KRYLOV-SUBSPACE BASED MODEL REDUCTION OF NONLINEAR CIRCUIT MODELS USING BILINEAR AND QUADRATIC-LINEAR APPROXIMATIONS

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MAX-PLANCK-GESELLSCHAFT

Outline

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2. Model Reduction via Bilinear Approximations
   - Bilinear Control Systems
   - Krylov-Based Model Reduction
   - Interpolatory Model Reduction
   - Numerical Examples
3. Model Reduction via Quadratic-Bilinearizations
   - Quadratic-Bilinear Control Systems
   - Reduction Techniques
   - Numerical Examples
4. Outlook

Nonlinear Model Reduction
Motivation

Consider a large-scale state-nonlinear control system of the form

$$\Sigma : \quad \begin{cases} \dot{x}(t) = f(x(t)) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ nonlinear, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$.

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MOR

$$\hat{\Sigma} : \quad \begin{cases} \dot{\hat{x}}(t) = \hat{f}(\hat{x}(t)) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t), \quad \hat{x}(0) = \hat{x}_0, \end{cases}$$

with $\hat{f} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}}$, $B \in \mathbb{R}^{\hat{n} \times m}$, $C \in \mathbb{R}^{p \times \hat{n}}$, $x \in \mathbb{R}^{\hat{n}}$, $u \in \mathbb{R}^m$, $\hat{y} \in \mathbb{R}^p$, $\hat{n} \ll n$..

Goal

$\hat{y} \approx y$ for all admissible u.

Nonlinear Model Reduction

Common Reduction Techniques

Proper Orthogonal Decomposition (POD)

- Take computed or experimental 'snapshots' of full model: $[x(t_1), x(t_2), \ldots, x(t_N)] =: X$,
- perform SVD of snapshot matrix: $X = VSW^T \approx V_{\hat{n}} S_{\hat{n}} W_{\hat{n}}^T$.
- Reduction by POD-Galerkin projection: $\dot{\hat{x}} = V_{\hat{n}}^T f(V_{\hat{n}} \hat{x}) + V_{\hat{n}}^T Bu$.
- Requires evaluation of f

  $\rightsquigarrow$ discrete empirical interpolation [Sorensen/Chaturantabut '09].
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Trajectory Piecewise Linear (TPWL)

- Linearize f along trajectory,
- reduce resulting linear systems,
- construct reduced model by weighting sum of linear systems.
- Requires simulation of original model and several linear reduction steps, many heuristics.

$\rightsquigarrow$ talk by J.P. Amorocho, SyreNe dissemination mini-symposium (Thu, 15h)

Outline

Bilinear Control Systems

State-Space and Output Representation

Consider continuous-time bilinear systems of the form

$$\Sigma_c : \quad \begin{cases} \dot{x}(t) = Ax(t) + \sum_{j=1}^{m} N_j x(t) u_j(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

where $A, N_j \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x \in \mathbb{R}^{n}$, $u \in \mathbb{R}^{m}$, $y \in \mathbb{R}^{p}$.

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Output Characterization: Volterra series

$$y(t) = \sum_{j=1}^{\infty} \int_0^t \int_0^{t_1} \ldots \int_0^{t_{j-1}} h(t_1, \ldots, t_j) u(t - t_1 - \ldots - t_j) \cdots u(t - t_j) dt_j \cdots dt_1,$$

with kernels $h(t_1, \ldots, t_j) = Ce^{At_j} N \cdots e^{At_2} N e^{At_1} B$.

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with kernels $h(t_1, \dots, t_j) = Ce^{At_j} N \cdots e^{At_2} N e^{At_1} B$.
Multivariable Laplace-transform:

$$H(s_1, \dots, s_j) = C(s_j I - A)^{-1} N \cdots (s_2 I - A)^{-1} N (s_1 I - A)^{-1} B.$$

Bilinear Control Systems

Carleman Bilinearization

Approximate nonlinear state evolution function f by Taylor polynomial, e.g.

$$\dot{x} = f(x) + Bu \approx A_1 x + A_2 (x \otimes x) + Bu.$$

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Construct enlarged bilinear system as

$$\frac{d}{dt}\begin{bmatrix} x \\ x \otimes x \end{bmatrix} \approx \underbrace{\begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \otimes I + I \otimes A_1 \end{bmatrix}}_{A^{\otimes}} \begin{bmatrix} x \\ x \otimes x \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ B \otimes I + I \otimes B & 0 \end{bmatrix}}_{N^{\otimes}} \begin{bmatrix} x \\ x \otimes x \end{bmatrix} u + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B^{\otimes}} u,$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C^{\otimes}} \begin{bmatrix} x \\ x \otimes x \end{bmatrix}.$$

