SIAM Conference on Optimization
Darmstadt, May 16–19, 2011

CONTROL-ORIENTED MODEL REDUCTION FOR PARABOLIC SYSTEMS

Peter Benner

Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg

Overview

Distributed Parameter Systems

Parabolic PDEs as infinite-dimensional systems

Given Hilbert spaces

$\mathcal{X}$ – state space,

$\mathcal{U}$ – control space,

$\mathcal{Y}$ – output space,

and linear operators

$$\begin{aligned} \mathbf{A} &: \quad \mathrm{dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}, \\ \mathbf{B} &: \quad \mathcal{U} \to \mathcal{X}, \\ \mathbf{C} &: \quad \mathcal{X} \to \mathcal{Y}. \end{aligned}$$

Linear Distributed Parameter System (DPS)

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}\mathbf{x}, \end{cases} \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X},$$

i.e., abstract evolution equation together with observation equation.

Distributed Parameter Systems

Parabolic PDEs as infinite-dimensional systems

Given Hilbert spaces

$\mathcal{X}$ – state space,

$\mathcal{U}$ – control space,

$\mathcal{Y}$ – output space,

and linear operators

$$\begin{aligned} \mathbf{A} :&\quad \mathrm{dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}, \\ \mathbf{B} :&\quad \mathcal{U} \to \mathcal{X}, \\ \mathbf{C} :&\quad \mathcal{X} \to \mathcal{Y}. \end{aligned}$$

Linear Distributed Parameter System (DPS)

$$\Sigma : \left\{ \begin{array}{rcl} \dot{\mathbf{x}} &=& \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} &=& \mathbf{C}\mathbf{x}, \end{array} \right. \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X},$$

i.e., abstract evolution equation together with observation equation.

Parabolic Systems

The state $x = x(t,\xi)$ is a (weak) solution of a parabolic PDE with $(t,\xi) \in [0,T] \times \Omega$, $\Omega \subset \mathbb{R}^d$:

$$\partial_t x - \nabla(a(\xi).\nabla x) + b(\xi).\nabla x + c(\xi)x = B_{pc}(\xi)u(t), \quad \xi \in \Omega, \ t > 0,$$

with initial and boundary conditions

$$\begin{aligned} \alpha(\xi)x + \beta(\xi)\partial_\eta x &= B_{bc}(\xi)u(t), && \xi \in \partial\Omega, \quad t \in [0,T], \\ x(0,\xi) &= x_0(\xi) \in \mathcal{X}, && \xi \in \Omega, \\ y(t) &= C(\xi)x, && \xi \in \Omega, \quad t \in [0,T]. \end{aligned}$$

- $B_{pc} = 0 \Longrightarrow$ boundary control problem
- $B_{bc} = 0 \Longrightarrow$ point control problem

Infinite-Dimensional Systems

Basic assumption:

The system $\Sigma(A, B, C)$ has a transfer function

$$\mathbf{G} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \in L_\infty.$$

If, in addition, $\mathbf{A}$ generates an exponentially stable C_0-semigroup, then $\mathbf{G}$ is in the Hardy space H_∞.

Infinite-Dimensional Systems

Basic assumption:

The system $\Sigma(A, B, C)$ has a transfer function

$$\mathbf{G} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \in L_\infty.$$

If, in addition, $\mathbf{A}$ generates an exponentially stable C_0-semigroup, then $\mathbf{G}$ is in the Hardy space H_∞.

Possible settings:

1. Basic setting in infinite-dimensional system theory:
   - $\mathbf{A}$ generates C_0-semigroup $T(t)$ on $\mathcal{X}$;
   - $(\mathbf{A}, \mathbf{B})$ is exponentially stabilizable, i.e., there exists $\mathbf{F} : \mathrm{dom}(\mathbf{A}) \mapsto \mathcal{U}$ s.t. $\mathbf{A} + \mathbf{BF}$ generates an exponentially stable C_0-semigroup $\mathbf{S}(\mathbf{t})$;
   - $(\mathbf{A}, \mathbf{C})$ is exponentially detectable, i.e., $(\mathbf{A}^*, \mathbf{C}^*)$ is exponentially stabilizable;
   - $\mathbf{B}, \mathbf{C}$ are finite-rank and bounded, e.g., $\mathcal{U} = \mathbb{R}^m$, $\mathcal{Y} = \mathbb{R}^p$.
2. $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is Pritchard-Salomon, allows certain unboundedness of $\mathbf{B}, \mathbf{C}$.
3. . . . ?

(Exponentially) Stable Systems

G is the Laplace transform of

$$\mathbf{h}(t) := \mathbf{C}T(t)\mathbf{B}$$

and symbol of the Hankel operator $\mathbf{H} : L_2(0,\infty;\mathbb{R}^m) \mapsto L_2(0,\infty;\mathbb{R}^p)$,

$$(\mathbf{Hu})(t) := \int_0^\infty \mathbf{h}(t+\tau)u(\tau)\,d\tau.$$

$\mathbf{H}$ is compact with countable many singular values σ_j, $j = 1,\ldots,\infty$, called the Hankel singular values (HSVs) of $\mathbf{G}$. Moreover,

$$\sum_{j=1}^{\infty} \sigma_j < \infty.$$

HSVs are system invariants, used for approximation similar to truncated SVD.
The 2-induced operator norm is the H_∞ norm; here,

$$\|\mathbf{G}\|_{H_\infty} = \sum_{j=1}^{\infty} \sigma_j.$$

(Exponentially) Stable Systems

G is the Laplace transform of

$$\mathbf{h}(t) := \mathbf{C}T(t)\mathbf{B}$$

and symbol of the Hankel operator $\mathbf{H} : L_2(0,\infty;\mathbb{R}^m) \mapsto L_2(0,\infty;\mathbb{R}^p)$,

$$(\mathbf{H}\mathbf{u})(t) := \int_0^\infty \mathbf{h}(t+\tau)u(\tau)\,d\tau.$$

H is compact with countable many singular values σ_j, $j = 1,\ldots,\infty$, called the Hankel singular values (HSVs) of **G**. Moreover,

$$\sum\nolimits_{j=1}^{\infty} \sigma_j < \infty.$$

HSVs are system invariants, used for approximation similar to truncated SVD.
The 2-induced operator norm is the H_∞ norm; here,

$$\|\mathbf{G}\|_{H_\infty} = \sum\nolimits_{j=1}^{\infty} \sigma_j.$$

(Exponentially) Stable Systems

G is the Laplace transform of

$$\mathbf{h}(t) := \mathbf{C}T(t)\mathbf{B}$$

and symbol of the Hankel operator $\mathbf{H} : L_2(0,\infty;\mathbb{R}^m) \mapsto L_2(0,\infty;\mathbb{R}^p)$,

$$(\mathbf{H}\mathbf{u})(t) := \int_0^\infty \mathbf{h}(t+\tau)u(\tau)\,d\tau.$$

H is compact with countable many singular values σ_j, $j = 1,\ldots,\infty$, called the Hankel singular values (HSVs) of **G**. Moreover,

$$\sum_{j=1}^{\infty} \sigma_j < \infty.$$

HSVs are system invariants, used for approximation similar to truncated SVD.
The 2-induced operator norm is the H_∞ norm; here,

$$\|\mathbf{G}\|_{H_\infty} = \sum_{j=1}^{\infty} \sigma_j.$$

Controller Design for Parabolic Systems

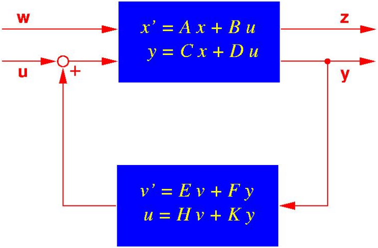
Designing a controller for parabolic control systems requires semi-discretization in space, control design for n-dim. system.

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$



Real-time control is only possible with controllers of low complexity.

$\rightsquigarrow$ Modern feedback control for parabolic systems w/o model reduction impossible due to large scale of discretized systems.

