is  $(W^T A V, W^T B, C V, D)$ .



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- Adaptive choice of  $r$  via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left( 2 \sum_{k=r+1}^n \sigma_k \right) \|u\|_2.$$



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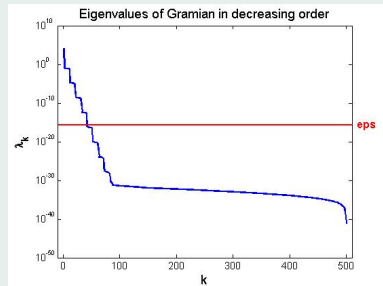
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New algorithmic ideas from numerical linear algebra:

- Instead of Gramians  $P, Q$  compute  $S, R \in \mathbb{R}^{n \times k}$ ,  $k \ll n$ , such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

- Compute  $S, R$  with problem-specific Lyapunov solvers of “low” complexity directly.







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New algorithmic ideas from numerical linear algebra:

### Parallelization:

- Efficient parallel algorithms based on matrix sign function.
- **Complexity  $\mathcal{O}(n^3/q)$**  on  $q$ -processor machine.
- Software library **PLICMR** with **WebComputing interface**.

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### Formatted Arithmetic:

For special problems from PDE control use implementation based on hierarchical matrices and matrix sign function method (BAUR/B.), **complexity  $\mathcal{O}(n \log^2(n)r^2)$** .

## Properties:

**General misconception: complexity  $\mathcal{O}(n^3)$**  – true for several implementations! (e.g., MATLAB, SLICOT).

New algorithmic ideas from numerical linear algebra:

**Sparse Balanced Truncation:**

- Sparse implementation using sparse Lyapunov solver (ADI+MUMPS/SuperLU).
- **Complexity  $\mathcal{O}(n(k^2 + r^2))$ .**
- Software:
  - + MATLAB toolbox **LYAPACK** (PENZL 1999),
  - + Software library **SPARED** with **WebComputing interface**.  
(BADÍA/B./QUINTANA-ORTÍ/QUINTANA-ORTÍ since 2003)

- Mathematical model: boundary control for linearized 2D heat equation.

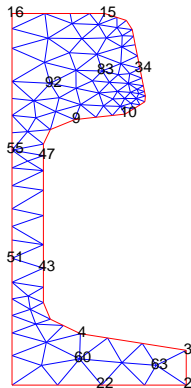
$$c \cdot \rho \frac{\partial}{\partial t} x = \lambda \Delta x, \quad \xi \in \Omega$$

$$\lambda \frac{\partial}{\partial n} x = \kappa (u_k - x), \quad \xi \in \Gamma_k, \quad 1 \leq k \leq 7,$$

$$\frac{\partial}{\partial n} x = 0, \quad \xi \in \Gamma_7.$$

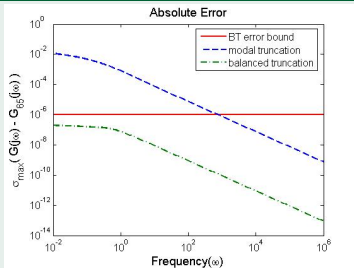
$$\implies m = 7, p = 6.$$

- FEM Discretization, different models for initial mesh ( $n = 371$ ),  
1, 2, 3, 4 steps of mesh refinement  $\implies$   
 $n = 1357, 5177, 20209, 79841$ .



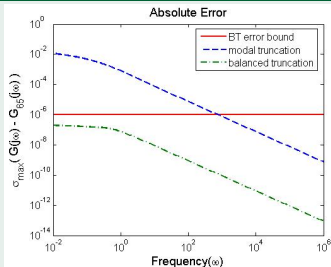
Source: Physical model: courtesy of Mannesmann/Demag.

Math. model: TRÖLTZSCH/UNGER 1999/2001, PENZL 1999, SAAK 2003.

