Model Reduction in Control and Simulation: Algorithms and Applications

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Overview

Joint work with

- Enrique Quintana-Ortí, Gregorio Quintana-Ortí, Rafa Mayo, José Manuel Badía, Alfredo Remón, Sergio Barrachina (Universidad Jaume I de Castellón, Spain).
- Ulrike Baur, Lihong Feng, Matthias Pester, Jens Saak ( ).
- Viatcheslav Sokolov (former).
- Heike Faßbender (TU Braunschweig).
- Infineon Technologies/Qimonda, IMTEK (U Freiburg), iwb (TU München), ...

Introduction

Problem

Given a physical problem with dynamics described by the states $x \in \mathbb{R}^n$, where n is the dimension of the state space.

Because of redundancies, complexity, etc., we want to describe the dynamics of the system using a reduced number of states.

This is the task of model reduction (also: dimension reduction, order reduction).

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Motivation: Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory: $4 \cdot n_x \cdot n_y$ bytes.

Theorem: (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum\nolimits_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U\Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\|X - \widehat{X}\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r$, $\sigma_1 v_1, \ldots, \sigma_r v_r$.
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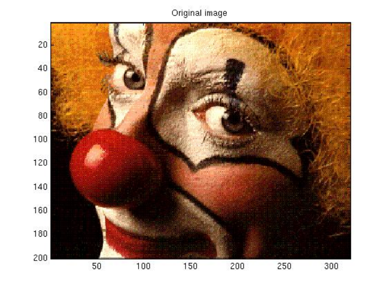
Example: Image Compression by Truncated SVD

Example: Clown



320×200 pixel
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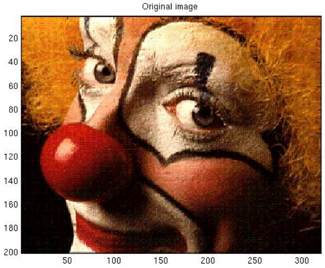

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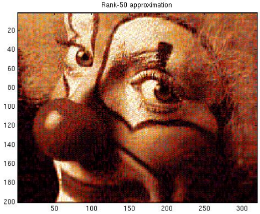
- rank $r = 50$, ≈ 104 kb



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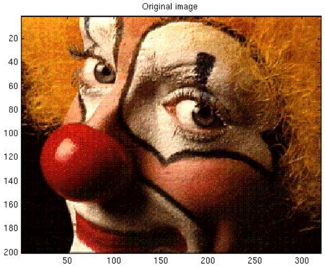

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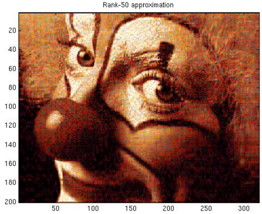
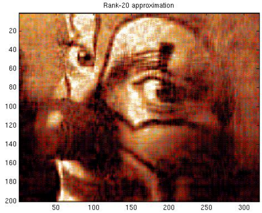


- rank $r = 20$, ≈ 42 kb



Dimension Reduction via SVD

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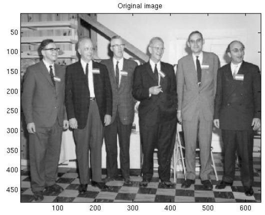

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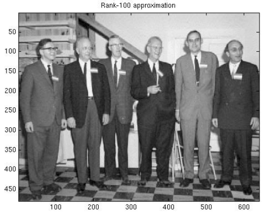

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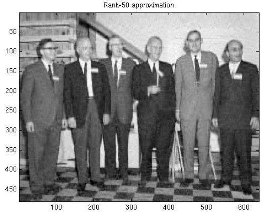

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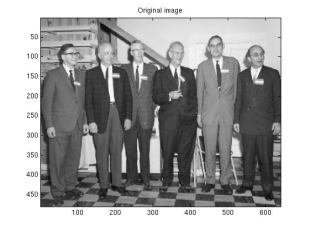

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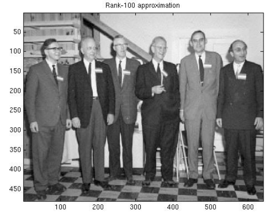

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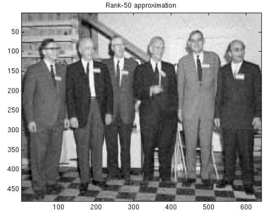

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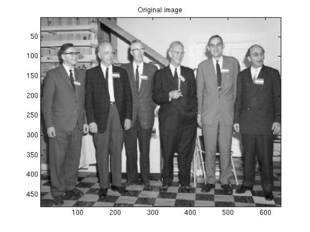

