A Newton-Galerkin-ADI Method for Large-Scale Algebraic Riccati Equations

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Outline

Introduction

Large-Scale Algebraic Lyapunov and Riccati Equations

General form of algebraic Riccati equation (ARE) for $A, G = G^T, W = W^T \in \mathbb{R}^{n \times n}$ given and $X \in \mathbb{R}^{n \times n}$ unknown:

$$0 = \mathcal{R}(X) := A^T X + XA - XGX + W.$$

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- G, W low-rank with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n \times m}$, $m \ll n$, $C \in \mathbb{R}^{p \times n}$, $p \ll n$.

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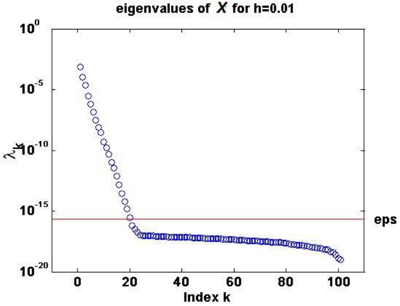
- Standard (eigenproblem-based) $\mathcal{O}(n^3)$ methods are not applicable!

Introduction
Low-Rank Approximation

Consider spectrum of ARE solution (analogous for Lyapunov equations).

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \implies n = 101$.



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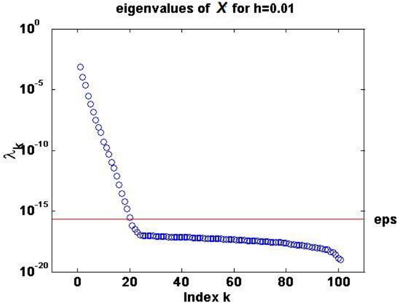

Idea: $X = X^T \geq 0 \implies$

$$X = ZZ^T = \sum_{k=1}^{n} \lambda_k z_k z_k^T \approx Z^{(r)}(Z^{(r)})^T = \sum_{k=1}^{r} \lambda_k z_k z_k^T.$$

$\implies$ Goal: compute $Z^{(r)} \in \mathbb{R}^{n \times r}$ directly w/o ever forming X!

Introduction

Review: LRCF-ADI for Lyapunov Equations

Consider $$FX + XF^T = -GG^T$$

ADI iteration for the Lyapunov equation (LE) [WACHSPRESS '95]

For $j = 1, \ldots, J$

$$\begin{aligned} X_0 &= 0 \\ (F + p_j I)X_{j-\frac{1}{2}} &= -GG^T - X_{j-1}(F^T - p_j I) \\ (F + p_j I)X_j^T &= -GG^T - X_{j-\frac{1}{2}}^T(F^T - p_j I) \end{aligned}$$

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Rewrite as one step iteration and factorize $X_i = Z_i Z_i^T$, $i = 0, \ldots, J$

$$\begin{aligned} Z_0 Z_0^T &= 0 \\ Z_j Z_j^T &= -2p_j(F + p_j I)^{-1} G G^T (F + p_j I)^{-T} \\ &\quad + (F + p_j I)^{-1}(F - p_j I) Z_{j-1} Z_{j-1}^T (F - p_j I)^T (F + p_j I)^{-T} \end{aligned}$$

Introduction

Review: LRCF-ADI for Lyapunov Equations

$$Z_j = [\sqrt{-2p_j}(F + p_j I)^{-1}G,\ (F + p_j I)^{-1}(F - p_j I)Z_{j-1}]$$

[Penzl '00]

Introduction

Review: LRCF-ADI for Lyapunov Equations

$$Z_j = [\sqrt{-2p_j}(F + p_j I)^{-1} G,\ (F + p_j I)^{-1}(F - p_j I) Z_{j-1}]$$

[Penzl '00]

Observing that $(F - p_i I)$, $(F + p_k I)^{-1}$ commute, we rewrite Z_J as

$$Z_J = [z_J,\ P_{J-1} z_J,\ P_{J-2}(P_{J-1} z_J),\ \ldots,\ P_1(P_2 \cdots P_{J-1} z_J)],$$

[Li/White '02]

where

$$z_J = \sqrt{-2p_J}(F + p_J I)^{-1} G$$

and

$$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}} \left[I - (p_i + p_{i+1})(F + p_i I)^{-1} \right].$$

Introduction

Review: LRCF-ADI for Lyapunov Equations

Algorithm 1 Low-rank Cholesky factor ADI iteration (LRCF-ADI)
[Penzl '97/'00, Li/White '99/'02, B./Li/Penzl '99/'08]

Input: F, G defining $FX + XF^T = -GG^T$ and shifts $\{p_1, \ldots, p_{i_{max}}\}$
Output: $Z = Z_{i_{max}} \in \mathbb{C}^{n \times t_{i_{max}}}$, such that $ZZ^H \approx X$

```
For V_1 solve (F + p_1 I) V_1 = sqrt(-2 Re(p_1)) G
Z_1 = V_1
for i = 2, 3, ..., i_max do
    For Ṽ solve (F + p_i I) Ṽ = V_{i-1}
    V_i = sqrt(Re(p_i) / Re(p_{i-1})) (V_{i-1} - (p_i + conj(p_{i-1})) Ṽ)
    Z_i = [Z_{i-1}  V_i]
end for
```