Krylov-Based Model Reduction

Multimoment-Matching

Common approach: expand j-th transfer function about $s_j = \sigma_j$:

$$H(s_1, \ldots, s_j) = \sum_{\ell_1, \ldots, \ell_j = 1}^{\infty} m_{\underline{\sigma}}(\ell_1, \ldots, \ell_j) s_1^{\ell_1 - 1} s_2^{\ell_2 - 1} \cdots s_j^{\ell_j - 1},$$

$$m_{\underline{\sigma}}(\ell_1, \ldots, \ell_k) = (-1)^j C(A - \sigma_j I)^{-\ell_j} N \cdots (A - \sigma_2 I)^{-\ell_2} N (A - \sigma_1 I)^{-\ell_1} B.$$

Theorem

Given Σ. Construct $\hat{\Sigma}$ by projection $P = VV^T$, where V is given as a basis of the union of the (block) Krylov subspaces

- $\operatorname{span}\{V^{(1)}\} = \mathcal{K}_{q_1}\left((A - \sigma_1 I)^{-1}, (A - \sigma_1 I)^{-1} B\right)$,
- $\operatorname{span}\{V^{(j)}\} = \mathcal{K}_{q_j}((A - \sigma_j I)^{-1}, (A - \sigma_j I)^{-1} N V^{(j-1)}), \quad j = 2, \ldots, r.$

Then for $j = 1, \ldots, r$ and $\ell_k = 1, \ldots, q_k$ $(k = 1, \ldots, j)$:

$$m_{\underline{\sigma}}(\ell_1, \ldots, \ell_j) = \hat{m}_{\underline{\sigma}}(\ell_1, \ldots, \ell_j).$$

Interpolatory Model Reduction

$\mathcal{H}_2$-Norm Computation via Generalized Lyapunov Equations

Definition [Zhang/Lam'02]

Assume that the solutions of the generalized Lyapunov equations

$$AP + PA^T + NPN^T + BB^T = 0,$$
$$A^TQ + QA + N^TQN + C^TC = 0,$$

associated with a bilinear system Σ, exist. Then the $\mathcal{H}_2$-norm of Σ is defined as

$$||\Sigma||_{\mathcal{H}_2} = \sqrt{CPC^T} = \sqrt{B^TQB}.$$

Question: is it possible to characterize the $\mathcal{H}_2$-norm in terms of the generalized transfer functions?

Interpolatory Model Reduction

$\mathcal{H}_2$-Norm Computation via Generalized Residues

Lemma

Let a bilinear system Σ be given and let $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ denote the spectrum of A. Then the $\mathcal{H}_2$-norm of Σ can be alternatively computed as follows

$$||\Sigma||_{\mathcal{H}_2}^2 = \sum_{j=1}^{\infty} \sum_{\ell_j=1}^{n} \cdots \sum_{\ell_1=1}^{n} \Phi_{\ell_1,\dots,\ell_j} \, H_j\left(-\lambda_{\ell_1}, \dots, -\lambda_{\ell_j}\right),$$

where

$$\Phi_{\ell_1,\dots,\ell_j} = \lim_{s_k \rightarrow \lambda_{\ell_k}} H_j(s_1, \dots, s_j)(s_1 - \lambda_{\ell_1}) \cdots (s_j - \lambda_{\ell_j})$$

denote generalized residues associated with the transfer functions.

Interpolatory Model Reduction

Error System and Optimal Interpolation Points

Consequently, the $\mathcal{H}_2$-norm of the error system is given as

$$||\Sigma - \hat{\Sigma}||^2_{\mathcal{H}_2} = \sum_{j=1}^{\infty} \sum_{\ell_1,\ldots,\ell_j}^{n} \Phi_{\lambda_{\ell_1},\ldots,\lambda_{\ell_j}} \left(H_j(-\lambda_{\ell_1},\ldots,-\lambda_{\ell_j}) - \hat{H}_j(-\lambda_{\ell_1},\ldots,-\lambda_{\ell_j}) \right)$$
$$+ \sum_{j=1}^{\infty} \sum_{\hat{\ell}_1,\ldots,\hat{\ell}_j}^{\hat{n}} \hat{\Phi}_{\hat{\lambda}_{\hat{\ell}_1},\ldots,\hat{\lambda}_{\hat{\ell}_j}} \left(\hat{H}_j(-\hat{\lambda}_{\hat{\ell}_1},\ldots,-\hat{\lambda}_{\hat{\ell}_j}) - H_j(-\hat{\lambda}_{\hat{\ell}_1},\ldots,-\hat{\lambda}_{\hat{\ell}_j}) \right).$$