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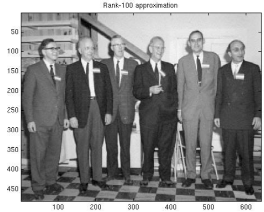

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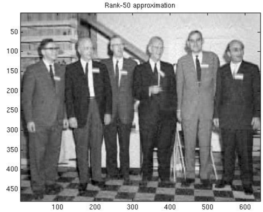
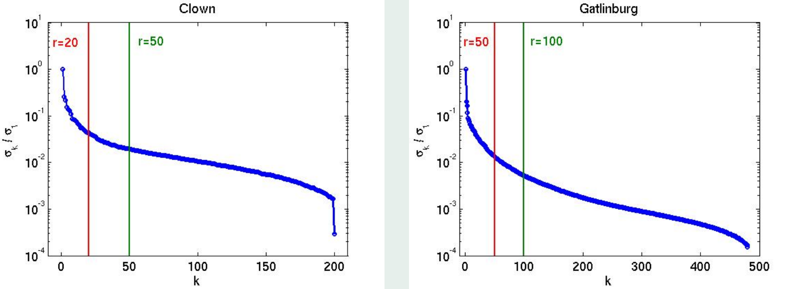


Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices



The Model Reduction Problem

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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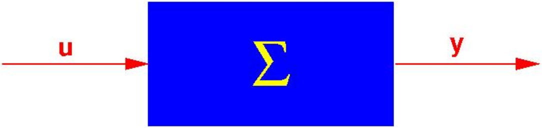

Model Reduction for Dynamical Systems

Original System

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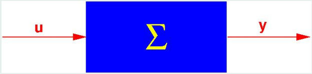

Reduced-Order System

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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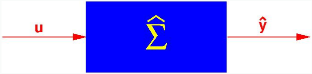

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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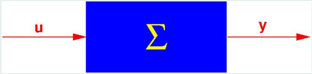

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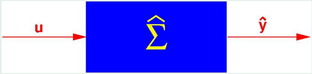

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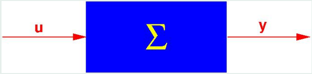

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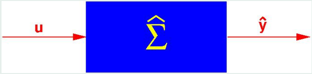

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Linear, Time-Invariant (LTI) Systems

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Model Reduction for Linear Systems

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State-Space Description for I/O-Relation ($D = 0$)

$$\mathcal{S} : u \mapsto y, \quad y(t) = (h \star u)(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\, d\tau,$$

where $h(s) = \begin{cases} Ce^{A(s)}B & \text{if } s > 0, \\ 0 & \text{if } s \le 0. \end{cases}$

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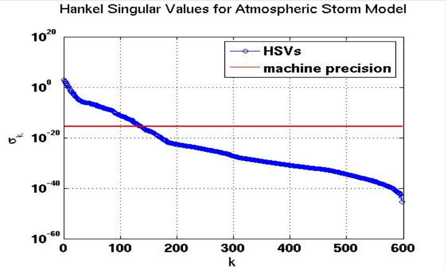

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$\Rightarrow$ solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

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But: computationally unfeasible for large-scale systems.

Linear Systems in Frequency Domain

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Laplace Transformation / Frequency Domain

Application of Laplace transformation ($x(t) \mapsto x(s)$, $\dot{x}(t) \mapsto sx(s)$) to linear system with $x(0) = 0$:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Bx(s) + Du(s),$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)} \Big) u(s)$$

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Model Reduction for Linear Systems

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Approximate the dynamical system

$$\begin{array}{rcl} \dot{x} &=& Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &=& Cx + Du, \qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}, \end{array}$$

by reduced-order system

$$\begin{array}{rcl} \dot{\hat{x}} &=& \hat{A}\hat{x} + \hat{B}u, \qquad \hat{A} \in \mathbb{R}^{r\times r}, \quad \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &=& \hat{C}\hat{x} + \hat{D}u, \qquad \hat{C} \in \mathbb{R}^{p\times r}, \quad \hat{D} \in \mathbb{R}^{p\times m}, \end{array}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\|\|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G})\leq r} \|G - \hat{G}\|$.

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Application Areas
(Optimal) Control

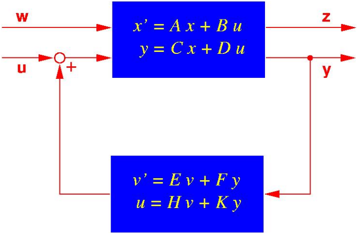
Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$

$\Rightarrow$ reduce order of original system.



