# A model predictive control approach for infinite-dimensional nonlinear systems with stochastic disturbance

Peter Benner Sabine Hein

Mathematik in Industrie und Technik
Fakultät für Mathematik
Technische Universität Chemnitz

April 23, 2010

# Outline

# Introduction/Motivation

## Goal

Nonlinear feedback strategy for instationary PDE control problems.

# Introduction/Motivation

## Goal

Nonlinear feedback strategy for instationary PDE control problems.

## Why?

Application of open-loop (optimization-based) control in practice often does not lead to desired performance due to unmodeled (stochastic) disturbances.

# Introduction/Motivation

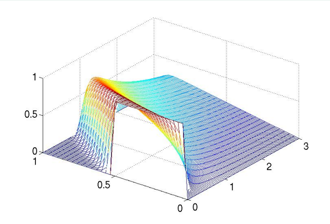
## Goal

Nonlinear feedback strategy for instationary PDE control problems.

## Why?

Application of open-loop (optimization-based) control in practice often does not lead to desired performance due to unmodeled (stochastic) disturbances.

## Example: Burgers equation with distributed control




uncontrolled

# Introduction/Motivation

## Goal

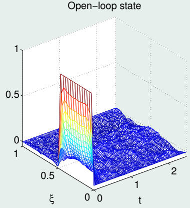
Nonlinear feedback strategy for instationary PDE control problems.

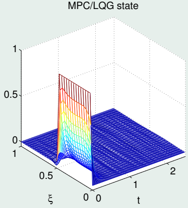
## Why?

Application of open-loop (optimization-based) control in practice often does not lead to desired performance due to unmodeled (stochastic) disturbances.

## Example: Burgers equation with distributed control

Application of optimal control to disturbed system:




open-loop (without feedback) with MPC (nonlinear feedback)

# Formulation of the Problem

## Nonlinear Optimal Control Problem

$$\min \int_{t_0}^{T_f} \langle Qy(t), y(t)\rangle + \langle Ru(t), u(t)\rangle \, dt + G(x(T_f)), \quad T_f \in (t_0, \infty],$$

subject to the semi-linear stochastic system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + B(t)u(t) + F(t)v(t), \quad t > t_0 \qquad (1) \\ x(t_0) &= x_0 + \eta_0, \quad u(t) \in \mathcal{U}, \quad x(t) \in \mathcal{X}. \end{aligned}$$

The output is given as $y(t) = C(t)x(t) + w(t), \quad y \in \mathcal{Y}$.

- $v(t), w(t)$ are unknown Gaussian disturbance processes
- If (1) is an ODE $\rightsquigarrow$ finite-dimensional problem
- If (1) is a PDE $\rightsquigarrow$ infinite-dimensional problem
  $\longrightarrow$ semi-discretization (space) $\rightsquigarrow$ ODE

# MPC/LQG Strategy

1. **Prediction step on $[t_i, t_i + T_p]$:**
   Linearize the nonlinear system dynamics around a reference $(x_r(t), u_r(t))$ to obtain the linear stochastic time-varying system

$$\begin{aligned} \dot{z}(t) &= A(t)\,z(t) + B(t)\,\tilde{u}(t) + F(t)v(t), \quad z(t_i) = z_{t_i}, \\ \tilde{y}(t) &= C(t)z(t) + w(t), \end{aligned}$$

   with $z(t) = x(t) - x_r(t)$, $\tilde{u}(t) = u(t) - u_r(t)$ and $A(t) := f'(x_r(t))$.
   If $B$, $F$, $C$, $Q$ and $R$ are time-invariant $\Rightarrow$ use an operating point $\bar{x}_r$ and $A := f'(\bar{x}_r)$ to obtain an LTI system.
2. **Optimization step on $[t_i, t_i + T_o]$, $T_o \leq T_p$:**
3. **Implementation step on $[t_i, t_i + T_c]$, $T_c \leq T_o$:**
4. **Receding horizon step:**

# MPC/LQG Strategy

1. **Prediction step on $[t_i, t_i + T_p]$:**
   Linearize the nonlinear system dynamics around a reference $(x_r(t), u_r(t))$ to obtain the linear stochastic time-varying system

$$\begin{aligned} \dot{z}(t) &= A(t)\,z(t) + B(t)\,\tilde{u}(t) + F(t)v(t), \quad z(t_i) = z_{t_i}, \\ \tilde{y}(t) &= C(t)z(t) + w(t), \end{aligned}$$

   with $z(t) = x(t) - x_r(t)$, $\tilde{u}(t) = u(t) - u_r(t)$ and $A(t) := f'(x_r(t))$.
   If $B$, $F$, $C$, $Q$ and $R$ are time-invariant $\Rightarrow$ use an operating point $\bar{x}_r$ and $A := f'(\bar{x}_r)$ to obtain an LTI system.
2. **Optimization step on $[t_i, t_i + T_o]$, $T_o \leq T_p$:**
3. **Implementation step on $[t_i, t_i + T_c]$, $T_c \leq T_o$:**
4. **Receding horizon step:**