$n = 1357$ , Absolute Error

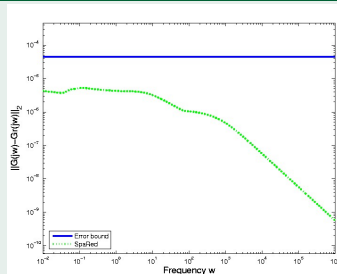
- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

### $n = 1357$ , Absolute Error



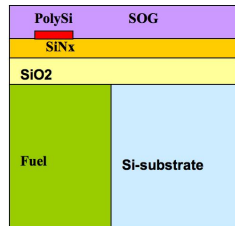
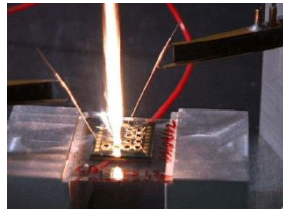
- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

### $n = 79841$ , Absolute error



- BT model computed using SpaRed,
- computation time: **8 min.**

- Co-integration of solid fuel with silicon micromachined system.
- Goal: Ignition of solid fuel cells by electric impulse.
- Application: nano satellites.
- Thermo-dynamical model, ignition via heating an electric resistance by applying voltage source.
- Design problem: reach ignition temperature of fuel cell w/o firing neighbouring cells.
- Spatial FEM discretization of thermo-dynamical model  $\rightsquigarrow$  linear system,  $m = 1$ ,  $p = 7$ .



Source: The Oberwolfach Benchmark Collection <http://www.intek.de/simulation/benchmark>

Courtesy of C. Rossi, LAAS-CNRS/EU project "Micropyros".



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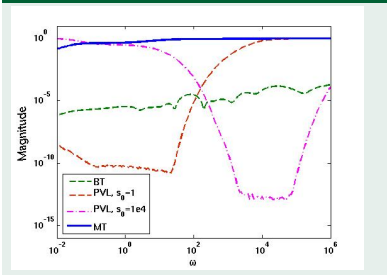
References

- axial-symmetric 2D model
- FEM discretisation using linear (quadratic) elements  $\rightsquigarrow n = 4,257$  (11,445)  $m = 1$ ,  $p = 7$ .
- Reduced model computed using SPARED. modal truncation using ARPACK, and Z. Bai's PVL implementation.



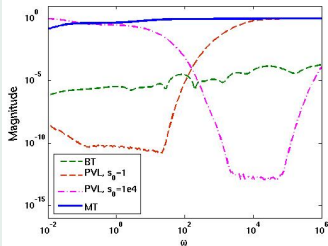
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## Relative error $n = 4,257$

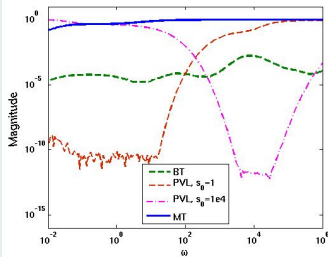


- axial-symmetric 2D model
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Relative error  $n = 4, 257$

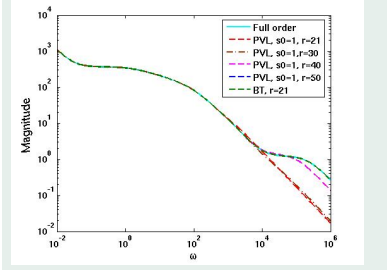


Relative error  $n = 11, 445$



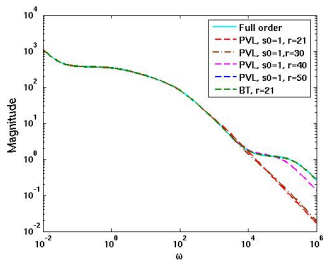
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## Frequency Response BT/PVL

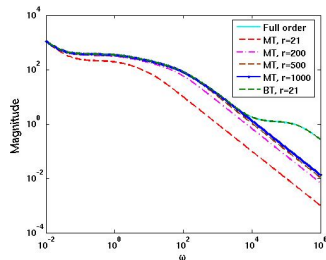


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## Frequency Response BT/PVL



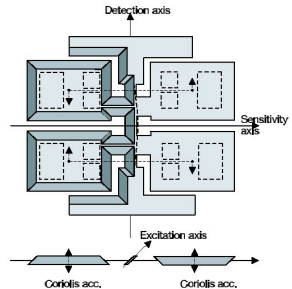
## Frequency Response BT/MT





- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.