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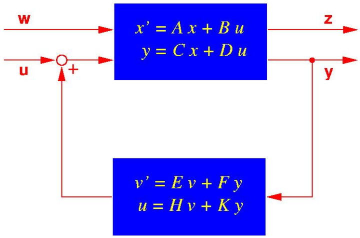

- Progressive miniaturization: **Moore's Law (1975)** states that the number of on-chip transistors doubles each 24 (now: ∼18) months.

Source: http://de.wikipedia.org/wiki/Mooresches_Gesetz

Application Areas
Micro Electronics

- Progressive miniaturization: **Moore's Law (1975)** states that the number of on-chip transistors doubles each 24 (now: $\sim$18) months.
- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- Increase in packing density requires modeling of interconncet to ensure that thermic/electro-magnetic effects do not disturb signal transmission.
- Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

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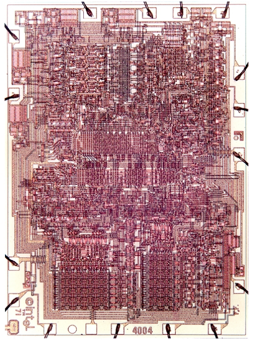
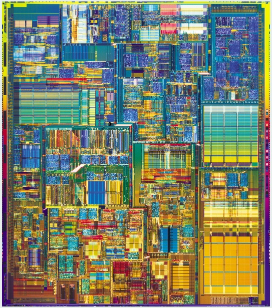
Intel 4004 (1971)



- 1 layer, 10μ technology,
- 2,300 transistors,
- 64 kHz clock speed.

Intel Pentium IV (2001)



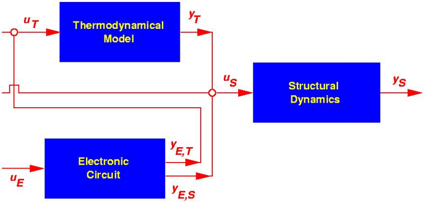
- 7 layers, 0.18μ technology,
- 42,000,000 transistors,
- 2 GHz clock speed,
- 2km of interconnect.

Latest Intel roll-out (last Monday, quad core CPUs):
45nm technology, $> 820,000,000$ transistors

Typical problem in MEMS simulation:
coupling of different models (thermic, structural, electric, electro-magnetic) during simulation.

Problems and Challenges:

- Reduce simulation times by replacing sub-systems with their reduced-order models.
- Stability properties of coupled system may deteriorate through model reduction even when stable sub-systems are replaced by stable reduced-order models.
- Multi-scale phenomena.



- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity ("system does not generate energy").

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Model Reduction

1. Modal Truncation
2. Padé-Approximation and Krylov Subspace Methods
3. Balanced Truncation
4. many more. . .

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2. Padé-Approximation and Krylov Subspace Methods
3. Balanced Truncation
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Joint feature of many methods: Galerkin or Petrov-Galerkin-type projection of state-space onto low-dimensional subspace $\mathcal{V}$ along $\mathcal{W}$: assume $x \approx VW^T x =: \tilde{x}$, where

$$\mathrm{range}\,(V) = \mathcal{V}, \quad \mathrm{range}\,(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ and

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|.$$

Modal Truncation

Idea:

Project state-space onto A-invariant subspace $\mathcal{V}$, where

$$\mathcal{V} = \operatorname{span}(v_1, \ldots, v_r),$$

$v_k =$ eigenvectors corresp. to "dominant" modes $\equiv$ eigenvalues of A.

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Properties:

- Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.
- Error bound:

$$\|G - \hat{G}\|_\infty \leq \operatorname{cond}_2(T)\, \|C_2\|_2 \|B_2\|_2 \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\operatorname{Re}(\lambda)|},$$

where $T^{-1}AT = \operatorname{diag}(A_1, A_2)$.

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Difficulties:

- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
  (Litz 1979: use Jordan canoncial form; otherwise merely heuristic criteria.
  Rommes 2007: dominant pole algorithm (two-sided RQI).)
- Error bound not computable for really large-scale probems.
- New direction: AMLS (automated multilevel substructuring) [Benninghof/Lehoucq '04, Elssel/Voss '05, Blömeling '06].