# MPC/LQG Strategy

1. **Prediction step on** $[t_i, t_i + T_p]$**:**
2. **Optimization step on** $[t_i, t_i + T_o]$, $T_o \leq T_p$**:**
   Find the optimal control for the linear problem via the solutions of Riccati equations when applying an LQG approach.
3. **Implementation step on** $[t_i, t_i + T_c]$, $T_c \leq T_o$**:**
4. **Receding horizon step:**

# MPC/LQG Strategy

1. **Prediction step on $[t_i, t_i + T_p]$:**
2. **Optimization step on $[t_i, t_i + T_o]$, $T_o \leq T_p$:**
   Find the optimal control for the linear problem via the solutions of Riccati equations when applying an LQG approach.
   Solve the DRE and FDRE
   $$\dot{X}(t) = -A^T(t)X(t) - X(t)A(t) + X(t)B(t)R^{-1}(t)B^T(t)X(t) - \tilde{Q}(t),$$
   with $X(t_i + T_p) = G$ and $\tilde{Q}(t) = C^T(t)Q(t)C(t)$,
   $$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A^T(t) - \Sigma(t)C^T(t)W^{-1}C(t)\Sigma(t) + F(t)VF^T(t),$$
   with $\Sigma(t_i) = \Sigma_i$.
3. **Implementation step on $[t_i, t_i + T_c]$, $T_c \leq T_o$:**
4. **Receding horizon step:**

# MPC/LQG Strategy

1. **Prediction step on $[t_i, t_i + T_p]$:**
2. **Optimization step on $[t_i, t_i + T_o]$, $T_o \leq T_p$:**
   Solve the DRE and FDRE
   $$\dot{X}(t) = -A^T(t)X(t) - X(t)A(t) + X(t)B(t)R^{-1}(t)B^T(t)X(t) - \tilde{Q}(t),$$
   with $X(t_i + T_p) = G$ and $\tilde{Q}(t) = C^T(t)Q(t)C(t)$,
   $$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A^T(t) - \Sigma(t)C^T(t)W^{-1}C(t)\Sigma(t) + F(t)VF^T(t),$$
   with $\Sigma(t_i) = \Sigma_i$.
   Optimal control on $[t_i, t_i + T_o]$:
   $$u_*(t) = u_r(t) - R^{-1}(t)B^T(t)X_*(t)(\hat{x}(t) - x_r(t)),$$
   where $\hat{x}(t)$ is the estimated state resulting from the Kalman filter
   $$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + L(t)(y(t) - C(t)\hat{x}(t)) + f(x_r(t)) - A(t)x_r(t)$$
   and $L(t) = \Sigma_*(t)C^T(t)W^{-1}$.
3. **Implementation step on $[t_i, t_i + T_c]$, $T_c \leq T_o$:**
4. **Receding horizon step:**

# MPC/LQG Strategy

1. **Prediction step on $[t_i, t_i + T_p]$:**
2. **Optimization step on $[t_i, t_i + T_o]$, $T_o \leq T_p$:**
   LTI case:
   Solve the ARE and FARE

$$0 = A^T X + XA - XBR^{-1}B^T X + C^T QC,$$

$$0 = A\Sigma + \Sigma A^T - \Sigma C^T W^{-1} C\Sigma + FVF^T.$$

   Optimal control on $[t_i, t_i + T_o]$ is given by

$$u_*(t) = u_r(t) - R^{-1}B^T X_*(\hat{x}(t) - x_r(t)).$$

3. **Implementation step on $[t_i, t_i + T_c]$, $T_c \leq T_o$:**
4. **Receding horizon step:**

# MPC/LQG Strategy

1. **Prediction step on $[t_i, t_i + T_p]$:**
2. **Optimization step on $[t_i, t_i + T_o]$, $T_o \leq T_p$:**
3. **Implementation step on $[t_i, t_i + T_c]$, $T_c \leq T_o$:**
   Feed the original system with

$$u_*(t) = u_r(t) - R^{-1}(t)B^T(t)X_*(t)(\hat{x}(t) - x_r(t)),$$

   using the measurements $y(t)$ for estimating $\hat{x}(t)$ (by solving the corresponding ODEs).
4. **Receding horizon step:**

# MPC/LQG Strategy

1. **Prediction step on** $[t_i, t_i + T_p]$**:**
2. **Optimization step on** $[t_i, t_i + T_o]$, $T_o \leq T_p$**:**
3. **Implementation step on** $[t_i, t_i + T_c]$, $T_c \leq T_o$**:**
4. **Receding horizon step:**
   Set $t_i := t_i + T_c$.