1. For V_1 solve $(F + p_1 I)\, V_1 = \sqrt{-2\,\mathrm{Re}\,(p_1)}\,G$
2. $Z_1 = V_1$
3. **for** $i = 2, 3, \ldots, i_{max}$ **do**
4. For $\tilde{V}$ solve $(F + p_i I)\tilde{V} = V_{i-1}$
5. $V_i = \sqrt{\mathrm{Re}\,(p_i)/\,\mathrm{Re}\,(p_{i-1})}\left(V_{i-1} - (p_i + \overline{p_{i-1}})\tilde{V}\right)$
6. $Z_i = [Z_{i-1} \;\; V_i]$
7. **end for**

Introduction

Review: LRCF-ADI for Lyapunov Equations

Algorithm 1 General. Low-rank Cholesky factor ADI iteration (G-LRCF-ADI)
[B. '04, B./Saak '09, S. '09]

Input: E,F,G defining $FXE^T + EXF^T = -GG^T$ and shifts $\{p_1, \ldots, p_{i_{max}}\}$
Output: $Z = Z_{i_{max}} \in \mathbb{C}^{n \times t_{i_{max}}}$, such that $ZZ^H \approx X$

For V_1 solve $(F + p_1 E)\, V_1 = \sqrt{-2\,\mathrm{Re}\,(p_1)}\,G$
$Z_1 = V_1$
for $i = 2, 3, \ldots, i_{max}$ **do**
    For $\tilde{V}$ solve $(F + p_i E)\tilde{V} = E\,V_{i-1}$
    $V_i = \sqrt{\mathrm{Re}\,(p_i)/\,\mathrm{Re}\,(p_{i-1})}\left(V_{i-1} - (p_i + \overline{p_{i-1}})\tilde{V}\right)$
    $Z_i = [Z_{i-1} \;\; V_i]$
end for

Introduction

Krylov Subspace Based Solvers for Lyapunov Equations

Consider Schur/singular value decomposition $X = U\Sigma U^T$,
$U \in \mathbb{R}^{n\times n}$, $U^T U = I$, $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$ and $|\sigma_1| \geq |\sigma_2| \geq \cdots \geq |\sigma_n|$.
The best rank-m Frobenius-norm approximation to X is thus given by

$$X_m := U \begin{bmatrix} \Sigma_m & 0 \\ 0 & 0 \end{bmatrix} U^T = U_m \Sigma_m U_m^T.$$

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Krylov Subspace Based Solvers for Lyapunov Equations

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$$X_m := U \begin{bmatrix} \Sigma_m & 0 \\ 0 & 0 \end{bmatrix} U^T = U_m \Sigma_m U_m^T.$$

Krylov projection idea [Saad '90, Jaimoukha/Kasenally '94]

Solve

$$(U_m^T F U_m) Y_m + Y_m (U_m^T F^T U_m) = -U_m^T G G^T U_m, \tag{1}$$

on $colspan(U_m)$ and get X_m as

$$X_m = U_m Y_m U_m^T.$$

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$$X_m := U \begin{bmatrix} \Sigma_m & 0 \\ 0 & 0 \end{bmatrix} U^T = U_m \Sigma_m U_m^T.$$

Note that a factorization

$$Z_m Z_m^T = X_m$$

can easily be computed from a Cholesky factorization of

$$Y_m = \tilde{Z}_m \tilde{Z}_m^T$$

as

$$Z_m = U_m \tilde{Z}_m.$$

Introduction

Krylov Subspace Based Solvers for Lyapunov Equations

Algorithm 2 Basic Krylov Subspace Method for the Lyapunov Equation

Input: F,G defining $FX + XF^T = -GG^T$, an initial Krylov subspace $\mathcal{V}$, e.g., $\mathcal{V} = \mathcal{K}_p(F, G)$ with orthogonal basis $V \in \mathbb{C}^{n \times p}$.

Output: $Z \in \mathbb{C}^{n \times t}$, such that $ZZ^H \approx X$

repeat

  if not first step **then**

    increase dimension of $\mathcal{V}$ and update V.

  end if

  Solve the "small" LE for $\tilde{Z}$ with a classical solver:

$$(V^T FV)\tilde{Z}\tilde{Z}^T + \tilde{Z}\tilde{Z}^T(V^T F^T V) = -V^T GG^T V,$$

  Lift $\tilde{Z}$ to the full space: $Z = U\tilde{Z}$

until res(Z)$<$ TOL

LRCF-ADI with Galerkin-Projection-Acceleration

ADI and Rational Krylov

[Li '00; Theorem 2] interprets the column span of the ADI solution as a certain rational Krylov subspace

$$\mathcal{L}(F,G,\mathbf{p}) := span\left\{ \ldots, \prod_{i=-j}^{-1}(F+p_iI)^{-1}G, \ldots, (F+p_{-2}I)^{-1}(F+p_{-1}I)^{-1}G, (F+p_{-1}I)^{-1}G,\ G,\ (F+p_1I)G, (F+p_2I)(F+p_1I)G, \ldots, \prod_{i=1}^{j}(F+p_iI)G \ \ldots \right\}$$

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$$(F+p_{-1}I)^{-1}G,\ G,\ (F+p_1I)G,$$
$$\left. (F+p_2I)(F+p_1I)G, \dots, \prod_{i=1}^{j}(F+p_iI)G \dots \right\}$$

Idea

Solve on current subspace of $\mathcal{L}(F,G,\mathbf{p})$ in the ADI step to increase the quality of the iterate.