$\Rightarrow$ Necessary $\mathcal{H}_2$-optimality conditions?

$$H_j(-\hat{\lambda}_{i_1},\ldots,-\hat{\lambda}_{i_j}) = \hat{H}_j(-\hat{\lambda}_{i_1},\ldots,-\hat{\lambda}_{i_j}),$$
$$\frac{\partial}{\partial s_k} H_j(-\hat{\lambda}_{i_1},\ldots,-\hat{\lambda}_{i_j}) = \frac{\partial}{\partial s_k} \hat{H}_j(-\hat{\lambda}_{i_1},\ldots,-\hat{\lambda}_{i_j}),$$

for $i_1,\ldots,i_j \leq n,\ k = 1,\ldots,j$ and $j = 1,2,3,\ldots$

Interpolatory Model Reduction

Generalized Rational Interpoation

Theorem

Let Σ be a bilinear system. Assume that V and W are given as bases of the unions of the column spaces

$$\begin{aligned}
V_1 &= \left[(\sigma_1 I - A)^{-1}B, \ldots, (\sigma_q I - A)^{-1}B\right],\\
W_1 &= \left[(\sigma_1 I - A^T)^{-1}C, \ldots, (\sigma_q I - A^T)^{-1}C\right],\\
V_j &= \left[(\sigma_1 I - A)^{-1}NV_{k-1}, \ldots, (\sigma_q I - A)^{-1}NV_{k-1}\right],\\
W_j &= \left[(\sigma_1 I - A^T)^{-1}N^T W_{k-1}, \ldots, (\sigma_q I - A^T)^{-1}N^T W_{k-1}\right],
\end{aligned}$$

for $j \leq r$. If $\hat{\Sigma}$ is constructed by the Petrov-Galerkin projection with $P = V(W^T V)^{-1}W^T$, it holds

$$\begin{aligned}
H_j(s_1, \ldots, s_j) &= \hat{H}_j(s_1, \ldots, s_j), \qquad j \leq 2r,\\
\frac{\partial}{\partial s_k} H_j(s_1, \ldots, s_j) &= \frac{\partial}{\partial s_k} \hat{H}_j(s_1, \ldots, s_j), \quad j = 1, \ldots, r, \quad k = 1, \ldots, j.
\end{aligned}$$

Interpolatory Model Reduction

Bilinear IRKA

Algorithm 1 Bilinear Iterative Rational Krylov Algorithm (Bilinear-IRKA)

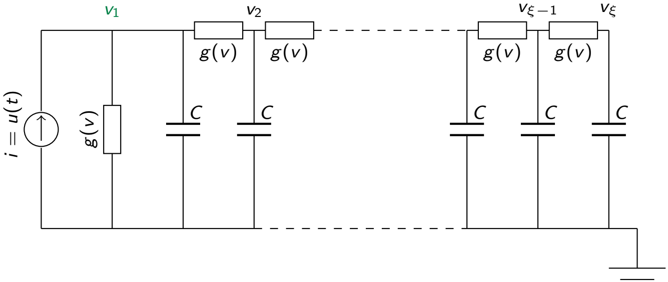
Input: A, N, B, C, r, q
Output: $\hat{A}, \hat{N}, \hat{B}, \hat{C}$

Make an initial selection $\{\sigma_1, \ldots, \sigma_q\}$.
while (change in $\sigma_i > \epsilon$) **do**
    Compute $V = [V_1, \ldots, V_r]$ and $W = [W_1, \ldots, W_r] \in \mathbb{R}^{n \times (q + \cdots + q^r)}$.
    Compute truncated SVD V_q and W_q of V and W.
    $\hat{A} = (W_q^T V_q)^{-1} W_q^T A V_q$
    $\sigma_i \leftarrow -\lambda_i(\hat{A})$
end while
$\hat{N} = (W_q^T V_q)^{-1} W_q^T N V_q, \ \hat{B} = (W_q^T V_q)^{-1} W_q^T B, \ \hat{C} = C V_q$

Remark: Exact interpolation properties are lost due to SVD.