Padé Approximation

Idea:

- Consider
$$E\dot{x} = Ax + Bu, \quad y = Cx$$
with rational transfer function $G(s) = C(sE - A)^{-1}B$.
- For $s_0 \notin \Lambda(A, E)$:
$$G(s) \quad = \quad m_0 + m_1(s - s_0) + m_2(s - s_0)^2 + \ldots$$
- As reduced-order model use rth Padé approximant $\hat{G}$ to G:
$$G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^{2r}),$$
i.e., $m_j = \widehat{m}_j$ for $j = 0, \ldots, 2r - 1$

$\rightsquigarrow$ moment matching if $s_0 < \infty$,

$\rightsquigarrow$ partial realization if $s_0 = \infty$.

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Padé-via-Lanczos Method (PVL)

- Moments need not be computed explicitly; moment matching is equivalent to projecting state-space onto

$$\mathcal{V} = \mathrm{span}(\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{r-1}B) = \mathcal{K}(\tilde{A}, \tilde{B}, r)$$

(where $\tilde{A} = (s_0E - A)^{-1}E,\ \tilde{B} = (s_0E - A)^{-1}B$) along

$$\mathcal{W} = \mathrm{span}(C^H, \tilde{A}^H C^H, \ldots, (\tilde{A}^H)^{r-1}C^H) = \mathcal{K}(\tilde{A}^H, C^H, r).$$

- Computation via unsymmetric Lanczos method, yields system matrices of reduced-order model as by-product.
- PVL applies w/o changes for singular E if $s_0 \notin \Lambda(A, E)$:
  - for $s_0 \neq \infty$: Gallivan/Grimme/Van Dooren 1994, Freund/Feldmann 1996, Grimme 1997
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Partial realization for descriptor systems: [B./Sokolov, SCL, 2006]

For nonsingular E

$$\text{moments} = \text{Markov parameters} = C(E^{-1}A)^j E^{-1}B, \quad j = 0, 1, \ldots$$

Question: for E singular, what is the correct generalized inverse here?

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Answer: {2}-inverse $E^{\{2\}} = Q \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} P$, where

$$sPEQ - PAQ = s \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

is the Weierstraß canonical form (WCF) of $sE - A$.

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- P, Q can be computed w/o WCF in many applications.
- Numerically, use Lanczos applied to $\{E^{\{2\}}A, E^{\{2\}}B, C\}$.

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Padé-via-Lanczos Method (PVL)

Difficulties:

- No computable error estimates/bounds for $\|y - \hat{y}\|_2$.
- Mostly heuristic criteria for choice of expansion points. Optimal choice for second-order systems with proportional/Rayleigh damping (Beattie/Gugercin 2005).
- Good approximation quality only locally.
- Preservation of physical properties only in very special cases; usually requires post processing which (partially) destroys moment matching properties.

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New direction: moment matching yields rational interpolation of $G^{(j)}(s)$ for $j = 0, \ldots, 2r - 1$ at $s = s_0$.
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Current work: where to put the s_j?

Balanced Truncation

Idea:

- A system Σ, realized by (A, B, C, D), is called balanced, if solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

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In balanced coordinates ... energy transfer from u_- to y_+:

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\, dt}{\int_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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$\Longrightarrow$ Truncate states corresponding to "small" HSVs

$\Longrightarrow$ complete analogy to best approximation via SVD!

Implementation: SR Method

1. Compute (Cholesky) factors of the solutions of the Lyapunov equations,
$$P = S^T S, \quad Q = R^T R.$$
2. Compute SVD
$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$
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$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$
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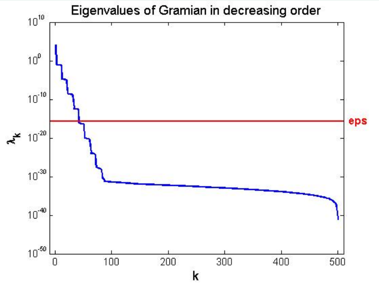
New algorithmic ideas from numerical linear algebra:

- Instead of Gramians P, Q compute $S, R \in \mathbb{R}^{n \times k}$, $k \ll n$, such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

- Compute S, R with problem-specific Lyapunov solvers of "low" complexity directly.



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New algorithmic ideas from numerical linear algebra:

Parallelization:

- Efficient parallel algorithms based on matrix sign function.
- Complexity $\mathcal{O}(n^3/q)$ on q-processor machine.
- Software library PLiCMR with WebComputing interface.

(B./Quintana-Ortí/Quintana-Ortí since 1999)

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For special problems from PDE control use implementation based on hierarchical matrices and matrix sign function method (Baur/B.), complexity $\mathcal{O}(n\log^2(n)r^2)$.