LRCF-ADI with Galerkin-Projection-Acceleration

Projected ADI Step

Projected ADI Step $\longrightarrow$ LRCF-ADI-GP [B./Li/Truhar'09, Saak'09, B./Saak'10]

1. Compute the LRCF-ADI iterate Z_i
2. Compute orthogonal basis via QR factorization: $Q_i R_i \Pi_i = Z_i$[a]
3. Solve (for $\tilde{Z}$) the projected Lyapunov equation

$$(Q_i^T F Q_i)\tilde{Z}\tilde{Z}^T + \tilde{Z}\tilde{Z}^T(Q_i^T F^T Q_i) = -Q_i^T G G^T Q_i$$

4. Update Z_i according to $Z_i := Q_i \tilde{Z}$

[a] economy size QR with column pivoting; crucial to compute correct subspace if Z_i rank deficient.

LRCF-ADI with Galerkin-Projection-Acceleration

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- Need to ensure that projected systems remain stable, e.g., $F + F^T < 0$
- may perform projected ADI step only every k-th step (e.g. $k = 5$) $\rightsquigarrow$ restarted ADI with shifts $\Lambda(Q_i^T F Q_i)$.

LRCF-ADI with Galerkin-Projection-Acceleration

Projected ADI Step

Projected ADI Step →G-LRCF-ADI-GP [B./Li/Truhar'09, Saak'09, B./Saak'10]

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LRCF-ADI with Galerkin-Projection-Acceleration

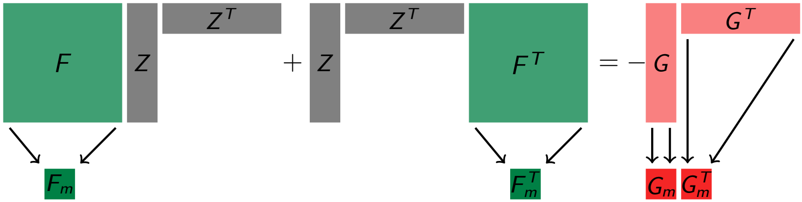
Projected ADI Step

Legend:

new factor original matrix projected matrix projected Cholesky factor
old factor original rhs projected rhs

LRCF-ADI with Galerkin-Projection-Acceleration

Projected ADI Step



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LRCF-ADI with Galerkin-Projection-Acceleration

Projected ADI Step

$$F_m \quad C_m C_m^T \quad + C_m C_m^T \quad F_m^T \quad = -G_m G_m^T$$

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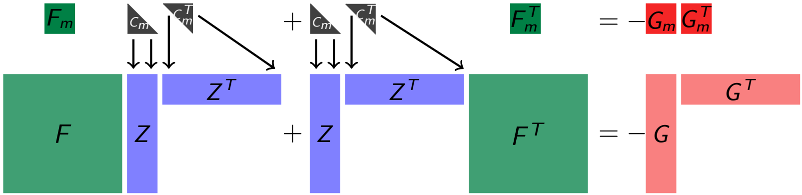

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LRCF-ADI with Galerkin-Projection-Acceleration

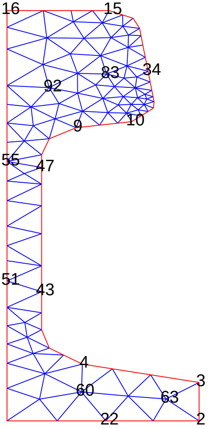
Test Example: Optimal Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

$$\begin{aligned} c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, & \xi \in \Omega \\ \lambda \frac{\partial}{\partial n} x &= \kappa (u_k - x), & \xi \in \Gamma_k,\ 1 \leq k \leq 7, \\ \frac{\partial}{\partial n} x &= 0, & \xi \in \Gamma_0. \end{aligned}$$

$\Longrightarrow q = 7, p = 6.$

- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1\,357, 5\,177, 20\,209, 79\,841.$

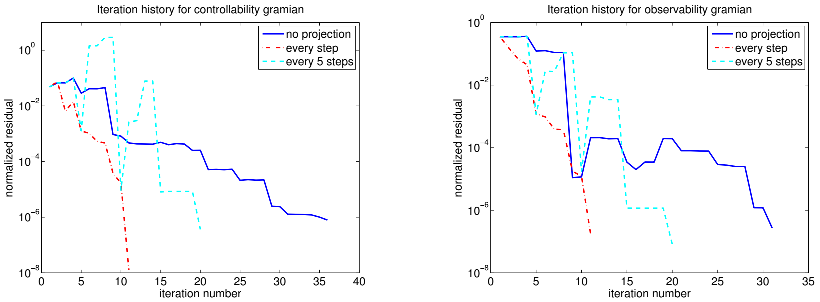


Source: Physical model: courtesy of Mannesmann/Demag.
Math. model: Tröltzsch/Unger '99/'01, Penzl '99, S. '03.