Controller Design for Parabolic Systems

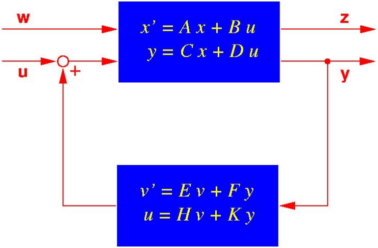
Designing a controller for parabolic control systems requires semi-discretization in space, control design for n-dim. system.

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$



Real-time control is only possible with controllers of low complexity.

$\rightsquigarrow$ Modern feedback control for parabolic systems w/o model reduction impossible due to large scale of discretized systems.

Controller Design for Parabolic Systems

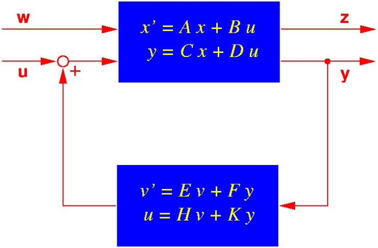
Designing a controller for parabolic control systems requires semi-discretization in space, control design for n-dim. system.

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$



Real-time control is only possible with controllers of low complexity.

$\rightsquigarrow$ Modern feedback control for parabolic systems w/o model reduction impossible due to large scale of discretized systems.

Controller Design for Parabolic Systems

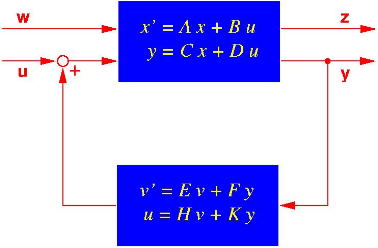
Designing a controller for parabolic control systems requires semi-discretization in space, control design for n-dim. system.

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$



Real-time control is only possible with controllers of low complexity.

$\rightsquigarrow$ Modern feedback control for parabolic systems w/o model reduction impossible due to large scale of discretized systems.

Controller Design for Parabolic Systems

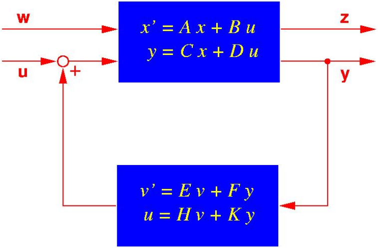
Designing a controller for parabolic control systems requires semi-discretization in space, control design for n-dim. system.

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$



Real-time control is only possible with controllers of low complexity.

$\rightsquigarrow$ Modern feedback control for parabolic systems w/o model reduction impossible due to large scale of discretized systems.

Control-Oriented Model Reduction?

If reduced-order model is to be used in (online) feedback control, the input function $u(t)$ is unknown a priori.

$\Longrightarrow$ Reduced-order models computed using snapshot-based methods or training sets (POD, RBM, TWPL, ANN, ...) might not catch dynamics induced by the control signals!

Discretizing the control space and including snapshots for all/many basis functions of $\mathcal{U}_h$ might work, but can become quite a challenging computation.

(Possible way out: cheap basis updates in online phase...)

Control-Oriented Model Reduction?

If reduced-order model is to be used in (online) feedback control, the input function $u(t)$ is unknown a priori.

$\Longrightarrow$ Reduced-order models computed using snapshot-based methods or training sets (POD, RBM, TWPL, ANN, ...) might not catch dynamics induced by the control signals!

Discretizing the control space and including snapshots for all/many basis functions of $\mathcal{U}_h$ might work, but can become quite a challenging computation.

(Possible way out: cheap basis updates in online phase...)

Control-Oriented Model Reduction?

If reduced-order model is to be used in (online) feedback control, the input function $u(t)$ is unknown a priori.

$\Longrightarrow$ Reduced-order models computed using snapshot-based methods or training sets (POD, RBM, TWPL, ANN, ...) might not catch dynamics induced by the control signals!

Discretizing the control space and including snapshots for all/many basis functions of $\mathcal{U}_h$ might work, but can become quite a challenging computation.

(Possible way out: cheap basis updates in online phase...)

$\Longrightarrow$ Aim at input-independent/simulation-free methods!

Balanced Truncation

Balanced Realization

Definition: [Curtain/Glover/(Partington) 1986,1988]

For $\mathbf{G} \in H_\infty$, $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a balanced realization of $\mathbf{G}$ if the controllability and observability Gramians, given by the unique self-adjoint positive semidefinite solutions of the Lyapunov equations

$$\begin{aligned} \mathbf{APz} + \mathbf{PA}^*\mathbf{z} + \mathbf{BB}^*\mathbf{z} &= 0 \quad \forall\, \mathbf{z} \in \mathrm{dom}(\mathbf{A}^*) \\ \mathbf{A}^*\mathbf{Qz} + \mathbf{QAz} + \mathbf{C}^*\mathbf{Cz} &= 0 \quad \forall\, \mathbf{z} \in \mathrm{dom}(\mathbf{A}) \end{aligned}$$

satisfy $\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\sigma_j) =: \mathbf{\Sigma}$.

Balanced Truncation

Model Reduction by Truncation

Abstract balanced truncation [Glover/Curtain/Partington 1988]

Given balanced realization with

$$\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\sigma_j) = \mathbf{\Sigma},$$

choose r with $\sigma_r > \sigma_{r+1}$ and partition $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ according to

$$\mathbf{P}_r = \mathbf{Q}_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r),$$

so that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_r & * \\ * & * \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_r \\ * \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_r & * \end{bmatrix},$$

then the reduced-order model is the stable system $\Sigma_r(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$ with transfer function $\mathbf{G}_r$ satisfying

$$\|\mathbf{G} - \mathbf{G}_r\|_{H_\infty} \leq 2 \sum_{j=r+1}^{\infty} \sigma_j.$$

LQG Balanced Truncation

LQG Balanced Realization

Balanced truncation only applicable for *stable* systems.
Now: unstable systems

Definition: [Curtain 2003].

For $\mathbf{G} \in L_\infty$, $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is an LQG-balanced realization of $\mathbf{G}$ if the unique self-adjoint, positive semidefinite, stabilizing solutions of the operator Riccati equations

$$\begin{aligned} \mathbf{APz} + \mathbf{PA}^*\mathbf{z} - \mathbf{PC}^*\mathbf{CPz} + \mathbf{BB}^*\mathbf{z} &= 0 \quad \text{for } \mathbf{z} \in \mathrm{dom}(\mathbf{A}^*) \\ \mathbf{A}^*\mathbf{Qz} + \mathbf{QAz} - \mathbf{QBB}^*\mathbf{Qz} + \mathbf{C}^*\mathbf{Cz} &= 0 \quad \text{for } \mathbf{z} \in \mathrm{dom}(\mathbf{A}) \end{aligned}$$

are bounded and satisfy $\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\gamma_j) =: \mathbf{\Gamma}$.
($\mathbf{P}$ stabilizing $\Leftrightarrow$ $\mathbf{A} - \mathbf{PC}^*\mathbf{C}$ generates exponentially stable C_0-semigroup.)

LQG Balanced Truncation

LQG Balanced Realization

Balanced truncation only applicable for *stable* systems.
Now: unstable systems

Definition: [Curtain 2003].

For $\mathbf{G} \in L_\infty$, $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is an LQG-balanced realization of $\mathbf{G}$ if the unique self-adjoint, positive semidefinite, stabilizing solutions of the operator Riccati equations

$$\begin{aligned} \mathbf{APz} + \mathbf{PA}^*\mathbf{z} - \mathbf{PC}^*\mathbf{CPz} + \mathbf{BB}^*\mathbf{z} &= 0 \quad \text{for } \mathbf{z} \in \mathrm{dom}(\mathbf{A}^*) \\ \mathbf{A}^*\mathbf{Qz} + \mathbf{QAz} - \mathbf{QBB}^*\mathbf{Qz} + \mathbf{C}^*\mathbf{Cz} &= 0 \quad \text{for } \mathbf{z} \in \mathrm{dom}(\mathbf{A}) \end{aligned}$$

are bounded and satisfy $\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\gamma_j) =: \mathbf{\Gamma}$.
($\mathbf{P}$ stabilizing $\Leftrightarrow$ $\mathbf{A} - \mathbf{PC}^*\mathbf{C}$ generates exponentially stable C_0-semigroup.)