Numerical Examples

A Nonlinear RC Circuit



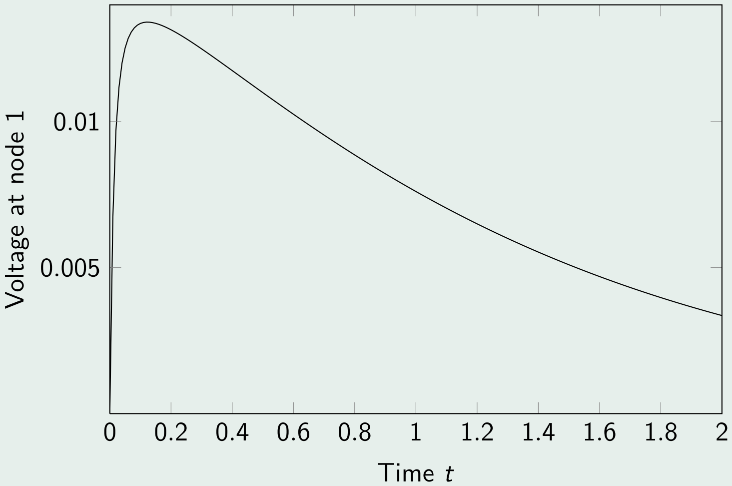
$$g(v) = e^{40v} + v - 1, \quad C = 1,$$
$$\dot{v}(t) = f(v(t), g(v(t))) + Bu(t)$$
$$y(t) = v_1(t)$$

- state-nonlinear control system
- bilinearization yields system dimension $\xi + \xi^2$

Numerical Examples

A Nonlinear RC Circuit

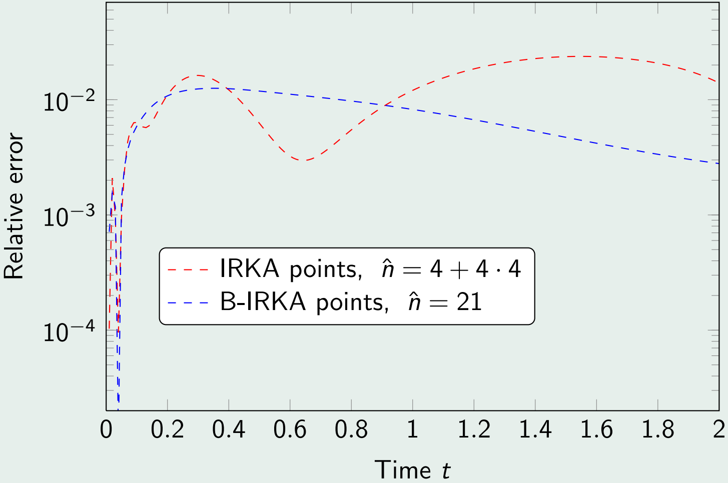
Transient response for $\xi = 100$ and $u(t) = e^{-t}$



Numerical Examples

A Nonlinear RC Circuit

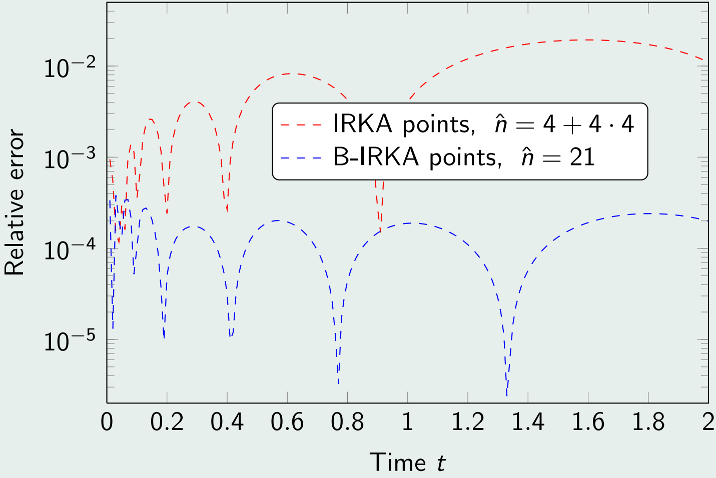
Relative errors for $\xi = 100$, $r = 2$ and $u(t) = e^{-t}$