LRCF-ADI with Galerkin-Projection-Acceleration

Numerical Results

steel profile n=20 209 good shifts



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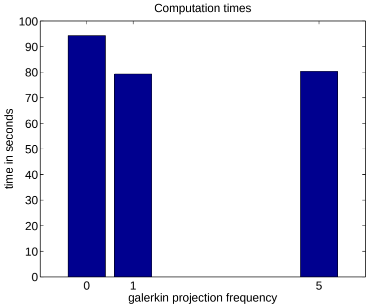

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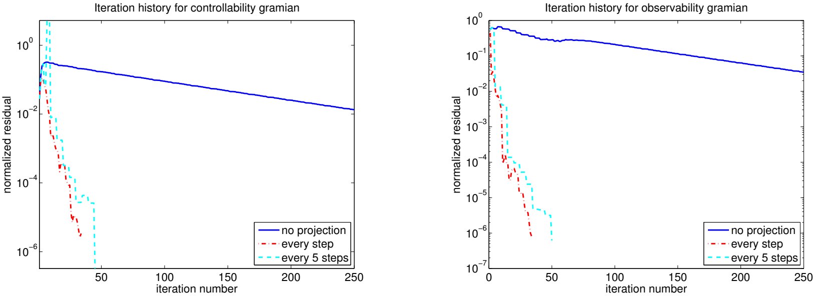

LRCF-ADI with Galerkin-Projection-Acceleration
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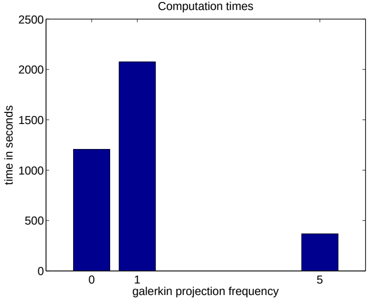

LRCF-NM for the ARE

1. Introduction
2. LRCF-ADI with Galerkin-Projection-Acceleration
3. LRCF-NM for the ARE
   - Newton's Method for AREs
   - Low-Rank Newton-ADI (LRCF-NM) for AREs
   - Test Examples
   - Test Results (ADI-loop)
   - Test Results (both-loops)
   - Computation Time Scaling with Problem Size

LRCF-NM for the ARE

Newton's Method for AREs

Consider $\mathfrak{R}(X) := C^TC + A^TX + XA - XBB^TX = 0$

Newton's Iteration for the ARE

$$\mathfrak{R}'|_X(N_\ell) = -\mathfrak{R}(X_\ell), \qquad X_{\ell+1} = X_\ell + N_\ell, \qquad \ell = 0, 1, \ldots$$

where the Frechét derivative of $\mathfrak{R}$ at X is the Lyapunov operator

$$\mathfrak{R}'|_X : \quad Q \mapsto (A - BB^TX)^TQ + Q(A - BB^TX),$$

i.e., in every Newton step solve a

Lyapunov Equation [Kleinman '68]

$$(A - BB^TX_\ell)^TX_{\ell+1} + X_{\ell+1}(A - BB^TX_\ell) = -C^TC - X_\ell BB^TX_\ell.$$

LRCF-NM for the ARE

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Lyapunov Equation [Kleinman '68]

$$F_\ell^TX_{\ell+1} + X_{\ell+1}F_\ell = -G_\ell G_\ell^T.$$

LRCF-NM for the ARE

Newton's Method for AREs

Consider $\mathfrak{R}(X) := C^T C + A^T X E + E^T X A - E^T X B B^T X E = 0$

Newton's Iteration for the ARE

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i.e., in every Newton step solve a

Lyapunov Equation [Kleinman '68]

$$F_\ell^T X_{\ell+1} E + E^T X_{\ell+1} F_\ell = -\tilde{G}_\ell \tilde{G}_\ell^T.$$

LRCF-NM for the ARE

Low-Rank Newton-ADI (LRCF-NM) for AREs

Factored Newton-Kleinman Iteration [Benner/Li/Penzl '99/'08]

$F_\ell = A - BB^T X_\ell =: A - BK_\ell$ is "sparse + low rank"

$G_\ell = [C^T \; K_\ell^T]$ is low rank factor

LRCF-NM for the ARE

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- apply LRCF-ADI in every Newton step
- exploit structure of F_ℓ using Sherman-Morrison-Woodbury formula

LRCF-NM for the ARE

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Factored Newton-Kleinman Iteration [Benner/Li/Penzl '99/'08]

$$F_\ell = A - BB^T X_\ell E =: A - BK_\ell \quad \text{is "sparse + low rank"}$$
$$G_\ell = [C^T \; K_\ell^T] \quad \text{is low rank factor}$$

- apply LRCF-ADI in every Newton step
- exploit structure of F_ℓ using Sherman-Morrison-Woodbury formula

LRCF-NM for the ARE

Low-Rank Newton-ADI (LRCF-NM) for AREs

Algorithm 3 Low-Rank Cholesky Factor Newton Method (LRCF-NM)

Input: A, B, C, $K^{(0)}$ for which $A - BK^{(0)^T}$ is stable
Output: $Z = Z^{(k_{max})}$, such that ZZ^H approximates the solution X of

$$C^T C + A^T X + XA - XBB^T X = 0.$$

1: **for** $k = 1, 2, \ldots, k_{max}$ **do**
2: Determine (sub)optimal ADI shift parameters $p_1^{(k)}, p_2^{(k)}, \ldots$ with respect to the matrix $F^{(k)} = A^T - K^{(k-1)}B^T$.
3: $G^{(k)} = \begin{bmatrix} C^T & K^{(k-1)} \end{bmatrix}$
4: Compute $Z^{(k)}$ using Algorithm 1 (LRCF-ADI) such that

$$F^{(k)}Z^{(k)}Z^{(k)^H} + Z^{(k)}Z^{(k)^H}F^{(k)^T} \approx -G^{(k)}G^{(k)^T}.$$