LQG Balanced Truncation

Model Reduction by Truncation

Abstract LQG Balanced Truncation [Curtain 2003]

Given balanced realization with

$$\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\gamma_j) = \mathbf{\Gamma},$$

choose r with $\gamma_r > \gamma_{r+1}$ and partition $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ according to

$$\mathbf{P}_r = \mathbf{Q}_r = \mathrm{diag}(\gamma_1, \ldots, \gamma_r),$$

so that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_r & * \\ * & * \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_r \\ * \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_r & * \end{bmatrix},$$

then the reduced-order model is the LQG balanced system $\Sigma_r(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$ with transfer function $\mathbf{G}_r$ satisfying

$$\text{``}\|\mathbf{G} - \mathbf{G}_r\|_{L_\infty}\text{''} \leq 2 \sum_{j=r+1}^{\infty} \frac{\gamma_j}{\sqrt{1+\gamma_j^2}}.$$

Computation of Reduced-Order Systems

Spatial discretization (FEM, FDM) $\rightsquigarrow$ finite-dimensional system on $\mathcal{X}_n \subset \mathcal{X}$ with $\dim \mathcal{X}_n = n$:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, with corresponding

- algebraic Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

- algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= \mathcal{R}_f(P) := AP + PA^T - PC^TCP + BB^T, \\ 0 &= \mathcal{R}_c(Q) := A^TQ + QA - QBB^TQ + C^TC. \end{aligned}$$

Computation of Reduced-Order Systems

Spatial discretization (FEM, FDM) $\rightsquigarrow$ finite-dimensional system on $\mathcal{X}_n \subset \mathcal{X}$ with $\dim \mathcal{X}_n = n$:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, with corresponding

- algebraic Lyapunov equations

$$AP + PA^T + BB^T \;=\; 0, \qquad A^TQ + QA + C^TC \;=\; 0,$$

- algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= \mathcal{R}_f(P) := AP + PA^T - PC^TCP + BB^T, \\ 0 &= \mathcal{R}_c(Q) := A^TQ + QA - QBB^TQ + C^TC. \end{aligned}$$

Computation of Reduced-Order Systems

Spatial discretization (FEM, FDM) $\rightsquigarrow$ finite-dimensional system on $\mathcal{X}_n \subset \mathcal{X}$ with $\dim \mathcal{X}_n = n$:

$$
\begin{aligned}
\dot{x} &= Ax + Bu, \quad x(0) = x_0, \\
y &= Cx,
\end{aligned}
$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, with corresponding

- algebraic Lyapunov equations

$$
AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,
$$

- algebraic Riccati equations (AREs)

$$
\begin{aligned}
0 &= \mathcal{R}_f(P) := AP + PA^T - PC^T CP + BB^T, \\
0 &= \mathcal{R}_c(Q) := A^T Q + QA - QBB^T Q + C^T C.
\end{aligned}
$$

Computation of Reduced-Order Systems

Convergence of Gramians

Theorem [Curtain 2003]

Under given assumptions for $\mathbf{\Sigma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, the solutions of the algebraic Lyapunov equations on $\mathcal{X}_n$ converge in the nuclear norm to the solutions of the corresponding operator equations and the transfer functions converge in the gap topology if the n-dimensional approximations satisfy the assumptions:

- $\exists$ orthogonal projector $\Pi_n : \mathcal{X} \mapsto \mathcal{X}_n$ such that

$$\Pi_n \mathbf{z} \rightarrow \mathbf{z} \ (n \rightarrow \infty) \quad \forall \mathbf{z} \in \mathcal{X}, \quad B = \Pi_n \mathbf{B}, \qquad C = \mathbf{C}|_{\mathcal{X}_n}.$$

- For all $\mathbf{z} \in \mathcal{X}$ and $n \rightarrow \infty$,

$$e^{At}\Pi_n \mathbf{z} \rightarrow T(t)\mathbf{z}, \qquad (e^{At})^* \Pi_n \mathbf{z} \rightarrow T(t)^* \mathbf{z},$$

  uniformly in t on bounded intervals.
- A is uniformly exponentially stable.

Computation of Reduced-Order Systems

Convergence of Gramians

Theorem [Curtain 2003]

Under given assumptions for $\mathbf{\Sigma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, the stabilizing solutions of the algebraic Riccati equations on $\mathcal{X}_n$ converge in the nuclear norm to the solutions of the corresponding operator equations and the transfer functions converge in the gap topology if the n-dimensional approximations satisfy the assumptions:

- $\exists$ orthogonal projector $\Pi_n : \mathcal{X} \mapsto \mathcal{X}_n$ such that

$$\Pi_n \mathbf{z} \to \mathbf{z} \ (n \to \infty) \quad \forall \mathbf{z} \in \mathcal{X}, \quad B = \Pi_n \mathbf{B}, \qquad C = \mathbf{C}|_{\mathcal{X}_n}.$$

- For all $\mathbf{z} \in \mathcal{X}$ and $n \to \infty$,

$$e^{At}\Pi_n \mathbf{z} \to T(t)\mathbf{z}, \qquad (e^{At})^* \Pi_n \mathbf{z} \to T(t)^* \mathbf{z},$$

  uniformly in t on bounded intervals.
- (A, B, C) is uniformly exponentially stabilizable and detectable.

Computation of Reduced-Order Systems

Computation of Reduced-Order Systems from Gramians

1. Given the Gramians P, Q of the n-dimensional system from either the Lyapunov equations or AREs in factorized form

$$P = S^T S, \quad Q = R^T R,$$

compute SVD

$$SR^T = [\, U_1,\; U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

2. Set $W = R^T V_1 \Sigma_1^{-1/2}$ and $V = S^T U_1 \Sigma_1^{-1/2}$.
3. Then the reduced-order model is

$$(A_r, B_r, C_r) = (W^T AV, W^T B, CV).$$

Thus, need to solve large-scale matrix equations—but need only factors!
$\rightsquigarrow$ Efficient solvers available:
- (Galerkin-)ADI/Newton-ADI (B., Li, Penzl, Saak,...1998–2011),
- K-PIK, rational Lanczos (Druskin, Heyouni, Jbilou, Simoncini,...2006–2011).

Computation of Reduced-Order Systems

Computation of Reduced-Order Systems from Gramians

1. Given the Gramians P, Q of the n-dimensional system from either the Lyapunov equations or AREs in factorized form

$$P = S^T S, \quad Q = R^T R,$$

compute SVD

$$SR^T = [\, U_1, \, U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

2. Set $\quad W = R^T V_1 \Sigma_1^{-1/2}$ and $V = S^T U_1 \Sigma_1^{-1/2}$.
3. Then the reduced-order model is

$$(A_r, B_r, C_r) = (W^T A V, W^T B, CV).$$

Thus, need to solve large-scale matrix equations—but need only factors!
$\rightsquigarrow$ Efficient solvers available:

- (Galerkin-)ADI/Newton-ADI (B., Li, Penzl, Saak,... 1998–2011),
- K-PIK, rational Lanczos (Druskin, Heyouni, Jbilou, Simoncini,... 2006–2011).

Computation of Reduced-Order Systems

Computation of Reduced-Order Systems from Gramians

1. Given the Gramians P, Q of the n-dimensional system from either the Lyapunov equations or AREs in factorized form

$$P = S^T S, \quad Q = R^T R,$$

compute SVD

$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

2. Set $\quad W = R^T V_1 \Sigma_1^{-1/2}$ and $V = S^T U_1 \Sigma_1^{-1/2}$.
3. Then the reduced-order model is

$$(A_r, B_r, C_r) = (W^T A V, W^T B, CV).$$

Thus, need to solve large-scale matrix equations—but need only factors!
$\rightsquigarrow$ Efficient solvers available:

- (Galerkin-)ADI/Newton-ADI (B., Li, Penzl, Saak,... 1998–2011),
- K-PIK, rational Lanczos (Druskin, Heyouni, Jbilou, Simoncini,... 2006–2011).