Numerical Examples

A Nonlinear RC Circuit

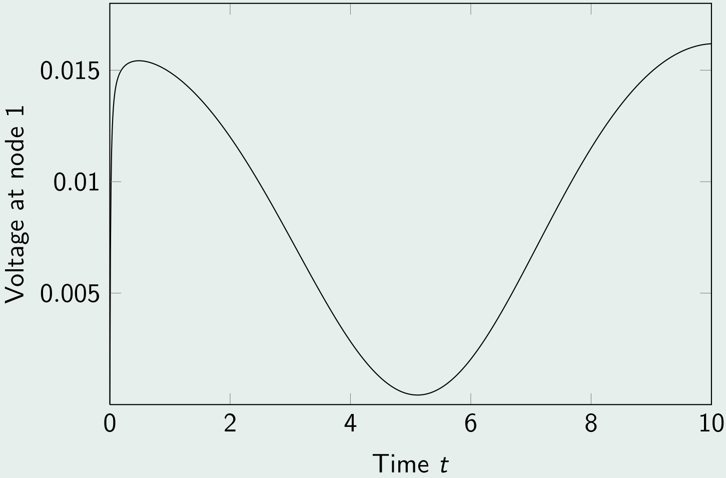
Relative errors for $n = 10100$, $r = 2$ and $u(t) = e^{-t}$



Numerical Examples

A Nonlinear RC Circuit

Transient response for $\xi = 100$ and $u(t) = \frac{1}{2}(\cos(\frac{1}{5}\pi t) + 1)$



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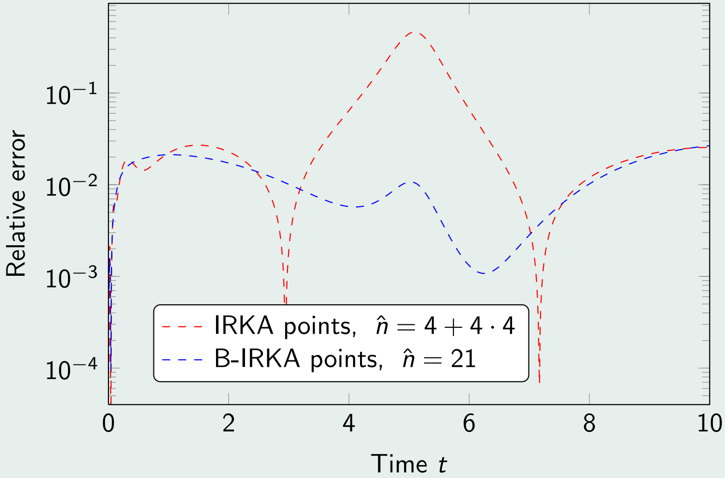

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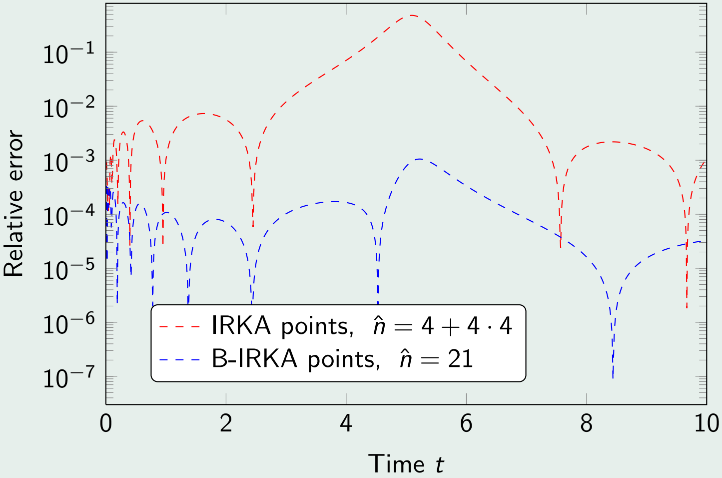

Outline

Quadratic-Bilinear Control Systems

State-Space Representation

Let us extend our focus to quadratic-bilinear systems of the form

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + A_2 x(t) \otimes x(t) + N x(t) u(t) + B u(t), \\ y(t) &= C x(t), \quad x(0) = x_0, \end{aligned}$$

where $A_1, N \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times n^2}, B, C^T \in \mathbb{R}^n$.

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- Additional quadratic term allows **exact** representations of a large class of nonlinear systems.
- Increase of state dimension, but significantly less than for Carleman.
- Transfer function approach will open up Krylov-based reduction techniques.

Quadratic-Bilinear Control Systems

Quadratic-Bilinearization

Theorem [Gu'09]

Assume that the state equation of a nonlinear system Σ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking Lie derivatives, Σ can be transformed into a quadratic-bilinear control system of dimension $N > n$.