# Formulation of the Problem - LTI Case

## Nonlinear Optimal Control Problem

$$\min \mathcal{J}(u) := \langle x_{T_f}, G x_{T_f} \rangle_{\mathcal{X}} + \int_0^{T_f} \langle x(t), C^* Q C x(t) \rangle_{\mathcal{X}} + \langle u(t), Ru(t) \rangle_{\mathcal{U}} \, dt,$$

$$\text{subject to} \quad \begin{aligned} \dot{x}(t) &= f(x(t)) + Bu(t) + Fv(t), \quad t > 0, \\ y(t) &= Cx(t) + w(t), \quad t > 0, \\ x(0) &= x_0 + \eta. \end{aligned}$$

- $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{U}$ are Hilbert spaces, $f : \mathcal{D}(f) \subseteq \mathcal{X} \to \mathcal{X}$ nonlinear map
- $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $F \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $G \in \mathcal{L}(\mathcal{X})$,
- $Q \in \mathcal{L}(\mathcal{Y})$ , $R, R^{-1} \in \mathcal{L}(\mathcal{U})$, all self-adjoint and nonnegative and $\langle \nu, R\nu \rangle \geq \alpha ||\nu||^2$ for all $\nu \in \mathcal{U}$ and some $\alpha > 0$,
- $x_0 \in \mathcal{X}$ and $\eta$ is a zero mean Gaussian random variable on $\mathcal{X}$ with covariance $\Sigma_0$,
- $v(t)$ and $w(t)$ are Wiener processes (Gaussian and zero mean) on Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ with incremental covariance operators $V \in \mathcal{L}(\mathcal{U})$ and $W, W^{-1} \in \mathcal{L}(\mathcal{Y})$, respectively.

# Linearization - LTI Case

- Assume that $f(x)$ is Fréchet-differentiable.
- Linearization on small intervals $[t_i, t_i + T_p]$ around a reference pair $(x_r(t), u_r(t))$ and partially replace $x_r(t)$ by a stationary operating point $\bar{x}_r$.

**LTI Problem in Differential Form on $[t_i, t_i + T_p]$**

$$
\begin{aligned}
dz(t) &= Az(t)dt + B\tilde{u}(t)dt + Fdv(t), \quad t_i < t < t_i + T_p, \\
d\tilde{y}(t) &= Cz(t)dt + dw(t), \quad t_i < t < t_i + T_p, \\
z(t_i) &= z_{t_i},
\end{aligned}
$$

with $z(t) := h(t) = x(t) - x_r(t)$, $\tilde{u}(t) = u(t) - u_r(t)$ and

$$
f(x_r + h)(t) \approx f(x_r(t)) + Ah(t),
$$

where $A := f'(\bar{x}_r)$ is the Fréchet derivative, evaluated at $x_r(t) = \bar{x}_r$.

# Linearization - LTI Case

To avoid problems of existence and uniqueness we use the

**Integral Form on $[t_i, t_i + T_p]$**

$$
\begin{aligned}
z(t) &= T_{t-t_i} z(t_i) + \int_{t_i}^{t} T_{t-s} B\tilde{u}(s)\, ds + \int_{t_i}^{t} T_{t-s} F\, dv(s), \\
&\quad t_i \le s \le t \le t_i + T_p, \\
\tilde{y}(t) &= \int_{t_i}^{t} Cz(s)\, ds + w(t), \quad t_i < t \le t_i + T_p, \\
z(t_i) &= z_{t_i},
\end{aligned}
$$

where $T_t$ is a strongly continuous semigroup on $\mathcal{X}$ generated by $A$ on $[t_i, t_i + T_p]$.