Computation of Reduced-Order Systems

Computation of Reduced-Order Systems from Gramians

1. Given the Gramians P, Q of the n-dimensional system from either the Lyapunov equations or AREs in factorized form

$$P = S^T S, \quad Q = R^T R,$$

compute SVD

$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

2. Set $\quad W = R^T V_1 \Sigma_1^{-1/2}$ and $V = S^T U_1 \Sigma_1^{-1/2}$.
3. Then the reduced-order model is

$$(A_r, B_r, C_r) = (W^T AV, W^T B, CV).$$

Thus, need to solve large-scale matrix equations—but need only factors!

$\rightsquigarrow$ Efficient solvers available:

- (Galerkin-)ADI/Newton-ADI (B., Li, Penzl, Saak,. . . 1998–2011),
- K-PIK, rational Lanczos (Druskin, Heyouni, Jbilou, Simoncini,. . . 2006–2011).

Computation of Reduced-Order Systems

Computation of Reduced-Order Systems from Gramians

1. Given the Gramians P, Q of the n-dimensional system from either the Lyapunov equations or AREs in factorized form

$$P = S^T S, \quad Q = R^T R,$$

compute SVD

$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

2. Set $\quad W = R^T V_1 \Sigma_1^{-1/2}$ and $V = S^T U_1 \Sigma_1^{-1/2}$.
3. Then the reduced-order model is

$$(A_r, B_r, C_r) = (W^T A V, W^T B, CV).$$

Thus, need to solve large-scale matrix equations—but need only factors!

$\rightsquigarrow$ Efficient solvers available:

– (Galerkin-)ADI/Newton-ADI (B., Li, Penzl, Saak,. . . 1998–2011),

– K-PIK, rational Lanczos (Druskin, Heyouni, Jbilou, Simoncini,. . . 2006–2011).

Error Bounds

For control applications, want to estimate/bound

$$\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^m)} \quad \text{or} \quad \|\mathbf{y}(t) - y_r(t)\|_2.$$

Error bound includes approximation errors caused by

- Galerkin projection/spatial FEM discretization,
- model reduction.

Ultimate goal

Balance the discretization and model reduction errors vs. each other in fully adaptive discretization scheme.

Output Error Bound

Corollary

Assume $\mathbf{C} \in \mathcal{L}(\mathcal{X}, \mathbb{R}^p)$ bounded ($c := \|C\|$), $C = \mathbf{C}|_{\mathcal{X}_n}$, $\mathcal{X}_n \subset \mathcal{X}$. Then:

Balanced truncation:

$\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^p)} \leq c\|\mathbf{x} - x\|_{L_2(0,T;\mathcal{X})} + 2\|u\|_{L_2(0,T;\mathbb{R}^p)} \sum_{j=r+1}^{n} \sigma_j.$

LQG balanced truncation:

$\|\mathbf{y} - y_r\|_{L_2(0,T;\mathbb{R}^p)} \leq c\|\mathbf{x} - x\|_{L_2(0,T;\mathcal{X})} + 2\|u\|_{L_2(0,T;\mathbb{R}^p)} \sum_{j=r+1}^{n} \frac{\gamma_j}{\sqrt{1+\gamma_j^2}}.$

Numerical Results

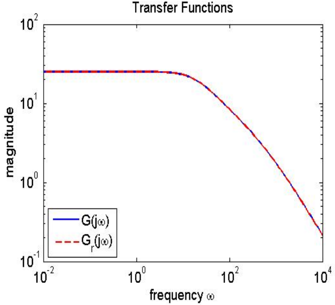
Model Reduction Performance

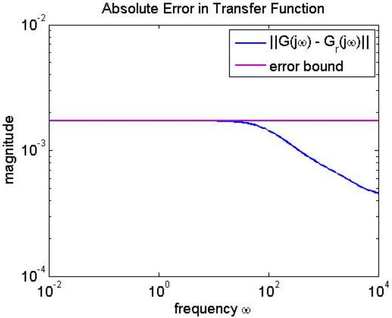
- Linear 2D heat equation with homogeneous Dirichlet boundary and point control/observation.
- FD discretization on uniform 150×150 grid.
- $n = 22.500$, $m = p = 1$, 10 shifts for ADI iterations.
- Convergence of large-scale matrix equation solvers:

Numerical Results

Model Reduction Performance

- Numerical ranks of Gramians are 31 and 26, respectively.
- Computed reduced-order model (BT): $r = 6$ ($\sigma_7 = 5.8 \cdot 10^{-4}$),
- BT error bound $\delta = 1.7 \cdot 10^{-3}$.

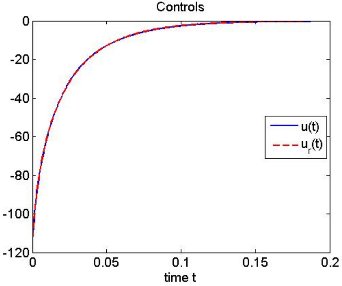


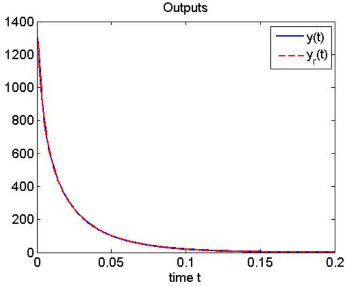


Numerical Results

Model Reduction Performance

- Computed reduced-order model (BT): $r = 6$, BT error bound $\delta = 1.7 \cdot 10^{-3}$.
- Solve LQR problem: quadratic cost functional, solution is linear state feedback.
- Computed controls and outputs (implicit Euler):

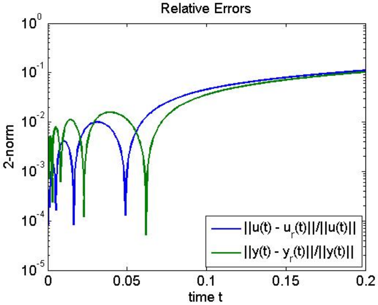




Numerical Results

Model Reduction Performance

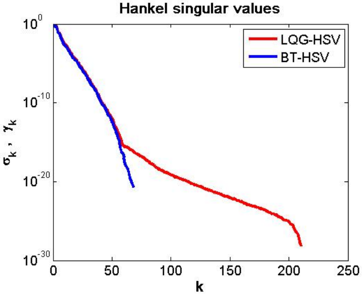
- Computed reduced-order model (BT): $r = 6$, BT error bound $\delta = 1.7 \cdot 10^{-3}$.
- Solve LQR problem: quadratic cost functional, solution is linear state feedback.
- Errors in controls and outputs:



Numerical Results

Model Reduction Performance: BT vs. LQG BT

- Boundary control problem for 2D heat flow in copper on rectangular domain; control acts on two sides via Robins BC.
- FDM $\rightsquigarrow$ $n = 4496$, $m = 2$; 4 sensor locations $\rightsquigarrow$ $p = 4$.
- Numerical ranks of BT Gramians are 68 and 124, respectively, for LQG BT both have rank 210.
- Computed reduced-order model: $r = 10$.

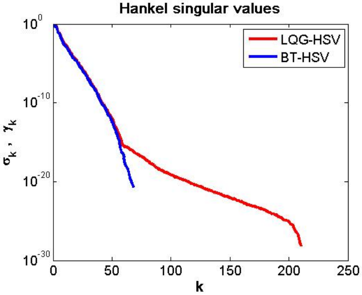


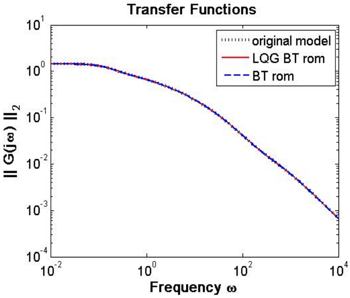
Source: *COMPl$_e$ib* v1.1, `www.compleib.de`.