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Example

- $\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \quad \dot{x}_2 = \sin x_2 + u$

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- $\dot{x}_1 = z_1 \cdot z_2,$

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Quadratic-Bilinear Control Systems

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Theorem [Gu'09]

Assume that the state equation of a nonlinear system Σ is given by

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Quadratic-Bilinear Control Systems

Variational Analysis and Linear Subsystems

Analysis of nonlinear systems by variational equation approach:

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- Comparison of terms $\alpha^i, i = 1, 2, \ldots$ leads to series of systems

$$\begin{aligned}
\dot{x}_1 &= A_1 x_1 + Bu, \\
\dot{x}_2 &= A_1 x_2 + A_2 x_1 \otimes x_1 + N x_1 u, \\
\dot{x}_3 &= A_1 x_3 + A_2 (x_1 \otimes x_2 + x_2 \otimes x_1) + N x_2 u \\
&\vdots
\end{aligned}$$

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- Although i-th subsystem is coupled nonlinearly to preceding systems, linear systems are obtained if terms $x_j,\ j < i$ are interpreted as pseudo inputs.

Quadratic-Bilinear Control Systems

Generalized Transfer Functions

In a similar way, a series of generalized symmetric transfer functions can be obtained via the growing exponential approach

$$H_1(s_1) = C \underbrace{(s_1 I - A_1)^{-1} B}_{G_1(s_1)},$$

$$H_2(s_1, s_2) = \frac{1}{2!} C \left((s_1 + s_2) I - A_1\right)^{-1} \left[N \left(G_1(s_1) + G_1(s_2)\right) + A_2 \left(G_1(s_1) \otimes G_1(s_2) + G_1(s_2) \otimes G_1(s_1)\right)\right],$$

$$\begin{aligned} H_3(s_1, s_2, s_3) = \frac{1}{3!} C \left((s_1 + s_2 + s_3) I - A_1\right)^{-1} & \Big[N(G_2(s_1, s_2) + G_2(s_2, s_3) + G_2(s_1, s_3)) \\ & + A_2\big(G_1(s_1) \otimes G_2(s_2, s_3) + G_1(s_2) \otimes G_2(s_1, s_3) \\ & \quad + G_1(s_3) \otimes G_2(s_1, s_3) + G_2(s_2, s_3) \otimes G_1(s_1) \\ & \quad + G_2(s_1, s_3) \otimes G_1(s_2) + G_2(s_1, s_2) \otimes G_1(s_3)\big)\Big]. \end{aligned}$$

Reduction Techniques

Approximating Transfer Functions

Let us now focus on the first two transfer functions.

- Similar to the biliner case, multimoments locally characterize transfer functions.
- In order to match derivatives up to order $q-1$, we will need the following Krylov spaces:

$$U = \mathcal{K}_q\left(A^{(\sigma)}, A^{(\sigma)}B\right)$$

for $i = 1 : q$

$$W_i = \mathcal{K}_{q-i+1}\left(A^{(2\sigma)}, A^{(2\sigma)}NU_i\right),$$

for $j = 1 : \min(q-i+1, i)$

$$Z_i = \mathcal{K}_{q-i-j+2}\left(A^{(2\sigma)}, A^{(2\sigma)}A_2(U_i \otimes U_j + U_j \otimes U_i)\right),$$

with $A^{(\sigma)} = (A_1 - \sigma I)^{-1}$ and U_i denoting the i-*th* column of U.

Reduction Techniques

Construction of Reduced System

The reduced system $\hat{\Sigma}$ is obtained by the Galerkin-Projection $P = VV^T$:

$$\hat{A}_1 = V^T A_1 V \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{A}_2 = V^T A_2 V \otimes V \in \mathbb{R}^{\hat{n}\times\hat{n}^2},$$

$$\hat{N} = V^T N V \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{B} = V^T B \in \mathbb{R}^{\hat{n}}, \quad \hat{C}^T = V^T C \in \mathbb{R}^{\hat{n}}.$$

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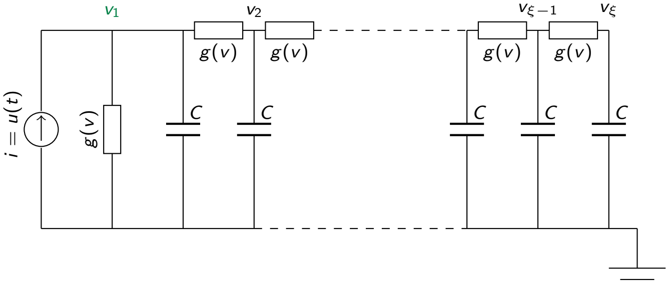
- Challenge of computing $\hat{A}_2$ has to be mastered
  $\rightarrow$ need for useful approximations.
- In contrast to original system, the above matrices are in general dense
  $\rightarrow$ computational complexity still reduced?