# Solution of the MPC/LQG/LTI Problem on $[t_i, t_i + T_p]$, $T_p < \infty$

Optimal control $u_*(t) = u_r(t) - R^{-1}B^*\Pi(t)\big(\hat{x}_*(t) - x_r(t)\big)$.
Estimated state is given by

$$\hat{x}_*(t) = U(t,t_i)\hat{x}(t_i) + \int_{t_i}^{t} U(t,s)\Sigma(s)C^*W^{-1}dy(s) + \int_{t_i}^{t} U(t,s)(f(x_r(s)) - Ax_r(s))ds,$$

where $U(t,s)$ is the quasi-evolution operator generated by

$$A - BR^{-1}B^*\Pi(t) - \Sigma(t)C^*W^{-1}C,$$

and $\Pi(t)$ and $\Sigma(t)$ are the unique solutions of the ODRE

$$\frac{d}{dt}\langle\Pi(t)\varphi,\psi\rangle =$$
$$\langle\Pi(t)BR^{-1}B^*\Pi(t)\varphi,\psi\rangle - \langle\Pi(t)\varphi, A\psi\rangle - \langle A\varphi, \Pi(t)\psi\rangle - \langle\varphi, C^*QC\psi\rangle,$$

for all $\varphi,\psi \in \mathcal{D}(A)$ and $\Pi(t_i + T_p) = G$ and the OFDRE

$$\frac{d}{dt}\langle\Sigma(t)\varphi,\psi\rangle =$$
$$\langle\Sigma(t)\varphi, A^*\psi\rangle + \langle A^*\varphi, \Sigma(t)\psi\rangle - \langle\Sigma(t)C^*W^{-1}C\Sigma(t)\varphi,\psi\rangle + \langle\varphi, FVF^*\psi\rangle,$$

for all $\varphi,\psi \in \mathcal{D}(A^*)$ and $\Sigma(t_i) = \Sigma_0$.

# Solution to the MPC/LQG/LTI Problem on $[t_i, t_i + T_p]$, $T_p = \infty$

The optimal control and corresponding estimated state on $[t_i, t_i + T_p]$ are given by

$$u_*(t) = u_r(t) - R^{-1}B^*\Pi_\infty\big(\hat{x}_*(t) - x_r(t)\big),$$

$$\hat{x}_*(t) = T_{t_i}\hat{x}(t_i) + \int_{t_i}^{t} T_{t-s}\Sigma_\infty C^* W^{-1}\, dy(s) + \int_{t_i}^{t} T_{t-s}(f(x_r(s)) - Ax_r(s))\, ds,$$

where $T_t$ is the strongly continuous semigroup generated by

$$A - BR^{-1}B^*\Pi_\infty - \Sigma_\infty C^* W^{-1} C,$$

and $\Pi_\infty$ and $\Sigma_\infty$ are the unique nonnegative, self-adjoint solutions of the OARE and OFARE

$$0 = A^*\Pi + \Pi A - \Pi BR^{-1}B^*\Pi + C^*QC\,,$$

$$0 = A\Sigma + \Sigma A^* - \Sigma C^* W^{-1} C\Sigma + FVF^*\,.$$

# Formulation of the Problem — LTV Case

## Integral Form after Linearization on $[t_i, t_i + T_p]$

$$
\begin{aligned}
z(t) &= U(t,t_i)z(t_i) + \int_{t_i}^{t} U(t,s)B(s)\tilde{u}(s)\,ds + \int_{t_i}^{t} U(t,s)F(s)\,dv(s),\\
&\quad t_i \leq s \leq t \leq t_i + T_p,\\
z(t_i) &= z_0 + \eta \text{ if } t = 0 \text{ or } z(t_i) \text{ is given from the last interval for } t > 0,\\
\tilde{y}(t) &= \int_{t_i}^{t} C(s)z(s)\,ds + w(t),
\end{aligned}
$$

where $U(t,s)$ is the mild evolution operator associated with $A(t)$.

- $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are real Hilbert spaces,
- $B \in \mathcal{B}^\infty(t_i, t_i + T_p; \mathcal{L}(\mathcal{U}, \mathcal{X}))$, $F \in \mathcal{B}^\infty(t_i, t_i + T_p; \mathcal{L}(\mathcal{U}, \mathcal{X}))$,
- $C \in \mathcal{B}^\infty(t_i, t_i + T_p; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$, $Q \in \mathcal{B}^\infty(t_i, t_i + T_p; \mathcal{L}(\mathcal{Y}))$,
- $R \in \mathcal{B}^\infty(t_i, t_i + T_p; \mathcal{L}(\mathcal{U}))$, $V \in \mathcal{L}(\mathcal{U})$, $W \in \mathcal{L}(\mathcal{Y})$ and $z_0 \in \mathcal{X}$

# Solution to the MPC/LQG/LTV Problem on $[t_i, t_i + T_p]$

The optimal control and corresponding estimated state on $[t_i, t_i + T_p]$ are given by

$$u_*(t) = u_r(t) - R^{-1}(t)B^*(t)\Pi(t)\big(\hat{x}_*(t) - x_r(t)\big),$$

$$\hat{x}_*(t) = U_{\Pi\Sigma}(t, t_i)\hat{x}(t_i) + \int_{t_i}^{t} U_{\Pi\Sigma}(t,s)\Sigma(s)C^*(s)W^{-1}\, d\tilde{y}(s) + \int_{t_i}^{t} U_{\Pi\Sigma}(t,s)(f(x_r(s)) - Ax_r(s))\, ds,$$

where $U_{\Pi\Sigma}(t,s)$ is the quasi-evolution operator generated by

$$A(t) - B(t)R^{-1}(t)B^*(t)\Pi(t) - \Sigma(t)C^*(t)W^{-1}C(t)$$

and $\Pi(t)$ and $\Sigma(t)$ are the unique solutions of the IRE and FIRE.