Numerical Results

Model Reduction Performance: BT vs. LQG BT

- Boundary control problem for 2D heat flow in copper on rectangular domain; control acts on two sides via Robins BC.
- FDM $\rightsquigarrow$ $n = 4496$, $m = 2$; 4 sensor locations $\rightsquigarrow$ $p = 4$.
- Numerical ranks of BT Gramians are 68 and 124, respectively, for LQG BT both have rank 210.
- Computed reduced-order model: $r = 10$.



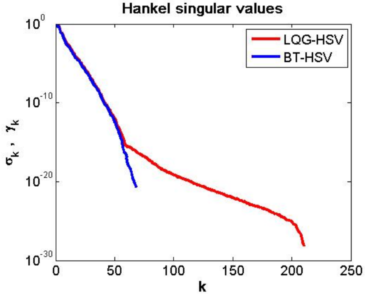


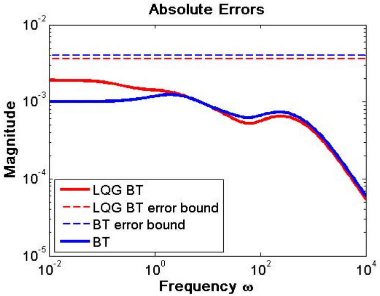
Source: *COMPl$_e$ib* v1.1, `www.compleib.de`.

Numerical Results

Model Reduction Performance: BT vs. LQG BT

- Boundary control problem for 2D heat flow in copper on rectangular domain; control acts on two sides via Robins BC.
- FDM $\rightsquigarrow$ $n = 4496$, $m = 2$; 4 sensor locations $\rightsquigarrow$ $p = 4$.
- Numerical ranks of BT Gramians are 68 and 124, respectively, for LQG BT both have rank 210.
- Computed reduced-order model: $r = 10$.





Source: *COMPl$_e$ib* v1.1, `www.compleib.de`.

Model Reduction Based on Rational Interpolation

Short Introduction

Computation of reduced-order model by projection

Given a linear (descriptor) system $E\dot{x} = Ax + Bu,\ y = Cx$ with transfer function $G(s) = C(sE - A)^{-1}B,$ a reduced-order model is obtained using projection matrices $V, W \in \mathbb{R}^{n\times r}$ with $W^T V = I_r$ ($\rightsquigarrow (VW^T)^2 = VW^T$ is projector) by computing

$$\hat{E} = W^T EV,\ \hat{A} = W^T AV,\ \hat{B} = W^T B,\ \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

Model Reduction Based on Rational Interpolation

Short Introduction

Computation of reduced-order model by projection

Given a linear (descriptor) system $E\dot{x} = Ax + Bu,\ y = Cx$ with transfer function $G(s) = C(sE - A)^{-1}B,$ a reduced-order model is obtained using projection matrices $V, W \in \mathbb{R}^{n\times r}$ with $W^T V = I_r$ ($\rightsquigarrow (VW^T)^2 = VW^T$ is projector) by computing

$$\hat{E} = W^T EV,\ \hat{A} = W^T AV,\ \hat{B} = W^T B,\ \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

Rational Interpolation/Moment-Matching

Choose V, W such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i} G(s_j) = \frac{d^i}{ds^i} \hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$

Model Reduction Based on Rational Interpolation

Short Introduction

Theorem (simplified) [Grimme 1997, Villemagne/Skelton 1987]

If

$$\begin{aligned} \mathrm{span}\left\{(s_1 E - A)^{-1} B, \ldots, (s_k E - A)^{-1} B\right\} &\subset \mathrm{Ran}(V), \\ \mathrm{span}\left\{(s_1 E - A)^{-T} C^T, \ldots, (s_k E - A)^{-T} C^T\right\} &\subset \mathrm{Ran}(W), \end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

Model Reduction Based on Rational Interpolation

Short Introduction

Theorem (simplified) [Grimme 1997, Villemagne/Skelton 1987]

If

$$\begin{aligned} \operatorname{span}\left\{(s_1E-A)^{-1}B,\ldots,(s_kE-A)^{-1}B\right\} &\subset \operatorname{Ran}(V), \\ \operatorname{span}\left\{(s_1E-A)^{-T}C^T,\ldots,(s_kE-A)^{-T}C^T\right\} &\subset \operatorname{Ran}(W), \end{aligned}$$

then

$$G(s_j)=\hat{G}(s_j), \quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j), \quad \text{for } j=1,\ldots,k.$$

Remarks:

computation of V, W from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme '97],
- Iterative Rational Krylov-Algo. [Antoulas/Beattie/Gugercin '07].

Model Reduction Based on Rational Interpolation

Short Introduction

Theorem (simplified) [Grimme 1997, Villemagne/Skelton 1987]

If

$$\begin{aligned} \operatorname{span}\left\{(s_1 E - A)^{-1} B, \ldots, (s_k E - A)^{-1} B\right\} &\subset \operatorname{Ran}(V), \\ \operatorname{span}\left\{(s_1 E - A)^{-T} C^T, \ldots, (s_k E - A)^{-T} C^T\right\} &\subset \operatorname{Ran}(W), \end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

Remarks:

using Galerkin/one-sided projection yields $G(s_j) = \hat{G}(s_j)$, but in general

$$\frac{d}{ds} G(s_j) \neq \frac{d}{ds} \hat{G}(s_j).$$

Model Reduction Based on Rational Interpolation

Short Introduction

Theorem (simplified) [Grimme 1997, Villemagne/Skelton 1987]

If

$$\begin{aligned} \mathrm{span}\left\{(s_1E-A)^{-1}B,\ldots,(s_kE-A)^{-1}B\right\} &\subset \mathrm{Ran}(V), \\ \mathrm{span}\left\{(s_1E-A)^{-T}C^T,\ldots,(s_kE-A)^{-T}C^T\right\} &\subset \mathrm{Ran}(W), \end{aligned}$$

then

$$G(s_j)=\hat{G}(s_j), \quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j), \quad \text{for } j=1,\ldots,k.$$

Remarks:

$k=1$, standard Krylov subspace(s) of dimension K $\rightsquigarrow$ moment-matching methods/Padé approximation,

$$\frac{d^i}{ds^i}G(s_1)=\frac{d^i}{ds^i}\hat{G}(s_1), \quad i=0,\ldots,K-1(+K).$$

Moment Matching using Quadratic-Bilinear Models

- Key observation: Many nonlinear dynamics can be modeled by quadratic bilinear differential algebraic equations (QBDAEs), i.e.

$$\begin{aligned} E\dot{x} &= A_1 x + A_2 x \otimes x + Nxu + bu, \\ y &= cx, \end{aligned}$$

where $E, A_1, N \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times n^2}, b, c^T \in \mathbb{R}^n$.

- Combination of quadratic and bilinear control systems.
- Variational analysis allows characterization of input-output behavior via generalized transfer functions, e.g.

$$H_1(s) = c \underbrace{(sE - A_1)^{-1} b}_{G(s)},$$

$$H_2(s_1, s_2) = \frac{1}{2} c \left((s_1 + s_2) E - A_1\right)^{-1} \left[A_2 (G(s_1) \otimes G(s_2) + G(s_2) \otimes G(s_1)) + N (G(s_1) + G(s_2))\right]$$

Moment Matching using Quadratic-Bilinear Models

Which systems can be transformed?

Theorem [Gu 2009]

Assume that the state equation of a nonlinear system Σ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of rational, exponential, logarithmic, trigonometric or root functions, respectively.
Then Σ can be transformed into a quadratic bilinear differential algebraic equation of dimension $N > n$.

- Transformation is not unique.
- Original system has to be increased before reduction is possible.
- Minimal dimension N?