5: $K^{(k)} = Z^{(k)}(Z^{(k)^H}B)$
6: **end for**

LRCF-NM for the ARE

Low-Rank Newton-ADI (LRCF-NM) for AREs

Algorithm 3 Low-Rank Cholesky Factor Newton Method (G-LRCF-NM)

Input: E, A, B, C, $K^{(0)}$ for which $A - BK^{(0)^T}$ is stable
Output: $Z = Z^{(k_{max})}$, such that ZZ^H approximates the solution X of

$$C^TC + A^TXE + E^TXA - E^TXBB^TXE = 0.$$

1: **for** $k = 1, 2, \ldots, k_{max}$ **do**
2: Determine (sub)optimal ADI shift parameters $p_1^{(k)}, p_2^{(k)}, \ldots$ with respect to the matrix $F^{(k)} = A^TE^{-T} - K^{(k-1)}B^TE^{-T}$.
3: $G^{(k)} = \begin{bmatrix} C^T & K^{(k-1)} \end{bmatrix}$
4: Compute $Z^{(k)}$ using Algorithm 1 (G-LRCF-ADI) such that

$$F^{(k)}Z^{(k)}Z^{(k)^H}E + E^TZ^{(k)}Z^{(k)^H}F^{(k)^T} \approx -G^{(k)}G^{(k)^T}.$$

5: $K^{(k)} = E^T(Z^{(k)}(Z^{(k)^H}B))$
6: **end for**

LRCF-NM for the ARE

Low-Rank Newton-ADI (LRCF-NM) for AREs

Algorithm 3 Low-Rank Cholesky Factor Newton Method (LRCF-NM)

Input: A, B, C, $K^{(0)}$ for which $A - BK^{(0)^T}$ is stable
Output: $Z = Z^{(k_{max})}$, such that ZZ^H approximates the solution X of

$$C^T C + A^T X + XA - XBB^T X = 0.$$

1: **for** $k = 1, 2, \ldots, k_{max}$ **do**
2: Determine (sub)optimal ADI shift parameters $p_1^{(k)}, p_2^{(k)}, \ldots$ with respect to the matrix $F^{(k)} = A^T - K^{(k-1)} B^T$.
3: $G^{(k)} = \left[\begin{array}{cc} C^T & K^{(k-1)} \end{array} \right]$
4: Compute $Z^{(k)}$ using Algorithm 1 (LRCF-ADI) or (LRCF-ADI-GP) such that $F^{(k)} Z^{(k)} Z^{(k)^H} + Z^{(k)} Z^{(k)^H} F^{(k)^T} \approx -G^{(k)} G^{(k)^T}$.
5: $K^{(k)} = Z^{(k)} (Z^{(k)^H} B)$
6: **end for**

LRCF-NM for the ARE

Low-Rank Newton-ADI (LRCF-NM) for AREs

Algorithm 4 Simpl. Low-Rank Cholesky Factor Newton Method (LRCF-NM-S)

Input: A, B, C, $K^{(0)}$ for which $A - BK^{(0)^T}$ is stable
Output: $Z = Z^{(k_{max})}$, such that ZZ^H approximates the solution X of

$$C^T C + A^T X + XA - XBB^T X = 0.$$

1: Determine (sub)optimal ADI shift parameters $p_1, p_2, \dots$ with respect to the matrix $F^{(k)} = A^T - K^{(0)} B^T$.
2: **for** $k = 1, 2, \dots, k_{max}$ **do**
3: $G^{(k)} = \left[\begin{array}{cc} C^T & K^{(k-1)} \end{array}\right]$
4: Compute $Z^{(k)}$ using Algorithm 1 (LRCF-ADI) or (LRCF-ADI-GP) such that $F^{(k)} Z^{(k)} Z^{(k)^H} + Z^{(k)} Z^{(k)^H} F^{(k)^T} \approx -G^{(k)} G^{(k)^T}$.
5: $K^{(k)} = Z^{(k)} (Z^{(k)^H} B)$
6: **end for**

LRCF-NM for the ARE

Low-Rank Newton-ADI (LRCF-NM) for AREs

Algorithm 5 Low-Rank Cholesky Factor Galerkin-Newton Method (LRCF-NM-GP)

Input: A, B, C, $K^{(0)}$ for which $A - BK^{(0)^T}$ is stable
Output: $Z = Z^{(k_{max})}$, such that ZZ^H approximates the solution X of

$$C^TC + A^TX + XA - XBB^TX = 0.$$

1: **for** $k = 1, 2, \dots, k_{max}$ **do**
2: Determine (sub)optimal ADI shift parameters $p_1^{(k)}, p_2^{(k)}, \dots$ with respect to the matrix $F^{(k)} = A^T - K^{(k-1)}B^T$.
3: $G^{(k)} = \left[\begin{array}{cc} C^T & K^{(k-1)} \end{array}\right]$
4: Compute $Z^{(k)}$ using Algorithm 1 (LRCF-ADI) or (LRCF-ADI-GP) such that $F^{(k)}Z^{(k)}Z^{(k)^H} + Z^{(k)}Z^{(k)^H}F^{(k)^T} \approx -G^{(k)}G^{(k)^T}$.
5: Project ARE, solve and prolongate solution
6: $K^{(k)} = Z^{(k)}(Z^{(k)^H}B)$
7: **end for**

LRCF-NM for the ARE

Test Examples

Example 1: 3d Convection-Diffusion Equation

- FDM for 3D convection-diffusion equation on $[0,1]^3$
- proposed in [Simoncini '07], $q = p = 1$
- non-symmetric $A \in \mathbb{R}^{n \times n}$, $n = 10\,648$

Example 2: 2d Convection-Diffusion Equation

- FDM for 2D convection-diffusion equations on $[0,1]^2$
- LyaPack benchmark, $q = p = 1$, e.g., `demo_l1`
- non-symmetric $A \in \mathbb{R}^{n \times n}$, $n = 22\,500$.