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New algorithmic ideas from numerical linear algebra:

Sparse Balanced Truncation:

- Sparse implementation using sparse Lyapunov solver (ADI+MUMPS/SuperLU).
- Complexity $\mathcal{O}(n(k^2 + r^2))$.
- Software:
  - + Matlab toolbox LyaPack (Penzl 1999),
  - + Software library SpaRed with WebComputing interface. (Badía/B./Quintana-Ortí/Quintana-Ortí since 2003)

Balanced Truncation for Unstable Systems

For A stable, Gramians are defined by

$$P = \int_0^\infty e^{At} BB^T e^{A^T t}\, dt, \quad Q = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt.$$

For unstable A, integrals diverge!

Frequency-domain definition of Gramians

$$\begin{aligned} P &:= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega - A)^{-1} BB^T (j\omega - A)^{-H}\, d\omega, \\ Q &:= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega - A)^{-H} C^T C (j\omega - A)^{-1}\, d\omega. \end{aligned}$$

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Balanced Truncation for Unstable Systems

Computation of Unstable Gramians

If (A, B) stabilizable, (A, C) detectable, and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then P, Q are solutions of the Lyapunov equations

$$\begin{aligned}(A - BB^T X)P + P(A - BB^T X)^T + BB^T &= 0,\\ (A - YC^T C)^T Q + Q(A - YC^T C) + C^T C &= 0,\end{aligned}$$

where X and Y are the stabilizing solutions of the dual algebraic Bernoulli equations

$$\begin{aligned}A^T X + XA - XBB^T X &= 0,\\ AY + YA^T - YC^T CY &= 0.\end{aligned}$$

Balanced Truncation for Unstable Systems

Theorem [B. 2006]

Let (A, B) be stabilizable, $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, and X_+ be the unique stabilizing solution of the ABE

$$A^T X + XA - XBB^T X = 0.$$

Then

a) $\mathrm{rank}\,(X_+) = k$, where k is the number of eigenvalues of A in $\mathbb{C}^+$.

b) A full-rank factor $Y_+ \in \mathbb{R}^{n \times k}$ of X_+ is given by

$$Y_+ = \sqrt{2} Q_Y R^{-1},$$

where colspan(Q_Y) is basis of anti-stable A-invariant subspace, R is defined via $\mathrm{sign}\left(\begin{bmatrix} A^T & BB^T \\ 0 & -A \end{bmatrix}\right)$.

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Efficient solution of ABEs: sign-function based computation of Y_+ [Barrachina/B./Quintana-Ortí].
Current work: solvers for large-scale ABEs with (data-)sparse A.

Examples

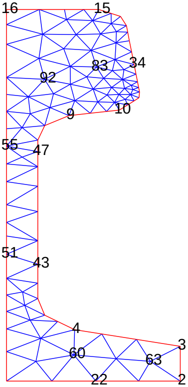
Optimal Control: Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, \qquad \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa (u_k - x), \quad \xi \in \Gamma_k, \ 1 \leq k \leq 7, \\
\frac{\partial}{\partial n} x &= 0, \qquad \xi \in \Gamma_7.
\end{aligned}
$$

$\Longrightarrow m = 7, p = 6.$

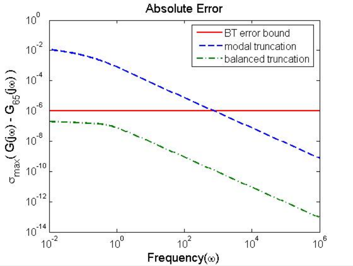
- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841$.



Source: Physical model: courtesy of Mannesmann/Demag.

Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

$n = 1357$, Absolute Error



- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

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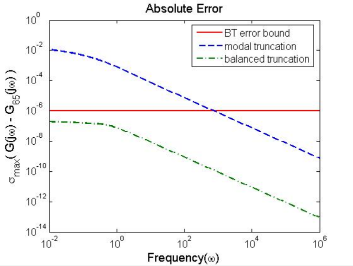

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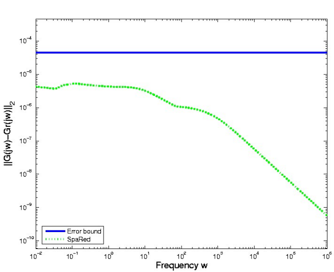
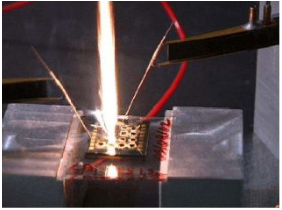
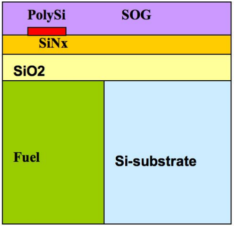
$n = 79841$, Absolute error



- BT model computed using SpaRed,
- computation time: 8 min.

