Balanced Truncation for Unstable Systems and Algebraic Bernoulli Equations

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Overview

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x}(t) &= Ax + Bu, \; x(0) = x_0, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y(t) &= Cx + Du, \qquad\qquad\qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

Linear Systems in Frequency Domain

Application of Laplace transformation $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s))$ to linear system with $x(0) = 0$:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Bx(s) + Du(s),$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)} \Big) u(s)$$

G is the transfer function of Σ.

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Problem

Approximate the dynamical system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y = Cx + Du, & C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}, \end{cases}$$

by reduced-order system

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r\times r}, \quad \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} = \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{p\times r}, \quad \hat{D} \in \mathbb{R}^{p\times m}, \end{cases}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \|u\| < \text{tolerance} \cdot \|u\|.$$

Idea:

- A system Σ, realized by (A, B, C, D), is called balanced, if solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned}\mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right)\end{aligned}$$

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Motivation:

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

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In balanced coordinates ...energy transfer from u_- to y_+:

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int\limits_0^\infty y(t)^T y(t)\, dt}{\int\limits_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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$\Longrightarrow$ Truncate states corresponding to “small” HSVs

$\Longrightarrow$ complete analogy to best approximation via SVD!

Implementation: SR Method

1. Compute Cholesky factors of the solutions of the Lyapunov equations,
$$P = S^T S, \quad Q = R^T R.$$
2. Compute SVD
$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$
3. Set
$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$
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Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

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Balanced Truncation for Unstable Systems

Basic Idea

For A stable, BT is based on balancing system Gramians, defined by

$$P = \int_0^\infty e^{At} BB^T e^{A^T t}\, dt, \quad Q = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt.$$

For unstable A, integrals diverge!

Frequency-domain definition of Gramians

$$\begin{aligned} P &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega - A)^{-1} BB^T (j\omega - A)^{-H}\, d\omega, \\ Q &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega - A)^{-H} C^T C (j\omega - A)^{-1}\, d\omega. \end{aligned}$$

Properties [Zhou/Salomon/Wu 1999]

- Well-defined if $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$; for stable A, definitions coincide.
- (A, B) controllable $\Leftrightarrow P > 0$; (A, C) observable $\Leftrightarrow Q > 0$.
- BT can be based on P, Q, BT error bound holds!

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Balanced Truncation for Unstable Systems

Basic Idea

Computation of Gramians for Unstable Systems

If (A, B) stabilizable, (A, C) detectable, and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then P, Q are solutions of the Lyapunov equations

$$\begin{aligned}(A - BB^T X)P + P(A - BB^T X)^T + BB^T &= 0,\\ (A - YC^T C)^T Q + Q(A - YC^T C) + C^T C &= 0,\end{aligned}$$

where X and Y are the stabilizing solutions of the dual algebraic Bernoulli equations

$$\begin{aligned}A^T X + XA - XBB^T X &= 0,\\ AY + YA^T - YC^T CY &= 0.\end{aligned}$$

Algebraic Bernoulli Equations (ABE)

$$A^T X + XA - XGX = 0, \qquad A \in \mathbb{R}^{n\times n},\ G = G^T \in \mathbb{R}^{n\times n},$$

as special case of

$$\mathcal{L}(X) + A_0 X \left(\prod_{j=1}^{k-1} A_j X \right) A_k = 0,$$

where $\mathcal{L}(X)$ is a linear operator and $A_j \in \mathbb{R}^{n\times n}$ for $j = 0, 1, \ldots, k$.

Why Bernoulli?

Bernoulli differential equation

$$\dot{y}(t) = p(t)y(t) + q(t)y(t)^k, \quad k \neq 0, 1.$$

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ABEs are ...

...almost *Lyapunov equations:*

if X were invertible: $X^{-1}(\text{ABE})X^{-1} \Longrightarrow$

$$YA^T + AY = G, \qquad \text{where} \quad Y = X^{-1};$$

...special *algebraic Riccati equations (ARE):*

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i.e., ABE is homogeneous ARE.

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Trivia about ABEs

$A^T X + XA - XGX = 0$

- ABE is homogeneous $\Longrightarrow X = 0$ is a (positive semidefinite) solution;
- as special ARE, solution can be obtained from invariant subspaces of corresponding Hamiltonian matrix: if

$$\underbrace{\begin{bmatrix} A & G \\ 0 & -A^T \end{bmatrix}}_{=:H} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} W, \quad U, V, W \in \mathbb{R}^{n \times n},$$

and U is invertible, then

$$X = -VU^{-1}$$

is a solution of the ABE, where $\Lambda(A - GX) = \Lambda(W) \subset \Lambda(H)$.