# Solution to the MPC/LQG/LTV Problem on $[t_i, t_i + T_p]$

IRE and FIRE:

$$\Pi(t)\varphi = \int_t^{t_i+T_p} U_\Pi^*(s,t)\Big[C^*(s)Q(s)C(s) + \Pi(s)B(s)R^{-1}(s)B^*(s)\Pi(s)\Big]U_\Pi(s,t)\varphi\,ds$$

$$+U_\Pi^*(t_i+T_p,t)GU_\Pi(t_i+T_p,t)\varphi,$$

$$\Sigma(t)\varphi = \int_{t_i}^{t} U_\Sigma(t,s)\Big[F(s)VF^*(s) + \Sigma(s)C^*(s)W^{-1}C(s)\Sigma(s)\Big]U_\Sigma^*(t,s)\varphi\,ds$$

$$+U_\Sigma(t,t_i)\Sigma_0 U_\Sigma^*(t,t_i)\varphi,$$

where $U_\Pi$ is the quasi-evolution operator generated by

$$A(t) - B(t)R^{-1}(t)B^*(t)\Pi(t)$$

and $U_\Sigma$ is the quasi-evolution operator generated by

$$A(t) - \Sigma(t)C^*(t)W^{-1}C(t).$$

# An Example: The Burgers Equation

## Burgers Equation

$$x_t(t,\xi) = \nu\, x_{\xi\xi}(t,\xi) - x(t,\xi)\, x_\xi(t,\xi), \quad \text{on } (0, T_f] \times (0,1)$$
$$x(t,0) = x(t,1) = 0, \quad t \in (0, T_f],$$
$$x(0,\xi) = x_0(\xi), \quad \xi \in (0,1)$$

Choose $\mathcal{X} = L^2(0,1)$ and define $D_\xi z = \frac{dz}{d\xi}$ with $\mathcal{D}(D_\xi) = \{z \in L^2(0,1) \,|\, z \text{ is absolutely continuous}, \frac{dz}{dx} \in L^2(0,1),\ z(0) = z(1) = 0\}$.

## Abstract Burgers Equation

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0, \quad \text{with} \quad f(x) = \nu D_\xi^2 x - x D_\xi x$$

- Linearization: $A(t)h = f'(x_r)h = \nu D_\xi^2 h - D_\xi(x_r h)$
- Replace $x_r(t)$ by stationary operating point $\bar{x}_r$:

$$Ah = f'(\bar{x}_r)h = \nu D_\xi^2 h - D_\xi(\bar{x}_r h)$$

# An Example: The Burgers Equation

## Abstract Burgers Equation

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0, \quad \text{with} \quad f(x) = \nu D_\xi^2 x - x D_\xi x$$

$$Ah = f'(\bar{x}_r)h = \nu D_\xi^2 h - D_\xi(\bar{x}_r h)$$

## Question

Does $A$ generates a strongly continuous semigroup?

## Lemma [Curtain/Zwart '95]

A closed, densely defined operator on a Hilbert space is an infinitesimal generator of a strongly continuous semigroup satisfying $||T_t|| \leq e^{\omega t}$, $\omega < 0$, if

$$\begin{aligned} \Re\langle Az, z\rangle &\leq \omega||z||^2 \quad \text{for } z \in \mathcal{D}(A), \\ \Re\langle A^* z, z\rangle &\leq \omega||z||^2 \quad \text{for } z \in \mathcal{D}(A). \end{aligned}$$

# An Example: The Burgers Equation

$$Ah = f'(\bar{x}_r)h = \nu D_\xi^2 h - D_\xi(\bar{x}_r h)$$

It can be shown that $D_\xi$ and $D_\xi^2$ are densely defined, closed operators, see [Curtain/Zwart '95 ].

$\Rightarrow$ $A$ is a densely defined, closed operator.