- 16 shift parameters
- Penzl's heuristic from 50/25 Ritz/harmonic Ritz values of A

LRCF-NM for the ARE

Test Results (ADI-loop): Example 1

Newton-ADI

NWT	rel. change	rel. residual	ADI
1	$9.97 \cdot 10^{-01}$	$9.27 \cdot 10^{-01}$	100
2	$3.67 \cdot 10^{-02}$	$9.58 \cdot 10^{-02}$	94
3	$1.36 \cdot 10^{-02}$	$1.09 \cdot 10^{-03}$	98
4	$3.48 \cdot 10^{-04}$	$1.01 \cdot 10^{-07}$	97
5	$6.41 \cdot 10^{-08}$	$1.34 \cdot 10^{-10}$	97
6	**$7.47 \cdot 10^{-16}$**	$1.34 \cdot 10^{-10}$	97

CPU time: 4 805.8 sec.

Newton-Galerkin-ADI

LRCF-ADI-GP(5)

NWT	rel. change	rel. residual	ADI
1	$9.97 \cdot 10^{-01}$	$9.29 \cdot 10^{-01}$	80
2	$3.67 \cdot 10^{-02}$	$9.60 \cdot 10^{-02}$	30
3	$1.36 \cdot 10^{-02}$	$1.09 \cdot 10^{-03}$	28
4	$3.47 \cdot 10^{-04}$	$1.01 \cdot 10^{-07}$	35
5	$6.41 \cdot 10^{-08}$	$1.03 \cdot 10^{-10}$	25
6	$1.23 \cdot 10^{-11}$	**$1.98 \cdot 10^{-11}$**	27

CPU time: 1 460.1 sec.

test system: Intel® Xeon® 5160 3.00GHz ; 16 GB RAM;
64Bit-MATLAB® (R2010a) using threaded BLAS (`romulus`)
stopping criterion tolerances: 10^{-10}

LRCF-NM for the ARE

Test Results (ADI-loop): Example 2

Newton-ADI

NWT	rel. change	rel. residual	ADI
1	1	$1.70 \cdot 10^{+02}$	46
2	$2.88 \cdot 10^{-01}$	$4.25 \cdot 10^{+01}$	39
3	$2.13 \cdot 10^{-01}$	$1.06 \cdot 10^{+01}$	43
4	$1.77 \cdot 10^{-01}$	$2.58 \cdot 10^{+00}$	46
5	$2.47 \cdot 10^{-01}$	$5.15 \cdot 10^{-01}$	43
6	$3.04 \cdot 10^{-01}$	$3.26 \cdot 10^{-02}$	52
7	$1.78 \cdot 10^{-02}$	$6.90 \cdot 10^{-05}$	50
8	$2.60 \cdot 10^{-05}$	$1.08 \cdot 10^{-10}$	46
9	$\mathbf{2.75 \cdot 10^{-11}}$	$1.07 \cdot 10^{-10}$	50

CPU time: 493.81 sec.

Newton-Galerkin-ADI

LRCF-ADI-GP(5)

NWT	rel. change	rel. residual	ADI
1	1	$1.70 \cdot 10^{+02}$	35
2	$2.88 \cdot 10^{-01}$	$4.25 \cdot 10^{+01}$	15
3	$2.13 \cdot 10^{-01}$	$1.06 \cdot 10^{+01}$	20
4	$1.77 \cdot 10^{-01}$	$2.58 \cdot 10^{+00}$	20
5	$2.47 \cdot 10^{-01}$	$5.15 \cdot 10^{-01}$	20
6	$3.04 \cdot 10^{-01}$	$3.26 \cdot 10^{-02}$	17
7	$1.78 \cdot 10^{-02}$	$6.90 \cdot 10^{-05}$	20
8	$2.60 \cdot 10^{-05}$	$1.10 \cdot 10^{-10}$	20
9	$2.75 \cdot 10^{-11}$	$\mathbf{1.92 \cdot 10^{-12}}$	20

CPU time: 280.55 sec.

test system: Intel® Core™2 Quad Q9400 2.66 GHz; 4 GB RAM;
64Bit-MATLAB® (R2009a) using threaded BLAS (`reynolds`)
stopping criterion tolerances: 10^{-10}

LRCF-NM for the ARE

Test Results (both-loops): Example 1

Newton-ADI

NWT	rel. change	rel. residual	ADI
1	$9.97 \cdot 10^{-01}$	$9.27 \cdot 10^{-01}$	100
2	$3.67 \cdot 10^{-02}$	$9.58 \cdot 10^{-02}$	94
3	$1.36 \cdot 10^{-02}$	$1.09 \cdot 10^{-03}$	98
4	$3.48 \cdot 10^{-04}$	$1.01 \cdot 10^{-07}$	97
5	$6.41 \cdot 10^{-08}$	$1.34 \cdot 10^{-10}$	97
6	$7.47 \cdot 10^{-16}$	$1.34 \cdot 10^{-10}$	97

CPU time: 4 805.8 sec.