- Co-integration of solid fuel with silicon micromachined system.
- Goal: Ignition of solid fuel cells by electric impulse.
- Application: nano satellites.
- Thermo-dynamical model, ignition via heating an electric resistance by applying voltage source.
- Design problem: reach ignition temperature of fuel cell w/o firing neighbouring cells.
- Spatial FEM discretizatoin of thermo-dynamical model $\rightsquigarrow$ linear system, $m = 1$, $p = 7$.





Source: The Oberwolfach Benchmark Collection `http://www.imtek.de/simulation/benchmark`

Courtesy of C. Rossi, LAAS-CNRS/EU project "Micropyros".

- axial-symmetric 2D model
- FEM discretisation using linear (quadratic) elements $\rightsquigarrow$ $n = 4,257$ $(11,445)$ $m = 1$, $p = 7$.
- Reduced model computed using SpaRed. modal truncation using ARPACK, and Z. Bai's PVL implementation.

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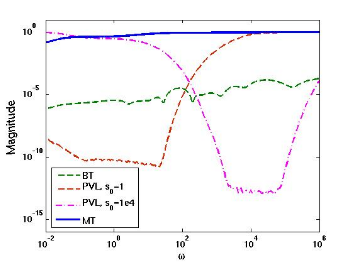

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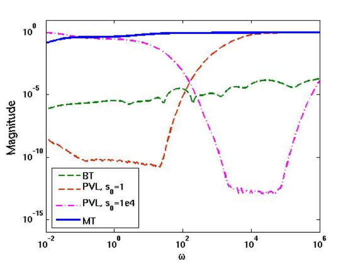

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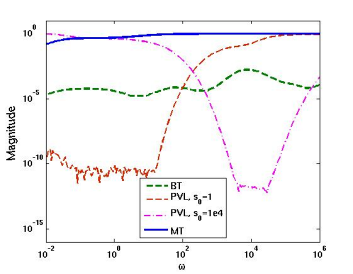

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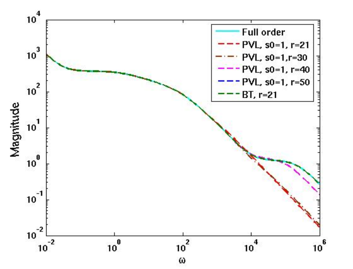

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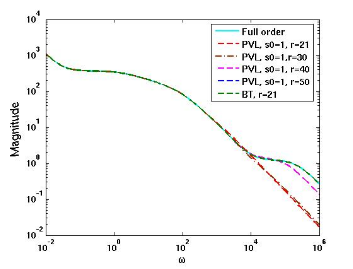
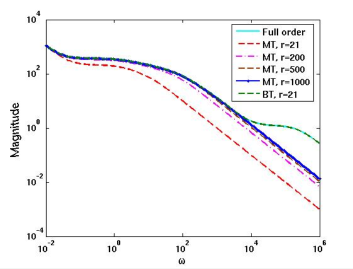


Frequency Response BT/MT

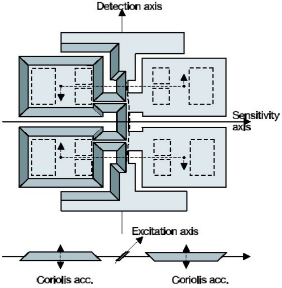


Examples

MEMS: Microgyroscope (Butterfly Gyro)

- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.



Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Courtesy of D. Billger (Imego Institute, Göteborg), Saab Bofors Dynamics AB.

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722$, $m = 1$, $p = 12$.
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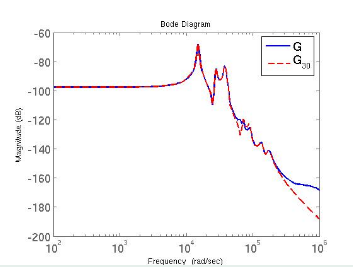

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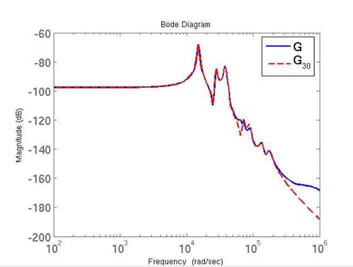
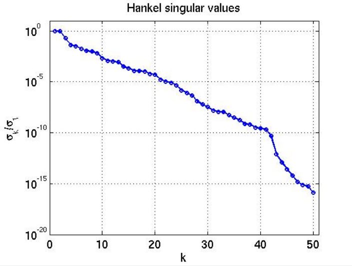