Numerical Examples
A Nonlinear RC-Circuit



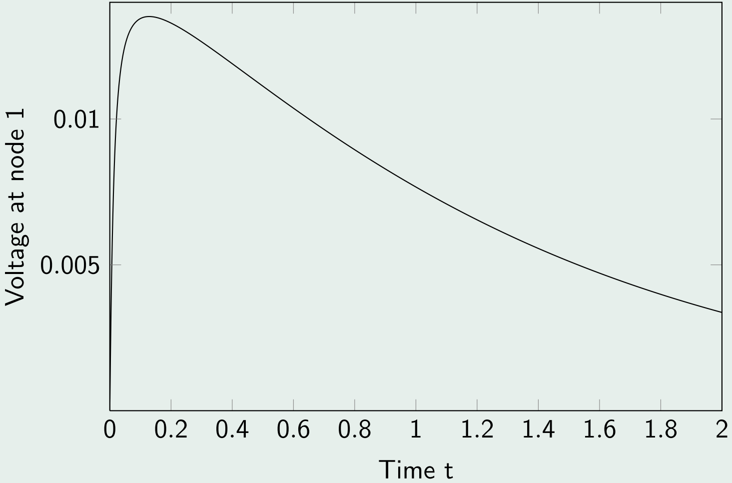
$$g(v) = e^{40v} + v - 1, \quad C = 1,$$
$$\dot{v}(t) = f(v(t), g(v(t))) + Bu(t)$$
$$y(t) = v_1(t)$$

- state-nonlinear control system
- 'clever' transformation only doubles the state dimension of the resulting quadratic-bilinear system

Numerical Examples

A Nonlinear RC-Circuit

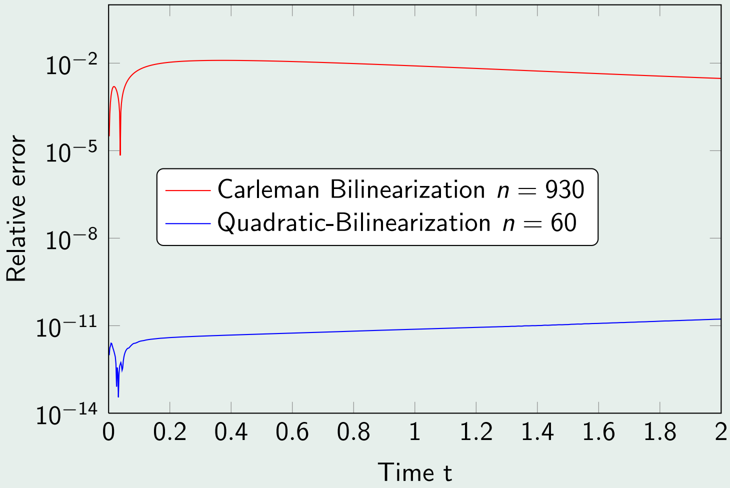
Transient response for $\xi = 30$ and $u(t) = e^{-t}$



Numerical Examples

A Nonlinear RC-Circuit

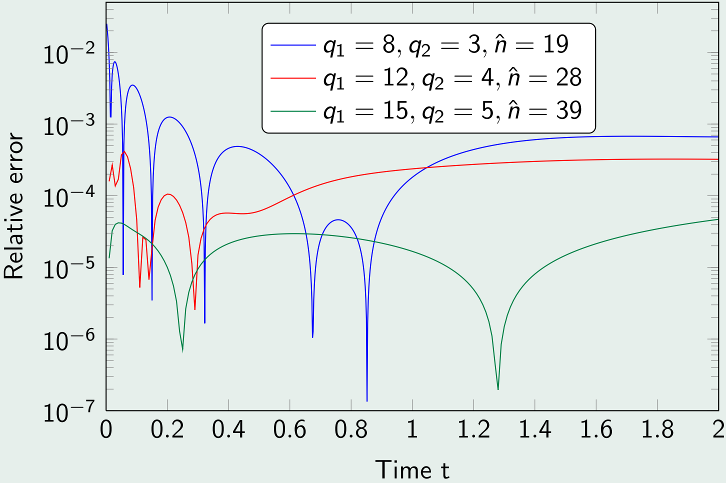
Bilinear vs Quadratic-Bilinear for $\xi = 30$ and $u(t) = e^{-t}$