If $\Lambda(W) = \Lambda(H) \cap \mathbb{C}^-$, then X is a stabilizing solution.

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Theorem

Consider the ABE

$$A^T X + XA - XBB^T X = 0 \qquad (1)$$

with (A, B) controllable. Then

- there exist symmetric solutions $X_+ \geq 0$, $X_- \leq 0$, with $X_- \leq X \leq X_+$ for all solutions X of the ABE;
- X_- is the unique solution satisfying $\Lambda(A - BB^T X_-) \subset \mathbb{C}^+ \cup \imath\mathbb{R}$;
- X_+ is the unique solution satisfying $\Lambda(A - BB^T X_+) \subset \mathbb{C}^- \cup \imath\mathbb{R}$.
- If $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then X_- is the unique anti-stabilizing solution and X_+ is the unique stabilizing solution of the ABE.

Corollary

If (A, B) is stabilizable and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then the ABE (1) has a unique stabilizing solution X_+ and $X_+ \geq 0$.

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Theorem

Consider again the ABE (1), i.e.,

$$A^T X + XA - XBB^T X = 0$$

with (A, B) stabilizable, $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, and its unique stabilizing solution X_+.
Then

$$\operatorname{rank}(X_+) = k,$$

where k is the number of eigenvalues of A in $\mathbb{C}^+$.

Corollary

If (A, B) is stabilizable and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then the ABE (1) has a unique stabilizing solution

$$X_+ = Y_+ Y_+^T, \quad \text{where} \quad Y_+ \in \mathbb{R}^{n \times k}.$$

Theorem

Consider again the ABE (1), i.e.,

$$A^T X + XA - XBB^T X = 0$$

with (A, B) stabilizable, $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, and its unique stabilizing solution X_+.
Then

$$\operatorname{rank}(X_+) = k,$$

where k is the number of eigenvalues of A in $\mathbb{C}^+$.

Corollary

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$$X_+ = Y_+ Y_+^T, \quad \text{where} \quad Y_+ \in \mathbb{R}^{n \times k}.$$

Recall:

- X can be obtained from invariant subspaces of
$$H = \begin{bmatrix} A & G \\ 0 & -A^T \end{bmatrix}.$$
- ABE is homogeneous ARE $\Longrightarrow$ ABE can be solved by any method for AREs, e.g., invariant subspace methods.

In particular, if $\mathcal{P}$ is a projector onto anti-stable, H-invariant subspace, then $\begin{bmatrix} I_n \\ -X_+ \end{bmatrix} \in \ker \mathcal{P}$

$\Longrightarrow$ solve consistent least-squares problem

$$0 = \mathcal{P} \begin{bmatrix} I_n \\ -X_+ \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I_n \\ -X_+ \end{bmatrix} \Leftrightarrow \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} X_+ = \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix}$$

$\rightsquigarrow$ compute projector from $\mathcal{P} = \frac{1}{2}(I_n + \operatorname{sign}(H))$.

Numerical Solution of ABEs

$A^TX + XA - XGX = 0$

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Definition

For $Z \in \mathbb{R}^{n \times n}$ with $\Lambda(Z) \cap \imath\mathbb{R} = \emptyset$ and Jordan canonical form

$$Z = S^{-1} \begin{bmatrix} J^+ & 0 \\ 0 & J^- \end{bmatrix} S$$

the matrix sign function is

$$\operatorname{sign}(Z) := S \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} S^{-1}.$$

Computation of $\operatorname{sign}(Z)$

$\operatorname{sign}(Z)$ is root of $I_n \Longrightarrow$ use Newton's method to compute it:

$$Z_0 \leftarrow Z, \qquad Z_{j+1} \leftarrow \frac{1}{2}\left(c_j Z_j + \frac{1}{c_j} Z_j^{-1}\right), \qquad j = 1, 2, \ldots$$

$\Longrightarrow \quad \operatorname{sign}(Z) = \lim_{j\to\infty} Z_j.$

($c_j > 0$ is scaling parameter for convergence acceleration and rounding error minimization.)