NG-ADI inner= 5, outer= 1

NWT	rel. change	rel. residual	ADI
1	$9.98 \cdot 10^{-01}$	$5.04 \cdot 10^{-11}$	80

CPU time: 497.6 sec.

NG-ADI inner= 1, outer= 1

NWT	rel. change	rel. residual	ADI
1	$9.98 \cdot 10^{-01}$	$7.42 \cdot 10^{-11}$	71

CPU time: 856.6 sec.

NG-ADI inner= 0, outer= 1

NWT	rel. change	rel. residual	ADI
1	$9.98 \cdot 10^{-01}$	$6.46 \cdot 10^{-13}$	100

CPU time: 506.6 sec.

test system: Intel® Xeon® 5160 3.00GHz ; 16 GB RAM;
64Bit-MATLAB® (R2010a) using threaded BLAS (`romulus`)
stopping criterion tolerances: 10^{-10}

LRCF-NM for the ARE

Test Results (both-loops): Example 2

Newton-ADI

NWT	rel. change	rel. residual	ADI
1	1	$1.70 \cdot 10^{+02}$	46
2	$2.88 \cdot 10^{-01}$	$4.25 \cdot 10^{+01}$	39
3	$2.13 \cdot 10^{-01}$	$1.06 \cdot 10^{+01}$	43
4	$1.77 \cdot 10^{-01}$	$2.58 \cdot 10^{+00}$	46
5	$2.47 \cdot 10^{-01}$	$5.15 \cdot 10^{-01}$	43
6	$3.04 \cdot 10^{-01}$	$3.26 \cdot 10^{-02}$	52
7	$1.78 \cdot 10^{-02}$	$6.90 \cdot 10^{-05}$	50
8	$2.60 \cdot 10^{-05}$	$1.08 \cdot 10^{-10}$	46
9	**$2.75 \cdot 10^{-11}$**	$1.07 \cdot 10^{-10}$	50

CPU time: 493.81 sec.

NG-ADI inner= 5, outer= 1

NWT	rel. change	rel. residual	ADI
1	1	$3.30 \cdot 10^{-11}$	35

CPU time: 24.1 sec.

NG-ADI inner= 1, outer= 1

NWT	rel. change	rel. residual	ADI
1	1	$1.31 \cdot 10^{-11}$	34

CPU time: 26.8 sec.

NG-ADI inner= 0, outer= 1

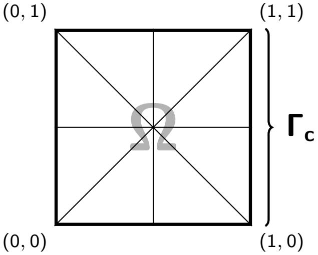
NWT	rel. change	rel. residual	ADI
1	1	$3.27 \cdot 10^{-15}$	46

CPU time: 24.0 sec.

test system: Intel® Core™2 Quad Q9400 2.66 GHz; 4 GB RAM;
64Bit-MATLAB® (R2009a) using threaded BLAS (`reynolds`)
stopping criterion tolerances: 10^{-10}

LRCF-NM for the ARE

Computation Time Scaling with Problem Size

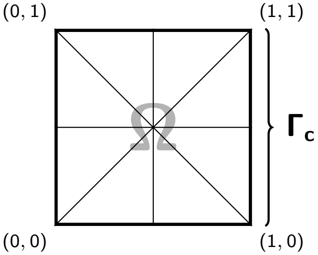


$$\begin{aligned} \partial_t x(\xi,t) &= \Delta x(\xi,t) && \text{in } \Omega \\ \partial_\nu x &= b(\xi)\cdot u(t) - x && \text{on } \Gamma_c \\ \partial_\nu x &= -x && \text{on } \partial\Omega \setminus \Gamma_c \\ x(\xi,0) &= 1 \end{aligned}$$

Note:
Here $b(\xi) = 4\,(1-\xi_2)\,\xi_2$ for $\xi \in \Gamma_c$ and 0 otherwise, thus $\forall t \in \mathbb{R}_{>0}$, we have $u(t) \in \mathbb{R}$.

LRCF-NM for the ARE

Computation Time Scaling with Problem Size



$$
\begin{aligned}
\partial_t x(\xi, t) &= \Delta x(\xi, t) && \text{in } \Omega \\
\partial_\nu x &= b(\xi) \cdot u(t) - x && \text{on } \Gamma_c \\
\partial_\nu x &= -x && \text{on } \partial\Omega \setminus \Gamma_c
\end{aligned}
$$

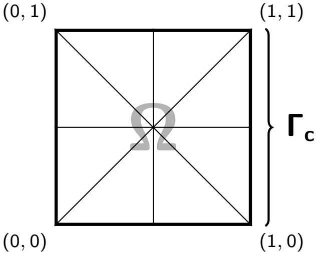
$$x(\xi, 0) = 1$$

Note:
Here $b(\xi) = 4\,(1 - \xi_2)\,\xi_2$ for $\xi \in \Gamma_c$ and 0 otherwise, thus $\forall t \in \mathbb{R}_{>0}$, we have $u(t) \in \mathbb{R}$.