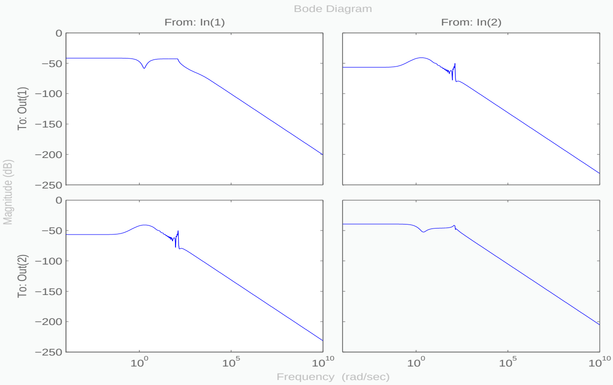
Hankel Singular Values



Interconnect

- RLC circuit, characteristic curve has falling edge at $\omega = 100\,\mathrm{Hz}$.
- $n = 1999$, $m = p = 2$, reduced model using PLiCMR: $r = 20$.

Accuracy of Reduced-Order Model



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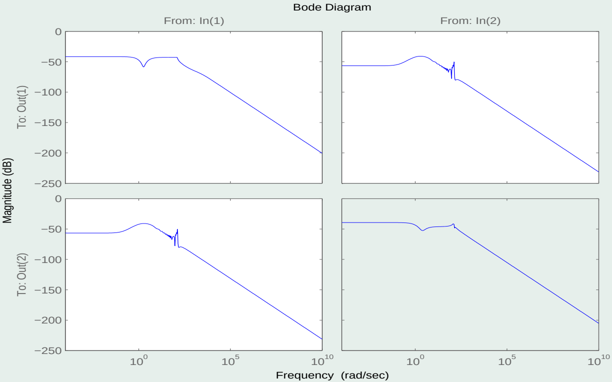
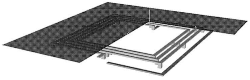


Examples

Micro Electronics: Spiral Inductor

- Passive device used for RF filters etc.
- $n = 500$, $m = 1$, $p = 1$.
- Numerical rank of Gramians is 34.
- $r = 11$ model computed by PLiCMR.



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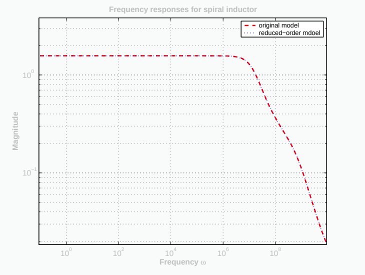
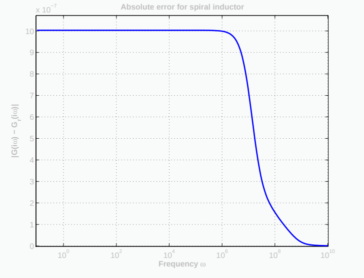


Absolute Error



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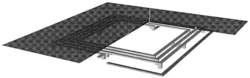

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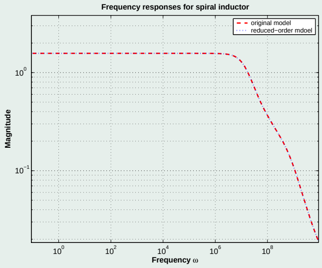

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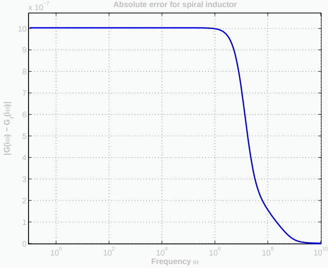

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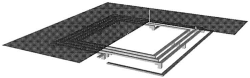
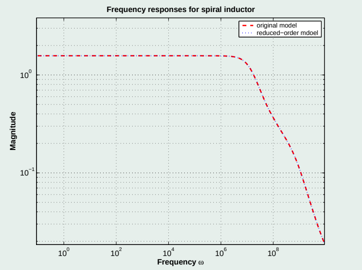
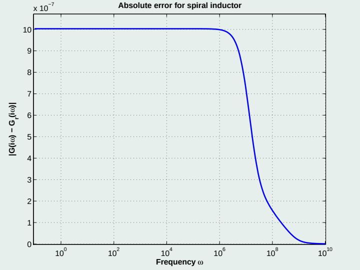


Frequency Repsonse Analysis



Absolute Error



Courtesy of MIT/Jing-Rebecca Li

BT is often criticized for its bias towards the input-output behavior of the system. But states can also be reconstructed using

$$x(t) \approx V x_r(t).$$

Example: 2D heat equation with localized heat source, 64×64 grid, $r = 6$ model by BT, simulation for $u(t) = 10 \cos(t)$.