Solving ABEs with the Matrix Sign Function Method

1. Apply sign function iteration $Z \leftarrow \frac{1}{2}(Z + Z^{-1})$ to H:

$$H + H^{-1} = \begin{bmatrix} A & G \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} A^{-1} & A^{-1}GA^{-T} \\ 0 & -A^{-T} \end{bmatrix}$$

$\Longrightarrow$ Sign function iteration for ABE:

$$\begin{aligned} A_0 &\leftarrow A, & A_{j+1} &\leftarrow \tfrac{1}{2}\left(A_j + A_j^{-1}\right), \\ G_0 &\leftarrow G, & G_{j+1} &\leftarrow \tfrac{1}{2}\left(G_j + A_j^{-1} G_j A_j^{-T}\right), \end{aligned} \qquad j = 0, 1, 2, \ldots.$$

Define $A_\infty := \lim_{j\to\infty} A_j$, $G_\infty := \lim_{j\to\infty} G_j$.

2. Solve linear least-squares problems

$$\begin{bmatrix} G_\infty \\ I_n - A_\infty^T \end{bmatrix} X_+ = \begin{bmatrix} A_\infty + I_n \\ 0_n \end{bmatrix}.$$

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Computing Solution Factors

Consider $A^T X + XA - XBB^T X = 0$ and recall that

$$X_+ = Y_+ Y_+^T, \quad Y_+ \in \mathbb{R}^{n \times k}.$$

Sign function iteration can be re-written as for Lyapunov equations:

$$\begin{aligned} B_{j+1} B_{j+1}^T = G_{j+1} &= \frac{1}{2}\left(G_j + A_j^{-1} G_j A_j^{-T}\right) \\ &= \frac{1}{2}\left(B_j B_j^T + A_j^{-1} B_j B_j^T A_j^{-T}\right) \\ &= \frac{1}{2}\left[\begin{array}{cc} B_j & A_j^{-1} B_j \end{array}\right]\left[\begin{array}{cc} B_j & A_j^{-1} B_j \end{array}\right]^T \end{aligned}$$

and use RRQR of $\left[\begin{array}{cc} B_j & A_j^{-1} B_j \end{array}\right]$ for column compression.

But: still need $G_\infty = \lim_{j \to \infty} B_j B_j^T$ for least-squares solution and Y_+ not directly obtained from least-squares problem!

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Computing Solution Factors

Theorem [B. 2006]

Let (A, B) be stabilizable, $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, and X_+ be the unique stabilizing solution of the ABE

$$A^T X + XA - XBB^T X = 0.$$

If $B_\infty = \lim_{k\to\infty} B_k$ is obtained from the factorized form of the sign function iteration for the ABE, then a full-rank factor $Y_+ \in \mathbb{R}^{n\times k}$ of X_+ is given by

$$Y_+ = \sqrt{2} Q_Y R^{-1}, \tag{2}$$

where the columns of Q_Y form a basis of $\ker(I - \operatorname{sign}(A)^T)$ and R is the upper triangular factor in the “skinny” QR factorization of $B_\infty^T Q_Y$.

Other Applications of ABEs

- Coprime factorization problems for rational transfer functions,
- computing (sub-)optimal H_∞ controllers using a mixed sensitivity approach ($\mathcal{S}$-over-$\mathcal{KS}$ design),
- state feedback stabilization of LTI systems.

Feedback stabilization problem

For $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, find $F \in \mathbb{R}^{m\times n}$ such that $\Lambda(A - BF) \subset \mathbb{C}^-$.

Corresponds to finding $u \in L_2(0,\infty;\mathbb{R}^m)$ such that the solution trajectory of

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n,$$

is asymptotically stable, i.e., $\lim_{t\to\infty} x(t;u) = 0$.

If F solves the feedback stabilization problem, then $u(t) = -Fx(t)$.

Solution by ABE

If X is a stabilizing solution of the ABE with $G = BB^T$, then

$$F := B^T X$$

is a stabilizing feedback matrix.

Other Applications of ABEs
Stabilization

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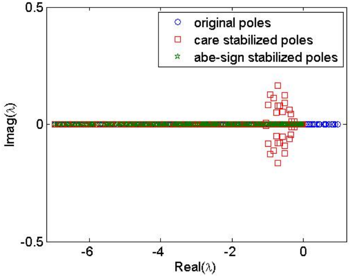
Stabilization
Example

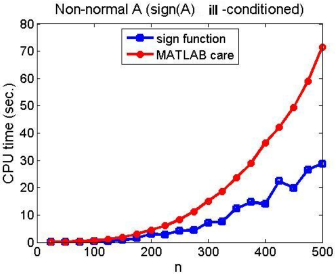
Here, mimic finite-differences discretization of control problem for linear reaction-diffusion problem on unit square: $A = -\Delta_h + I,\ B = I_{n,m}$, where $h = 1/21,\ m = 20$.