Moment Matching using Quadratic-Bilinear Models

Example

- Consider the following two dimensional nonlinear control system:

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1},$$
$$\dot{x}_2 = \sin x_2 + u.$$

- Introduce useful new state variables, e.g.

$$x_3 := \exp(-x_2), \quad x_4 := \sqrt{x_1^2 + 1}, \quad x_5 := \sin x_2, \quad x_6 := \cos x_2.$$

- System can be replaced by a QBDAE of dimension 6:

$$\begin{aligned} \dot{x}_1 &= x_3 \cdot x_4, & \dot{x}_2 &= x_5 + u, \\ \dot{x}_3 &= -x_3 \cdot (x_5 + u), & \dot{x}_4 &= \frac{2 \cdot x_1 \cdot x_3 \cdot x_4}{2 \cdot x_4} = x_1 \cdot x_3, \\ \dot{x}_5 &= x_6 \cdot (x_5 + u), & \dot{x}_6 &= -x_5 \cdot (x_5 + u). \end{aligned}$$

Moment Matching using Quadratic-Bilinear Models

Example

- Consider the following two dimensional nonlinear control system:

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1},$$
$$\dot{x}_2 = \sin x_2 + u.$$

- Introduce useful new state variables, e.g.

$$x_3 := \exp(-x_2), \quad x_4 := \sqrt{x_1^2 + 1}, \quad x_5 := \sin x_2, \quad x_6 := \cos x_2.$$

- System can be replaced by a QBDAE of dimension 6:

$$\dot{x}_1 = x_3 \cdot x_4, \qquad \dot{x}_2 = x_5 + u,$$
$$\dot{x}_3 = -x_3 \cdot (x_5 + u), \qquad \dot{x}_4 = \frac{2 \cdot x_1 \cdot x_3 \cdot x_4}{2 \cdot x_4} = x_1 \cdot x_3,$$
$$\dot{x}_5 = x_6 \cdot (x_5 + u), \qquad \dot{x}_6 = -x_5 \cdot (x_5 + u).$$

Moment Matching using Quadratic-Bilinear Models

Example

- Consider the following two dimensional nonlinear control system:

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1},$$
$$\dot{x}_2 = \sin x_2 + u.$$

- Introduce useful new state variables, e.g.

$$x_3 := \exp(-x_2), \quad x_4 := \sqrt{x_1^2 + 1}, \quad x_5 := \sin x_2, \quad x_6 := \cos x_2.$$

- System can be replaced by a QBDAE of dimension 6:

$$\dot{x}_1 = x_3 \cdot x_4, \qquad \dot{x}_2 = x_5 + u,$$
$$\dot{x}_3 = -x_3 \cdot (x_5 + u), \qquad \dot{x}_4 = \frac{2 \cdot x_1 \cdot x_3 \cdot x_4}{2 \cdot x_4} = x_1 \cdot x_3,$$
$$\dot{x}_5 = x_6 \cdot (x_5 + u), \qquad \dot{x}_6 = -x_5 \cdot (x_5 + u).$$

Moment Matching using Quadratic-Bilinear Models

Example

- Consider the following two dimensional nonlinear control system:

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1},$$
$$\dot{x}_2 = \sin x_2 + u.$$

- Introduce useful new state variables, e.g.

$$x_3 := \exp(-x_2), \quad x_4 := \sqrt{x_1^2 + 1}, \quad x_5 := \sin x_2, \quad x_6 := \cos x_2.$$

- System can be replaced by a QBDAE of dimension 6:

$$\dot{x}_1 = x_3 \cdot x_4, \qquad \dot{x}_2 = x_5 + u,$$
$$\dot{x}_3 = -x_3 \cdot (x_5 + u), \qquad \dot{x}_4 = \frac{2 \cdot x_1 \cdot x_3 \cdot x_4}{2 \cdot x_4} = x_1 \cdot x_3,$$
$$\dot{x}_5 = x_6 \cdot (x_5 + u), \qquad \dot{x}_6 = -x_5 \cdot (x_5 + u).$$

Moment Matching using Quadratic-Bilinear Models

Example

- Consider the following two dimensional nonlinear control system:

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1},$$
$$\dot{x}_2 = \sin x_2 + u.$$

- Introduce useful new state variables, e.g.

$$x_3 := \exp(-x_2), \quad x_4 := \sqrt{x_1^2 + 1}, \quad x_5 := \sin x_2, \quad x_6 := \cos x_2.$$

- System can be replaced by a QBDAE of dimension 6:

$$\dot{x}_1 = x_3 \cdot x_4, \qquad \dot{x}_2 = x_5 + u,$$
$$\dot{x}_3 = -x_3 \cdot (x_5 + u), \qquad \dot{x}_4 = \frac{2 \cdot x_1 \cdot x_3 \cdot x_4}{2 \cdot x_4} = x_1 \cdot x_3,$$
$$\dot{x}_5 = x_6 \cdot (x_5 + u), \qquad \dot{x}_6 = -x_5 \cdot (x_5 + u).$$

Moment Matching using Quadratic-Bilinear Models

Example

- Consider the following two dimensional nonlinear control system:

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1},$$
$$\dot{x}_2 = \sin x_2 + u.$$

- Introduce useful new state variables, e.g.

$$x_3 := \exp(-x_2), \quad x_4 := \sqrt{x_1^2 + 1}, \quad x_5 := \sin x_2, \quad x_6 := \cos x_2.$$

- System can be replaced by a QBDAE of dimension 6:

$$\begin{aligned} \dot{x}_1 &= x_3 \cdot x_4, & \dot{x}_2 &= x_5 + u, \\ \dot{x}_3 &= -x_3 \cdot (x_5 + u), & \dot{x}_4 &= \frac{2 \cdot x_1 \cdot x_3 \cdot x_4}{2 \cdot x_4} = x_1 \cdot x_3, \\ \dot{x}_5 &= x_6 \cdot (x_5 + u), & \dot{x}_6 &= -x_5 \cdot (x_5 + u). \end{aligned}$$

Moment Matching using Quadratic-Bilinear Models

Example

- Consider the following two dimensional nonlinear control system:

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1},$$
$$\dot{x}_2 = \sin x_2 + u.$$

- Introduce useful new state variables, e.g.

$$x_3 := \exp(-x_2), \quad x_4 := \sqrt{x_1^2 + 1}, \quad x_5 := \sin x_2, \quad x_6 := \cos x_2.$$

- System can be replaced by a QBDAE of dimension 6:

$$\begin{aligned} \dot{x}_1 &= x_3 \cdot x_4, & \dot{x}_2 &= x_5 + u, \\ \dot{x}_3 &= -x_3 \cdot (x_5 + u), & \dot{x}_4 &= \frac{2 \cdot x_1 \cdot x_3 \cdot x_4}{2 \cdot x_4} = x_1 \cdot x_3, \\ \dot{x}_5 &= x_6 \cdot (x_5 + u), & \dot{x}_6 &= -x_5 \cdot (x_5 + u). \end{aligned}$$

Moment Matching using Quadratic-Bilinear Models

Multi-Moment-Matching for QBDAEs

- Construct reduced order model by projection:

$$\hat{E} = Z^T E Z, \quad \hat{A}_1 = Z^T A_1 Z, \quad \hat{N} = Z^T N Z,$$
$$\hat{A}_2 = Z^T A_2 Z \otimes Z, \quad \hat{b} = Z^T b, \quad \hat{c} = cZ$$

- Approximate values and derivatives ("multi-moments") of transfer functions about an expansion point σ using Krylov spaces, e.g.

$$\begin{aligned}
\operatorname{span}\{V\} &= \mathcal{K}_6\left(A_\sigma E, A_\sigma b\right)\\
\operatorname{span}\{W_1\} &= \mathcal{K}_3\left(A_{2\sigma} E, A_{2\sigma}(A_2 V_1 \otimes V_1 - N_1 V_1)\right)\\
\operatorname{span}\{W_2\} &= \mathcal{K}_2\left(A_{2\sigma} E, A_{2\sigma}(A_2(V_2 \otimes V_1 + V_1 \otimes V_2) - N_1 V_2)\right)\\
\operatorname{span}\{W_3\} &= \mathcal{K}_1\left(A_{2\sigma} E, A_{2\sigma}(A_2(V_2 \otimes V_2 + V_2 \otimes V_2))\right)\\
\operatorname{span}\{W_4\} &= \mathcal{K}_1\left(A_{2\sigma} E, A_{2\sigma}(A_2(V_3 \otimes V_1 + V_1 \otimes V_3) - N_1 V_3)\right),
\end{aligned}$$

with $A_\sigma = (A_1 - \sigma E)^{-1}$ and V_i denoting the i-*th* column of V,
$\operatorname{span} Z = \operatorname{span}[V, W_1, \ldots]$.
$\rightarrow$ derivatives match up to order 5 (H_1) and 2 (H_2), respectively.