$$\Rightarrow B_h = M_{\Gamma,h} \cdot b.$$

LRCF-NM for the ARE

Computation Time Scaling with Problem Size



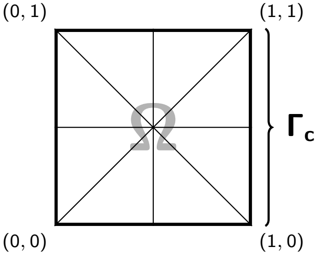
$$\begin{aligned} \partial_t x(\xi,t) &= \Delta x(\xi,t) && \text{in } \Omega \\ \partial_\nu x &= b(\xi)\cdot u(t) - x && \text{on } \Gamma_c \\ \partial_\nu x &= -x && \text{on } \partial\Omega \setminus \Gamma_c \\ x(\xi,0) &= 1 \end{aligned}$$

Consider: output equation $y = Cx$, where

$$\begin{aligned} C : \mathcal{L}^2(\Omega) &\rightarrow \mathbb{R} \\ x(\xi,t) &\mapsto y(t) = \int_\Omega x(\xi,t)\,d\xi. \end{aligned}$$

LRCF-NM for the ARE

Computation Time Scaling with Problem Size



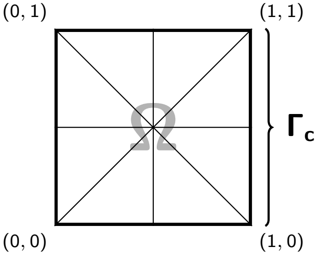
$$\begin{aligned} \partial_t x(\xi, t) &= \Delta x(\xi, t) && \text{in } \Omega \\ \partial_\nu x &= b(\xi) \cdot u(t) - x && \text{on } \Gamma_c \\ \partial_\nu x &= -x && \text{on } \partial\Omega \setminus \Gamma_c \\ x(\xi, 0) &= 1 \end{aligned}$$

Consider: output equation $y = Cx$, where

$$\begin{aligned} C : \mathcal{L}^2(\Omega) &\to \mathbb{R} \\ x(\xi, t) &\mapsto y(t) = \int_\Omega x(\xi, t)\, d\xi, \end{aligned} \quad \Rightarrow C_h = \underline{1} \cdot M_h.$$

LRCF-NM for the ARE

Computation Time Scaling with Problem Size



$$\begin{aligned} \partial_t x(\xi,t) &= \Delta x(\xi,t) && \text{in } \Omega \\ \partial_\nu x &= b(\xi)\cdot u(t) - x && \text{on } \Gamma_c \\ \partial_\nu x &= -x && \text{on } \partial\Omega \setminus \Gamma_c \end{aligned}$$

$$x(\xi,0) = 1$$

Cost Function:

$$\mathcal{J}(u) = \int_0^\infty y^2(t) + u^2(t)\,dt.$$

LRCF-NM for the ARE

Computation Time Scaling with Problem Size

simplified Low Rank Newton-Galerkin ADI

- generalized state space form implementation
- Penzl shifts (16/50/25) with respect to initial matrices
- projection acceleration in every outer iteration step
- projection acceleration in every 5-th inner iteration step

test system: Intel®Xeon® 5160 @ 3.00 GHz; 16 GB RAM;
64Bit-MATLAB® (R2010a) using threaded BLAS (`romulus`)
stopping criterion tolerances: 10^{-10}

LRCF-NM for the ARE

Computation Time Scaling with Problem Size

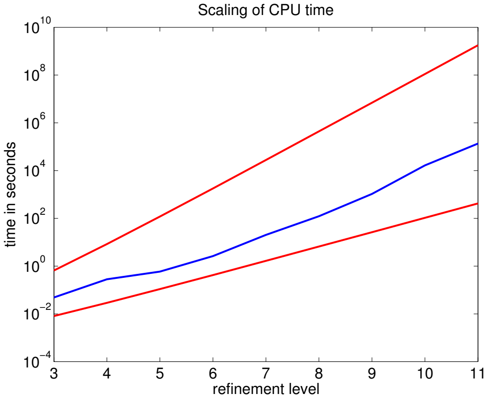
Computation Times

discretization level	problem size	time in seconds
3	81	$4.87\,10^{-2}$
4	289	$2.81\,10^{-1}$
5	1 089	$5.87\,10^{-1}$
6	4 225	2.63
7	16 641	$2.03\,10^{+1}$
8	66 049	$1.22\,10^{+2}$
9	263 169	$1.05\,10^{+3}$
10	1 050 625	$1.65\,10^{+4}$
11	4 198 401	$1.35\,10^{+5}$

test system: Intel®Xeon® 5160 @ 3.00 GHz; 16 GB RAM;
64Bit-MATLAB® (R2010a) using threaded BLAS (`romulus`)
stopping criterion tolerances: 10^{-10}

LRCF-NM for the ARE

Computation Time Scaling with Problem Size



test system: Intel®Xeon® 5160 @ 3.00 GHz; 16 GB RAM; 64Bit-MATLAB® (R2010a) using threaded BLAS (`romulus`)
stopping criterion tolerances: 10^{-10}