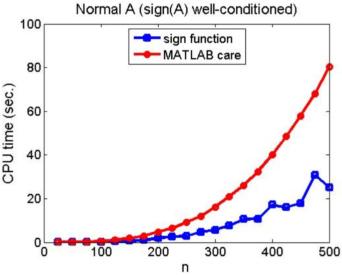


Results for different solvers:

	$\frac{\|B(X_+)\|_1}{\|X_+\|_1}$	CPU time
`care`:	$3.5 \cdot 10^{-5}$	13.8 sec
`sign`:	$4.8 \cdot 10^{-5}$	8.0 sec
`sign_fac`:	$5.2 \cdot 10^{-10}$	7.4 sec

Random ABE with $n = 25 : 25 : 500$, $m = n/5$.



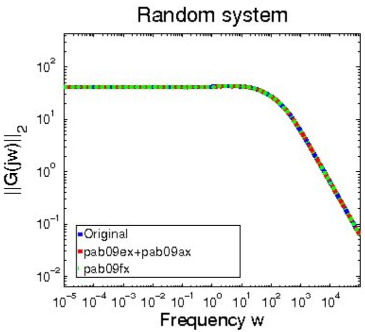


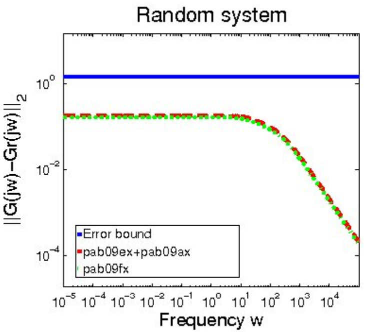
Numerical Examples

Numerical Accuracy: $\frac{\|A^T X_+ + X_+ A - X_+ BB^T X_+\|_1}{\|X_+\|_1}$.

Example	n	care	#Iter.	sign	sign_fac
CAREX 1.1	2	8.08e−22	2	8.08e−22	8.08e−22
CAREX 1.2	2	1.18e−15	3	1.26e−14	1.14e−14
CAREX 2.1	2	0.00e+00	3	0.00e+00	9.77e−16
CAREX 2.3	2	6.68e−46	4	1.00e+06	0.00e+00
CAREX 2.4	2	0.00e+00	3	1.61e−09	8.24e−12
CAREX 2.5	2	1.78e−15	3	1.78e−15	2.49e−15
CAREX 2.6	3	1.98e−09	5	2.82e−09	3.22e−09
CAREX 2.7	4	1.85e−10	5	7.92e−12	5.30e−10
CAREX 2.8	4	3.40e−12	3	5.55e−11	5.55e−11
CAREX 3.1	39	7.61e−16	4	9.71e−11	6.14e−16
CAREX 3.2	64	6.75e−15	24	1.73e−09	1.72e−14
CAREX 4.1a	21	1.97e−20	6	1.00e+00	1.00e+00
CAREX 4.1b	21	3.30e+00	6	8.66e−01	6.54e−08
CAREX 4.3	60	6.78e−15	21	2.85e−12	8.16e−15
RLC	199	2.72e−16	30	6.08e−11	1.41e−15

Random unstable system with $n = 500$, $m = p = 50$, 25 unstable poles, and $\ell = 50$.





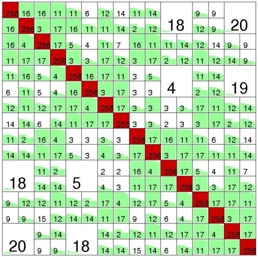
Solving Large-Scale ABEs

Reduced complexity/storage by use of hierarchical matrices

$\mathcal{H}$-matrices provide data-sparse representation for certain densely populated matrices (FEM^{-1}, BEM, ...) [*Hackbusch 98*]

- approximation by matrices of rank $k(\epsilon)$
- storage for $A_{\mathcal{H}} \in \mathbb{R}^{n\times n}$: $\mathcal{O}(n\,\log_2(n)\,k(\epsilon))$
- formatted arithmetic with complexity:

$$
\begin{aligned}
A_{\mathcal{H}}\,x &: \quad \mathcal{O}(n\,\log_2(n)\,k(\epsilon)) \\
\oplus &: \quad \mathcal{O}(n\,\log_2(n)\,k(\epsilon)^2) \\
\odot,\ \mathrm{Inv}_{\mathcal{H}} &: \quad \mathcal{O}(n\,\log_2^2(n)\,k(\epsilon)^2)
\end{aligned}
$$