Could show (using the Poincaré inequality and the Cauchy inequality with $\epsilon = \frac{\nu}{2}$):

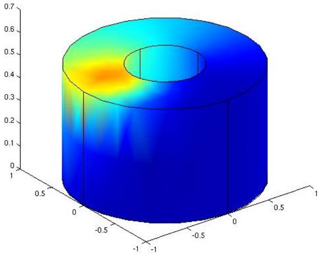
$$\langle Az, z\rangle \leq \left(\frac{||x_r||_\infty^2}{2\nu} - \frac{\nu}{2}\lambda_0\right)||z||^2.$$

**Corollary**

If $||x_r||_\infty^2 \leq 2\omega\nu + \nu^2\lambda_0$ the requirement $\langle Az, z\rangle \leq \omega||z||^2$ can be fulfilled and $A$ generates a strongly continuous semigroup. In the case of $\omega = 0$ and $||x_r||_\infty^2$ satisfying $||x_r||_\infty^2 \leq \nu^2\lambda_0$, the operator $A$ is dissipative and generates a contraction semigroup.

The same can be shown for the adjoint operator.

# Numerical Results: 3D-Reaction-Diffusion System




- Aim: model a chemical or biological process where the species involved are subjected to diffusion and reaction among each other.
- Modeled by a coupled system of reaction-diffusion equations ($i = 1, 2$):

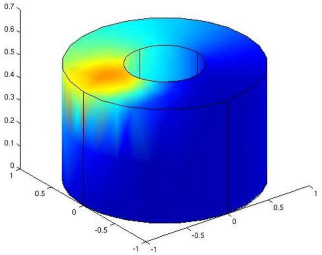
$$
\begin{aligned}
(c_i)_t(x,t) &= D_i \Delta c_i(x,t) - k c_1(x,t) c_2(x,t) \text{ on } \Omega \times (0,T), \\
c_i(x,0) &= c_{i0}(x) + \eta_i(x) \text{ on } \Omega, \\
\frac{\partial}{\partial n} c_1(x,t) &= 0 \text{ on } \delta\Omega \times (0,T),\ \frac{\partial}{\partial n} c_2(x,t) = 0 \text{ on } (\delta\Omega \setminus \delta\Omega_u) \times (0,T), \\
\frac{\partial}{\partial n} c_2(x,t) &= \alpha(x,t) u(t) \text{ on } \delta\Omega_u \times (0,T).
\end{aligned}
$$

  - $\alpha$ models a counter-clockwise revolving nozzle around the upper annular surface.
  - $u(t)$ describes the intensity of the spray.

Source: Griesse/Volkwein, *SIAM J. Cont. Optim.*, 44(2), 2005.

# Numerical Results: 3D-Reaction-Diffusion System

- Aim: model a chemical or biological process where the species involved are subjected to diffusion and reaction among each other.
- Modeled by a coupled system of reaction-diffusion equations ($i = 1, 2$):



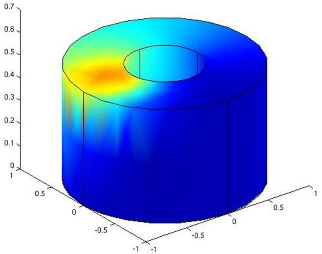

$$
\begin{aligned}
(c_i)_t(x,t) &= D_i \Delta c_i(x,t) - k c_1(x,t) c_2(x,t) \text{ on } \Omega \times (0,T),\\
c_i(x,0) &= c_{i0}(x) + \eta_i(x) \text{ on } \Omega,\\
\frac{\partial}{\partial n} c_1(x,t) &= 0 \text{ on } \delta\Omega \times (0,T),\ \frac{\partial}{\partial n} c_2(x,t) = 0 \text{ on } (\delta\Omega \setminus \delta\Omega_u) \times (0,T),\\
\frac{\partial}{\partial n} c_2(x,t) &= \alpha(x,t) u(t) \text{ on } \delta\Omega_u \times (0,T).
\end{aligned}
$$

- $\alpha$ models a counter-clockwise revolving nozzle around the upper annular surface.
- $u(t)$ describes the intensity of the spray.

Source: Griesse/Volkwein, *SIAM J. Cont. Optim.*, 44(2), 2005.

# Numerical Results: 3D-Reaction-Diffusion System

- Aim: model a chemical or biological process where the species involved are subjected to diffusion and reaction among each other.
- Modeled by a coupled system of reaction-diffusion equations ($i = 1, 2$):

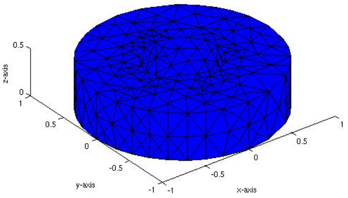



$$
\begin{aligned}
(c_i)_t(x,t) &= D_i \Delta c_i(x,t) - k c_1(x,t) c_2(x,t) \text{ on } \Omega \times (0,T),\\
c_i(x,0) &= c_{i0}(x) + \eta_i(x) \text{ on } \Omega,\\
\frac{\partial}{\partial n} c_1(x,t) &= 0 \text{ on } \delta\Omega \times (0,T),\ \frac{\partial}{\partial n} c_2(x,t) = 0 \text{ on } (\delta\Omega \setminus \delta\Omega_u) \times (0,T),\\
\frac{\partial}{\partial n} c_2(x,t) &= \alpha(x,t) u(t) \text{ on } \delta\Omega_u \times (0,T).
\end{aligned}
$$

Goal: control intensity $u(t)$ to achieve desired terminal concentrations of the substances.