Numerical Examples

A Nonlinear RC-Circuit

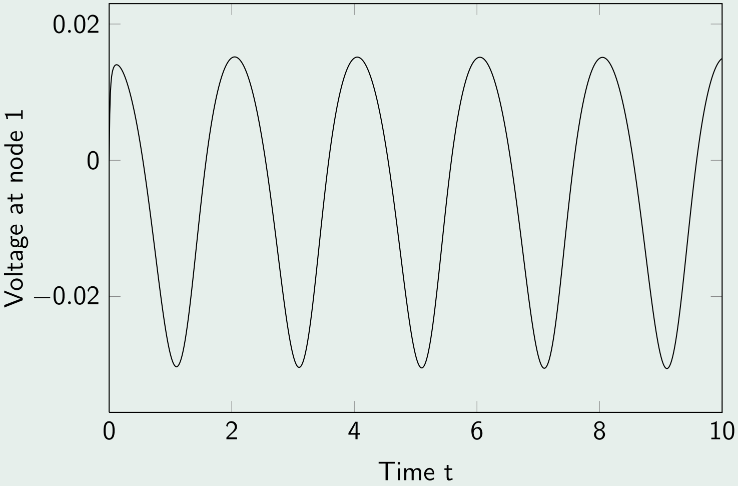
Red. Quadratic-Bilinear for $\xi = 3000, \sigma = 1$ and $u(t) = e^{-t}$



Numerical Examples

A Nonlinear RC-Circuit

Transient response for $\xi = 30$ and $u(t) = \cos(\pi t)$



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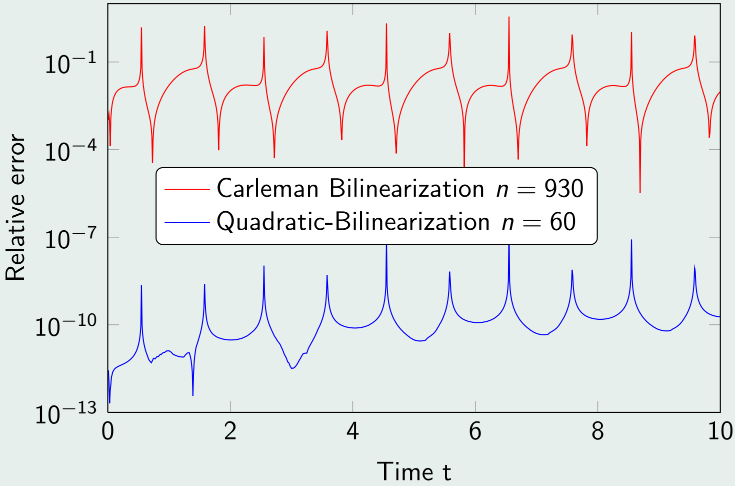

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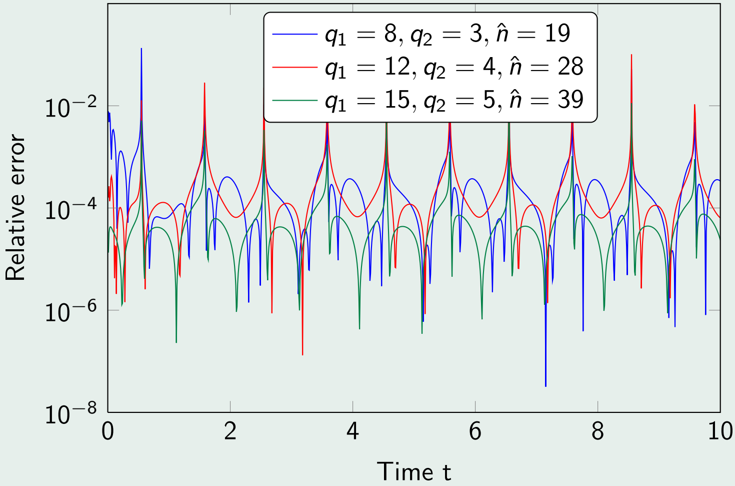

Outlook

To do:

- improve $\mathcal{H}_2$-model reduction approach for bilinear systems.
- Characterization of quadratic-bilinear systems via asymmetric transfer functions.
- Investigate possible two-sided reduction methods for quadratic-bilinear systems.
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Thank you for your attention!