$n = 4096 \;\; \epsilon = 10^{-4}$

$\Longrightarrow$ $\mathcal{H}$-sign function iteration

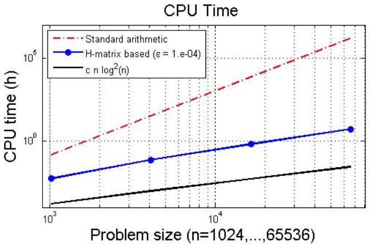
$$
A_{\mathcal{H},k+1} = \frac{1}{2}\left(A_{\mathcal{H},k} \oplus \mathrm{Inv}_{\mathcal{H}}(A_{\mathcal{H},k})\right) : \quad \mathcal{O}(n\,\log_2^2(n)\,k(\epsilon)^2)
$$

$$
\begin{aligned}
\frac{\partial \mathbf{x}}{\partial t}(t,\xi) &= \Delta \mathbf{x}(t,\xi) + b(\xi)u(t), \qquad \xi \in \Omega = [0,1]^2, \\
b(\xi) &= \chi_{\Omega_u}(\xi).
\end{aligned}
$$

- Finite element discretization with n inner grid points:

$$
\frac{d}{dt}x(t) = A\,x(t) + B\,u(t)
$$

- use HLib 1.3 [Börm/Grasedyck/Hackbusch]
- shifted $A_{\mathcal{H}} \rightsquigarrow$ one eigenvalue in $\mathbb{C}^+$



CPU time of sign function solver

Thanks to Ulrike Baur.

Accuracy results for different parameter choices and $n = 16,384$:

$$\textit{relative residual} = \frac{\|A^T X + XA - XBB^T X\|_2}{2(\|A\|_2 \|X\|_2) + \|X\|_2^2 \|BB^T\|_2}$$

ϵ is the blockwise accuracy in $\mathcal{H}$-matrix approximation.

τ determines the numerical rank r of B_{k+1}: $\sigma_{r+1} < \sigma_1 \cdot \tau \leq \sigma_r$

ϵ	τ	# it.	r	time[sec]	rel. residual
10^{-4}	10^{-4}	27	34	2376	2×10^{-5}
10^{-6}	10^{-4}	26	15	4235	5×10^{-7}
10^{-8}	10^{-4}	22	14	7136	6×10^{-9}
10^{-6}	10^{-6}	26	42	4273	5×10^{-7}
10^{-8}	10^{-6}	22	23	7150	6×10^{-9}
10^{-8}	10^{-8}	22	42	7183	6×10^{-9}

Thanks to Ulrike Baur.

Summary and Outlook

- Balanced truncation for unstable systems can be implemented based on efficient ABE solver.
- Sign function-based ABE solver
  - is sometimes more accurate than Schur decomposition-based approach,
  - is often faster than Schur decomposition-based approach,
  - can compute solution factor without forming X_+.
  - can be iomplemented for large-scale problems using $\mathcal{H}$-matrix arithmetic, using normal equations for solving the least-squares problem.
- **Work in progress:**
  - Poor man's balanced truncation implementation using suitable quadrature rules.
  - ADI-like or doubling-type iteration for large and sparse ABEs.
  - Hammarling-style algorithm for computing Y_+?

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References

1. Sergio Barrachina, Peter Benner, Enrique S. Quintana-Ortí, and Gregorio Quintana-Ortí: *Parallel Algorithms for Balanced Truncation of Large-Scale Unstable Systems*, in Proceedings of 44th IEEE Conference on Decision and European Control Conference ECC 2005, pp. 2248-2253, 2005.
2. Sergio Barrachina, Peter Benner, and Enrique S. Quintana-Ortí: *Parallel Solution of Large-Scale Algebraic Bernoulli Equations with the Matrix Sign Function Method*, International Journal of Computational Science and Engineering, to appear.
3. Peter Benner, Enrique S. Quintana-Ortí, and Gregorio Quintana-Ortí: *Solving Large-Scale Generalized Algebraic Bernoulli Equations via the Matrix Sign Function*, Preprint, TU Chemnitz, March 2006.

Thanks for your attention!