# Numerical Results: 3D-Reaction-Diffusion System

- Semi-discretization in space by using piecewise linear and globally continuous ($P_1$) finite elements on tetrahedra.
- After linearization on each interval we obtain the linear system




$$\mathbf{M}\dot{z}(t) = \mathbf{A}(\mathbf{t})z(t) + \mathbf{B}(\mathbf{t})(\tilde{u}(t) + v(t)), \quad z(t_i) = z_{t_i}, \text{ on } [t_i, t_i + T_p],$$

with

$$\mathbf{A} = \begin{bmatrix} -D_1 K - kM\mathrm{diag}(c_{r2}(t)) & -kM\mathrm{diag}(c_{r1}(t)) \\ -kM\mathrm{diag}(c_{r2}(t)) & -D_2 K - kM\mathrm{diag}(c_{r1}(t)) \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ G(t_i) \end{bmatrix}, \quad z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}.$$

# Numerical Results: 3D-Reaction-Diffusion System

- **LTI/ARE** time-invariant system matrices on each horizon, nozzle is fixed in the middle of the control interval, solve AREs
- **LTI/DRE** time-invariant system matrices on each horizon, nozzle is fixed in the middle of the control interval, solve DREs
- **LTV/DRE-At** time-varying $A$, nozzle is fixed in the middle of the control interval, solve DREs
- **LTV/DRE-AtBt** time-varying system matrices on each horizon, nozzle position changes in each time step, solve DREs

**Parameters:**

$D_1 = 0.15,\ D_2 = 0.2,\ k = 1,\ c_{10} = 1,\ c_{20} = 0,\ T = 1,\ dt = 0.01,$
$C = Q = I_{594},\ R = 10,\ \sigma(v) = \sigma(w) = 0.5,\ \eta = 0$

**Aim:** Steer $c_1$ to zero by spraying the second substance onto the reactor.

**Software:**

MATLAB: basic routines, FEMLAB: FEM, LyaPack 1.8: AREs
DREs were solved with an adapted BDF code [Mena 07]

# Numerical Results: 3D-Reaction-Diffusion System

- **LTI/ARE** time-invariant system matrices on each horizon, nozzle is fixed in the middle of the control interval, solve AREs
- **LTI/DRE** time-invariant system matrices on each horizon, nozzle is fixed in the middle of the control interval, solve DREs
- **LTV/DRE-At** time-varying $A$, nozzle is fixed in the middle of the control interval, solve DREs
- **LTV/DRE-AtBt** time-varying system matrices on each horizon, nozzle position changes in each time step, solve DREs

**Parameters:**

$D_1 = 0.15,\ D_2 = 0.2,\ k = 1,\ c_{10} = 1,\ c_{20} = 0,\ T = 1,\ dt = 0.01,$
$C = Q = I_{594},\ R = 10,\ \sigma(v) = \sigma(w) = 0.5,\ \eta = 0$

**Aim:** Steer $c_1$ to zero by spraying the second substance onto the reactor.

**Software:**

MATLAB: basic routines, FEMLAB: FEM, LyaPack 1.8: AREs
DREs were solved with an adapted BDF code [Mena 07]

# Numerical Results: 3D-Reaction-Diffusion System

- **LTI/ARE** time-invariant system matrices on each horizon, nozzle is fixed in the middle of the control interval, solve AREs
- **LTI/DRE** time-invariant system matrices on each horizon, nozzle is fixed in the middle of the control interval, solve DREs
- **LTV/DRE-At** time-varying $A$, nozzle is fixed in the middle of the control interval, solve DREs
- **LTV/DRE-AtBt** time-varying system matrices on each horizon, nozzle position changes in each time step, solve DREs

**Parameters:**

$D_1 = 0.15,\ D_2 = 0.2,\ k = 1,\ c_{10} = 1,\ c_{20} = 0,\ T = 1,\ dt = 0.01,$
$C = Q = I_{594},\ R = 10,\ \sigma(v) = \sigma(w) = 0.5,\ \eta = 0$