Numerical Examples

FitzHugh-Nagumo System

- Simple model for neuron (de-)activation.

$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1] \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

where
$\epsilon = 0.015, h = 0.5, \gamma = 2, g = 0.05, i_0(t) = 50000t^3 \exp(-15t)$.

[Chaturantabut/Sorensen 2009]

- Parameter g handled as an additional input.
- Original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$.

Numerical Examples

FitzHugh-Nagumo System

3d Phase Space

[B./Breiten 2010]

Numerical Examples

Jet Diffusion Flame Model [Galbally/Willcox 2009]

Consider a nonlinear PDE arising in jet-diffusion flame models

$$\frac{\partial w}{\partial t} + U \cdot \nabla w - \nabla(\kappa \nabla w) + f(w) = 0, \quad (x,t) \in (0,1) \times (0,T),$$

with Arrhenius type term $f(w) = Aw(c-w)e^{-\frac{E}{d-w}}$ and constant parameters U, A, E, c, d, κ.

Numerical Examples

Jet Diffusion Flame Model [Galbally/Willcox 2009]

Consider a nonlinear PDE arising in jet-diffusion flame models

$$\frac{\partial w}{\partial t} + U \cdot \nabla w - \nabla(\kappa \nabla w) + f(w) = 0, \quad (x,t) \in (0,1) \times (0,T),$$

with Arrhenius type term $f(w) = Aw(c-w)e^{-\frac{E}{d-w}}$ and constant parameters U, A, E, c, d, κ.

Initial and boundary conditions:

$$\begin{aligned} w(x,0) &= 0, \quad x \in [0,1], \\ w(0,t) &= u(t), \quad t \geq 0, \\ w(1,t) &= 0, \quad t \geq 0, \\ w_{center} &= \begin{bmatrix} \mathbf{0} & 1 & \mathbf{0} \end{bmatrix}. \end{aligned}$$

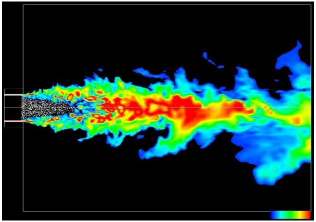


Figure: [Kurose]

Numerical Examples

Jet Diffusion Flame Model [Galbally/Willcox 2009]

Consider a nonlinear PDE arising in jet-diffusion flame models

$$\frac{\partial w}{\partial t} + U \cdot \nabla w - \nabla(\kappa \nabla w) + f(w) = 0, \quad (x,t) \in (0,1) \times (0,T),$$

with Arrhenius type term $f(w) = Aw(c-w)e^{-\frac{E}{d-w}}$ and constant parameters U, A, E, c, d, κ.

After spatial discretization of order k, define new state variables

$$z_i := -\frac{\beta}{\delta - w_i}, \quad q_i := e^{z_i},$$

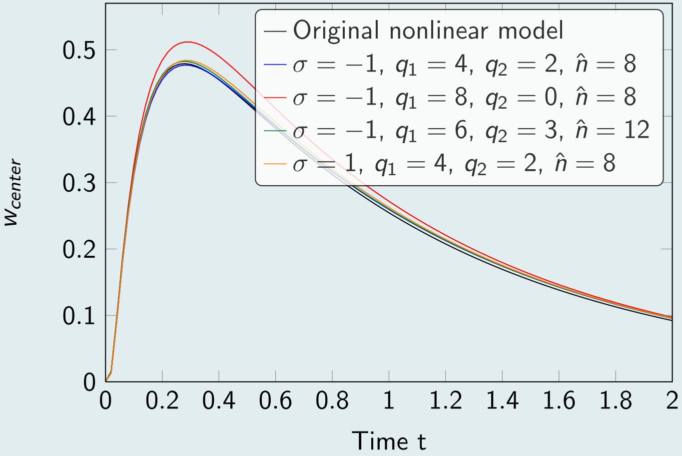
and iteratively construct a system of QBDAEs

$\rightsquigarrow$ state dimension increases to $n = 8 \cdot k$.

Numerical Examples

Jet Diffusion Flame Model [Galbally/Willcox 2009]

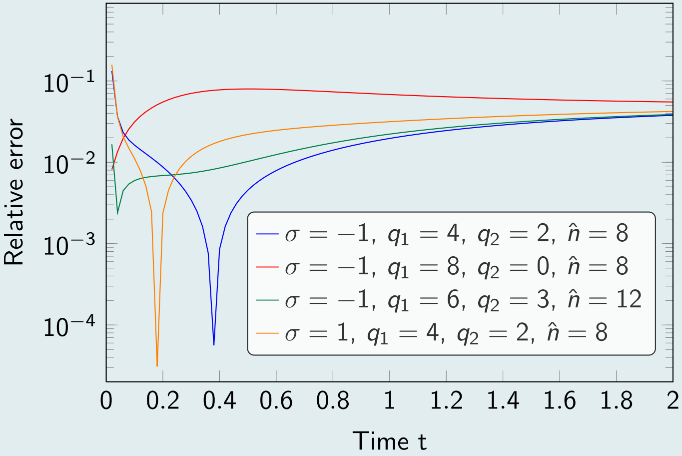
Transient responses for $k = 1500$ and $u(t) = e^{-t}$



Numerical Examples

Jet Diffusion Flame Model [Galbally/Willcox 2009]

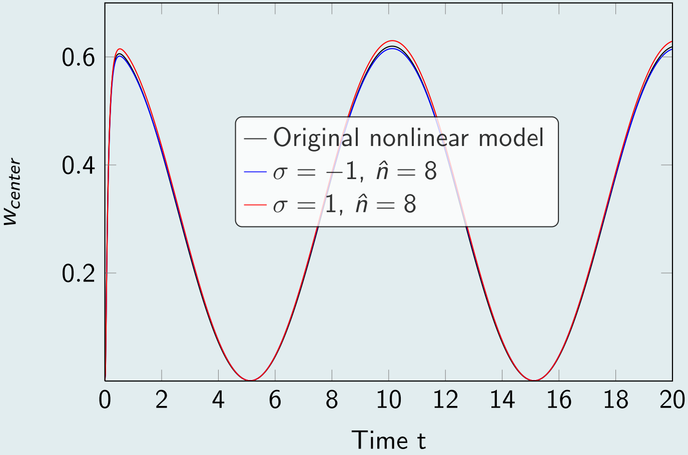
Relative errors for $k = 1500$ and $u(t) = e^{-t}$



Numerical Examples

Jet Diffusion Flame Model [Galbally/Willcox 2009]

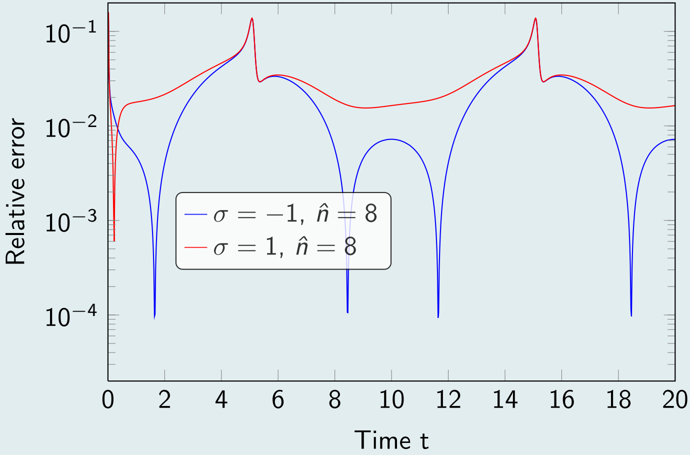
Transient responses for $k = 1500$ and $u(t) = \frac{1}{2}\cos(\frac{\pi t}{5} + 1)$



Numerical Examples

Jet Diffusion Flame Model [Galbally/Willcox 2009]

Relative errors for $k = 1500$ and $u(t) = \frac{1}{2}\cos(\frac{\pi t}{5} + 1)$



Conclusions and Open Problems

Linear Control Systems

- BT (and LQG) BT perform well for model reduction of (as of yet, simple) parabolic PDE control problems.
- Robust control design can be based on LQG BT (see CURTAIN 2004).
- State reconstruction using (LGQ)BT modes possible.
- Need more numerical tests.
- Open Problems:
  - Optimal combination of FEM and BT error estimates/bounds — use convergence of Hankel singular values for control of mesh refinement?
  - Application to nonlinear problems: for some semilinear problems, BT approaches seem to work well.
  - Rather than Discretize-then-reduced use reduce-then-discretize?