Source: The Oberwolfach Benchmark Collection <http://www.intek.de/simulation/benchmark>

Courtesy of D. Billger (Imego Institute, Göteborg), Saab Bofors Dynamics AB.



# Examples

## MEMS: Butterfly Gyro

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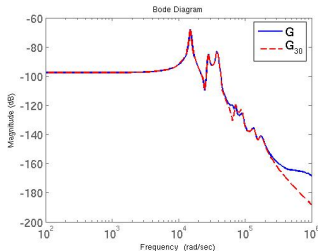
Future Work

References

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)  
 $\rightsquigarrow n = 34,722, m = 1, p = 12.$
- Reduced model computed using SPARED,  $r = 30.$

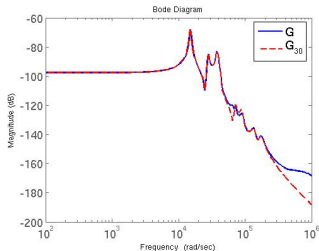
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## Frequency Repsonse Analysis

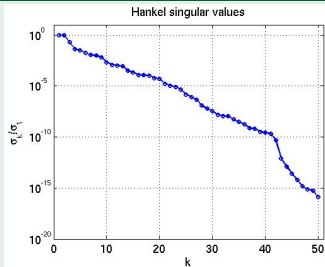


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### Frequency Repsonse Analysis



### Hankel Singular Values







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## Micro-electronics: Interconnect

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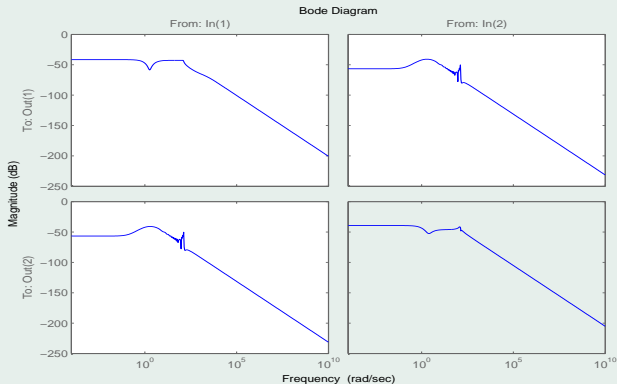
Future Work

References

- RLC circuit, characteristic curve has falling edge at  $\omega = 100$  Hz.
- $n = 1999$ ,  $m = p = 2$ , reduced model using PLICMR:  $r = 20$ .

- RLC circuit, characteristic curve has falling edge at  $\omega = 100$  Hz.
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## Accuracy of Reduced-Order Model





# Current and Future Work

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## Parametric Models

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x + D(p)u,$$

where  $p \in \mathbb{R}^s$  are free parameters which should be preserved in the reduced-order model.

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- Frequently:  $B, C, D$  parameter independent,

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## Nonlinear Systems

- Linear projection

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- Need specific methods
  - POD + balanced truncation  $\rightsquigarrow$  empirical Gramians (LALL/MARSDEN/GLAVASKI 1999/2002),
  - Approximate inertial manifold method ( $\sim$  static condensation for nonlinear systems).

## Nonlinear Systems

- Linear projection

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- Exploit structure of nonlinearities, e.g., in optimal control of linear PDEs with nonlinear BCs  $\rightsquigarrow$ 
  - bilinear control systems  $\dot{x} = Ax + \sum_j N_j x u_j + Bu$ ,
  - formal linear systems (cf. FÖLLINGER 1982)

$$\dot{x} = Ax + N g(Hx) + Bu = Ax + \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} u \\ g(z) \end{bmatrix},$$

where  $z := Hx \in \mathbb{R}^\ell$ ,  $\ell \ll n$ .



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Thanks for your attention!