Examples
Reconstruction of the State

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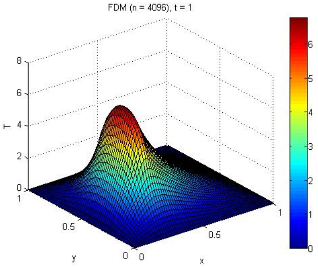

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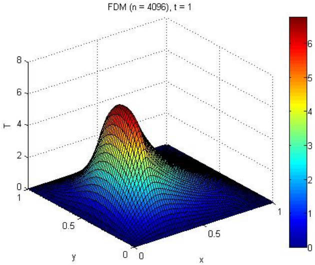
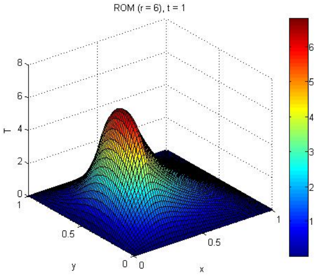
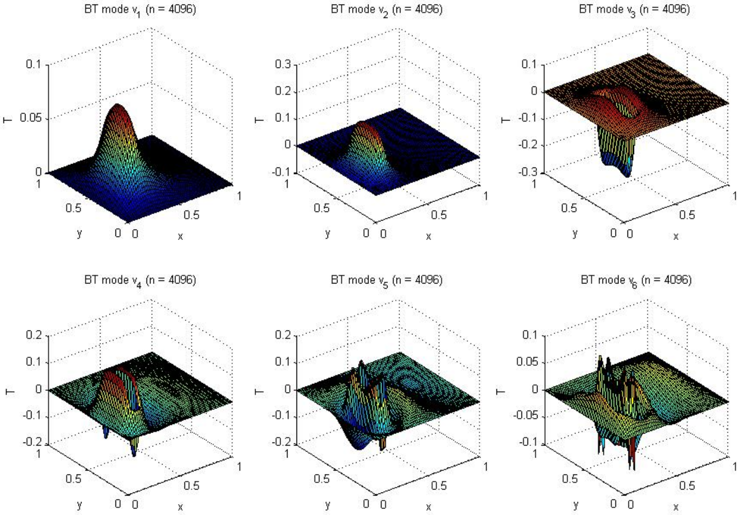




Examples

BT modes are intelligent ansatz functions for Galerkin projection



Parametric Models

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x + D(p)u,$$

where $p \in \mathbb{R}^s$ are free parameters which should be preserved in the reduced-order model.

Current and Future Work

Parametric Models

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where $p \in \mathbb{R}^s$ are free parameters which should be preserved in the reduced-order model.

- Frequently: B, C, D parameter independent,

$$A(p) = A_0 + p_1 A_1 + \ldots + p_s A_s.$$

  $\Rightarrow$ (Modified) linear model reduction methods applicable.
- Multipoint expansion combined with Padé-type approx. possible.
- New idea: BT for reference parameters combined with interpolation yields parametric reduced-order models.

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Nonlinear Systems

- Linear projection

$$x \approx V\hat{x}, \quad \dot{\hat{x}} = W^T f(V\hat{x}, u)$$

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Nonlinear Systems

- Linear projection

$$x \approx V\hat{x}, \quad \dot{\hat{x}} = W^T f(V\hat{x}, u)$$

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- Need specific methods
  - POD + balanced truncation $\rightsquigarrow$ empirical Gramians (Lall/Marsden/Glavaski 1999/2002),
  - Approximate inertial manifold method ($\sim$ static condensation for nonlinear systems).

Nonlinear Systems

- Linear projection

$$x \approx V\hat{x}, \quad \dot{\hat{x}} = W^T f(V\hat{x}, u)$$

  is in general not model reduction!

- Exploit structure of nonlinearities, e.g., in optimal control of linear PDEs with nonlinear BCs $\rightsquigarrow$
  - bilinear control systems $\dot{x} = Ax + \sum_j N_j x u_j + Bu$,
  - formal linear systems (cf. FÖLLINGER 1982)

$$\dot{x} = Ax + N\, g(Hx) + Bu = Ax + \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} u \\ g(z) \end{bmatrix},$$

    where $z := Hx \in \mathbb{R}^\ell$, $\ell \ll n$.

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Thanks for your attention!