    [Reis 2010:] BT in function space. Extension to LQG BT?

    Interpolation in function space:

    $$\mathbf{G} = \mathbf{C}(s_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \hat{\mathbf{C}}(s_k\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} =: \hat{\mathbf{G}},\ k = 1, \ldots, r,$$

    where $\hat{\mathbf{A}} : \mathcal{X}_r \to \mathcal{X}_r, \quad \mathcal{X}_r \subset \mathcal{X}$, etc.
    $\rightsquigarrow$ solve r Helmholtz-type problems $\mathbf{L}(\mathbf{x}) - s_k\mathbf{x} = -\mathbf{B}\mathbf{u}$.

Conclusions and Open Problems

Linear Control Systems

- BT (and LQG) BT perform well for model reduction of (as of yet, simple) parabolic PDE control problems.
- Robust control design can be based on LQG BT (see CURTAIN 2004).
- State reconstruction using (LGQ)BT modes possible.
- Need more numerical tests.
- Open Problems:
  - Optimal combination of FEM and BT error estimates/bounds — use convergence of Hankel singular values for control of mesh refinement?
  - Application to nonlinear problems: for some semilinear problems, BT approaches seem to work well.
  - Rather than Discretize-then-reduced use reduce-then-discretize?

    [Reis 2010:] BT in function space. Extension to LQG BT?

    Interpolation in function space:

    $$\mathbf{G} = \mathbf{C}(s_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \hat{\mathbf{C}}(s_k\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} =: \hat{\mathbf{G}}, \; k = 1, \ldots, r,$$

    where $\hat{\mathbf{A}} : \mathcal{X}_r \to \mathcal{X}_r, \quad \mathcal{X}_r \subset \mathcal{X}$, etc.
    $\rightsquigarrow$ solve r Helmholtz-type problems $\mathbf{L}(\mathbf{x}) - s_k\mathbf{x} = -\mathbf{Bu}$.

Conclusions and Open Problems

Linear Control Systems

- BT (and LQG) BT perform well for model reduction of (as of yet, simple) parabolic PDE control problems.
- Robust control design can be based on LQG BT (see CURTAIN 2004).
- State reconstruction using (LGQ)BT modes possible.
- Need more numerical tests.
- Open Problems:
  - Optimal combination of FEM and BT error estimates/bounds — use convergence of Hankel singular values for control of mesh refinement?
  - Application to nonlinear problems: for some semilinear problems, BT approaches seem to work well.
  - Rather than Discretize-then-reduced use reduce-then-discretize?

    [Reis 2010:] BT in function space. Extension to LQG BT?

    Interpolation in function space:

    $$\mathbf{G} = \mathbf{C}(s_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \hat{\mathbf{C}}(s_k\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} =: \hat{\mathbf{G}}, \; k = 1, \ldots, r,$$

    where $\hat{\mathbf{A}} : \mathcal{X}_r \to \mathcal{X}_r, \quad \mathcal{X}_r \subset \mathcal{X}$, etc.
    $\rightsquigarrow$ solve r Helmholtz-type problems $\mathbf{L}(\mathbf{x}) - s_k\mathbf{x} = -\mathbf{B}\mathbf{u}$.

Conclusions and Open Problems

Linear Control Systems

- BT (and LQG) BT perform well for model reduction of (as of yet, simple) parabolic PDE control problems.
- Robust control design can be based on LQG BT (see CURTAIN 2004).
- State reconstruction using (LGQ)BT modes possible.
- Need more numerical tests.
- Open Problems:
  - Optimal combination of FEM and BT error estimates/bounds — use convergence of Hankel singular values for control of mesh refinement?
  - Application to nonlinear problems: for some semilinear problems, BT approaches seem to work well.
  - Rather than Discretize-then-reduced use reduce-then-discretize?

    [Reis 2010:] BT in function space. Extension to LQG BT?

    Interpolation in function space:

    $$\mathbf{G} = \mathbf{C}(s_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \hat{\mathbf{C}}(s_k\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} =: \hat{\mathbf{G}},\ k = 1, \ldots, r,$$

    where $\hat{\mathbf{A}} : \mathcal{X}_r \to \mathcal{X}_r, \quad \mathcal{X}_r \subset \mathcal{X}$, etc.
    $\rightsquigarrow$ solve r Helmholtz-type problems $\mathbf{L}(\mathbf{x}) - s_k\mathbf{x} = -\mathbf{B}\mathbf{u}$.

Conclusions and Open Problems

Linear Control Systems

- BT (and LQG) BT perform well for model reduction of (as of yet, simple) parabolic PDE control problems.
- Robust control design can be based on LQG BT (see CURTAIN 2004).
- State reconstruction using (LGQ)BT modes possible.
- Need more numerical tests.
- Open Problems:
  - Optimal combination of FEM and BT error estimates/bounds — use convergence of Hankel singular values for control of mesh refinement?
  - Application to nonlinear problems: for some semilinear problems, BT approaches seem to work well.
  - Rather than Discretize-then-reduced use reduce-then-discretize?

    [Reis 2010:] BT in function space. Extension to LQG BT?

    Interpolation in function space:

    $$\mathbf{G} = \mathbf{C}(s_k \mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \hat{\mathbf{C}}(s_k \mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} =: \hat{\mathbf{G}}, \; k = 1, \ldots, r,$$

    where $\hat{\mathbf{A}} : \mathcal{X}_r \to \mathcal{X}_r, \quad \mathcal{X}_r \subset \mathcal{X}$, etc.
    $\rightsquigarrow$ solve r Helmholtz-type problems $\mathbf{L}(\mathbf{x}) - s_k \mathbf{x} = -\mathbf{B}\mathbf{u}$.

Conclusions and Open Problems

Nonlinear Control Systems

- QBDAE approach allows input-independent reduced-order models, no training set/snapshots needed.
- QBDAE approach very suitable for systems with homogeneous nonlinearity, but also possible for other types of problems (e.g., biogas reactor model at MPI Magdeburg).

Work in Progress:

- Computation of Krylov spaces involves tensor products, requires efficient tensor calculus.
- Two-sided projection methods (interpolate twice as many derivatives with same reduced order).
- Optimal expansion points (greedy-type algorithm) (with B. Haasdonk).
- Automatic generation of QBDAE system using computer algebra?
- Optimal QBDAE model?

Conclusions and Open Problems

Nonlinear Control Systems

- QBDAE approach allows input-independent reduced-order models, no training set/snapshots needed.
- QBDAE approach very suitable for systems with homogeneous nonlinearity, but also possible for other types of problems (e.g., biogas reactor model at MPI Magdeburg).

Work in Progress:

- Computation of Krylov spaces involves tensor products, requires efficient tensor calculus.
- Two-sided projection methods (interpolate twice as many derivatives with same reduced order).
- Optimal expansion points (greedy-type algorithm) (with B. Haasdonk).
- Automatic generation of QBDAE system using computer algebra?
- Optimal QBDAE model?