**Aim:** Steer $c_1$ to zero by spraying the second substance onto the reactor.

**Software:**

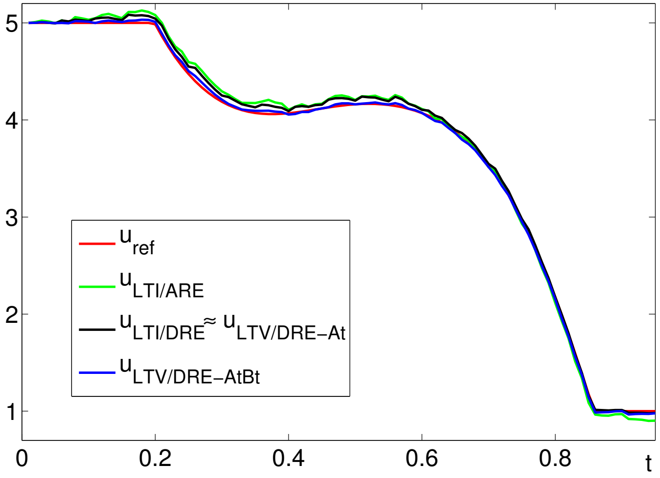
MATLAB: basic routines, FEMLAB: FEM, LyaPack 1.8: AREs
DREs were solved with an adapted BDF code [Mena 07]

# Numerical Results: 3D-Reaction-Diffusion System

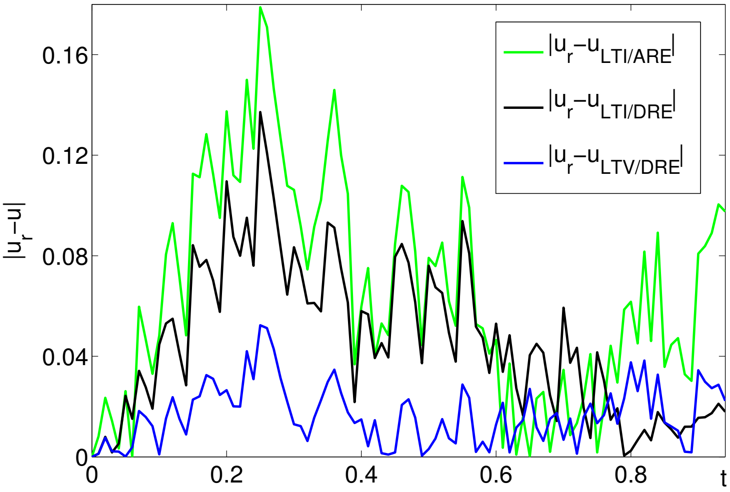
| $T_p$ | $T_c$ | Type | $J$ | $\int_0^{\tilde{T}} z_1^T z_1\, dt$ | $\int_0^{\tilde{T}} z_2^T z_2\, dt$ | $\int_0^{\tilde{T}} \tilde{u}^T \tilde{u}\, dt$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.05 | LTI/ARE | 0.644872 | 0.067804 | 0.521638 | 0.005543 |
| | | LTI/DRE | 0.623733 | 0.070703 | 0.524184 | 0.002885 |
| | | LTV-At | 0.624833 | 0.070120 | 0.525954 | 0.002876 |
| | | LTV-AtBt | 0.129287 | 0.068377 | 0.057168 | 0.000374 |
| 0.05 | 0.05 | LTI/ARE | 0.646785 | 0.067504 | 0.523364 | 0.005592 |
| | | LTI/DRE | 0.612729 | 0.068944 | 0.529253 | 0.001453 |
| | | LTV-At | 0.612223 | 0.068680 | 0.529031 | 0.001451 |
| | | LTV-AtBt | 0.131773 | 0.068104 | 0.061985 | 0.000168 |
| 0.1 | 0.1 | LTI/ARE | 0.823303 | 0.061546 | 0.687169 | 0.007459 |
| | | LTI/DRE | 0.812116 | 0.061004 | 0.724103 | 0.002701 |
| | | LTV-At | 0.809999 | 0.060147 | 0.722932 | 0.002692 |
| | | LTV-AtBt | 0.145055 | 0.067758 | 0.073827 | 0.000347 |

$$J = \int_0^{\tilde{T}} z^T Q z + \tilde{u}^T R \tilde{u}\, dt, \qquad \tilde{T} = 0.91$$

# Numerical Results: 3D-Reaction-Diffusion System



# Numerical Results: 3D-Reaction-Diffusion System



# Numerical Results: 3D-Reaction-Diffusion System

**Optimized trajectory, no disturbances**

# Numerical Results: 3D-Reaction-Diffusion System

**Disturbed trajectory, with MPC/LQG feedback**