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Numerical Computation of Robust Controllers for Parabolic Systems

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Overview

Parabolic Systems

Parabolic PDEs as distributed parameter systems

Given Hilbert spaces

$\mathcal{X}$ – state space,

$\mathcal{U}$ – control space,

$\mathcal{Y}$ – output space,

and linear operators

$$\begin{aligned} \mathbf{A}: &\quad \mathrm{dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}, \\ \mathbf{B}: &\quad \mathcal{U} \to \mathcal{X}, \\ \mathbf{C}: &\quad \mathcal{X} \to \mathcal{Y}. \end{aligned}$$

Linear Distributed Parameter System (DPS)

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}\mathbf{x}, \end{cases} \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X},$$

i.e., abstract evolution equation together with observation equation.

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Parabolic Systems

Examples

The state $x = x(t, \xi)$ is a weak solution of a parabolic PDE with $(t, \xi) \in [0, T] \times \Omega$, $\Omega \subset \mathbb{R}^d$:

$$\partial_t x - \nabla(a(\xi).\nabla x) + b(\xi).\nabla x + c(\xi)x = B_{pc}(\xi)u(t), \quad \xi \in \Omega, \ t > 0$$

with initial and boundary conditions

$$\begin{aligned} \alpha(\xi)x + \beta(\xi)\partial_\eta x &= B_{bc}(\xi)u(t), & \xi \in \partial\Omega, \quad & t \in [0, T], \\ x(0, \xi) &= x_0(\xi) \in \mathcal{X}, & \xi \in \Omega, \quad & \\ y(t) &= C(\xi)x, & \xi \in \Omega, \quad & t \in [0, T]. \end{aligned}$$

- $B_{pc} = 0 \implies$ boundary control problem
- $B_{bc} = 0 \implies$ point control problem

Infinite-Dimensional Systems Theory

Assume

- $\mathbf{A}$ generates C_0-semigroup $\mathbf{T}(t)$ on $\mathcal{X}$;
- $(\mathbf{A}, \mathbf{B})$ is exponentially stabilizable, i.e., there exists $\mathbf{F} : \mathrm{dom}(\mathbf{A}) \mapsto \mathcal{U}$ such that $\mathbf{A} - \mathbf{BF}$ generates an exponentially stable C_0-semigroup;
- $(\mathbf{A}, \mathbf{C})$ is exponentially detectable, i.e., there exists $\mathbf{G} : \mathrm{dom}(\mathbf{A}) \mapsto \mathcal{U}$ such that $\mathbf{A} - \mathbf{GC}$ generates an exponentially stable C_0-semigroup;
- $\mathbf{B}, \mathbf{C}$ are finite-rank and bounded.

Then the system $\Sigma(A, B, C)$ has a transfer function

$$\mathbf{G} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \in L_\infty.$$

If, in addition, $\mathbf{A}$ is exponentially stable, $\mathbf{G}$ is in the Hardy space H_∞.

Weaker assumptions:
$\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ defines a nuclear Hankel operator

$$\mathbf{H} : L_2([0, \infty), \mathcal{U}) \to L_2([0, \infty), \mathcal{Y}),$$

this allows for boundary control and observation!

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Robust Control

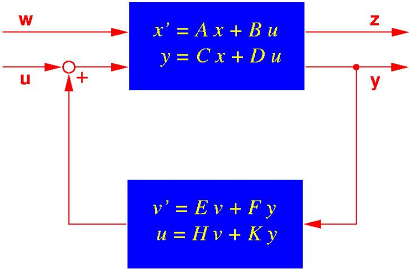
H_∞ Control

Linear time-invariant systems (finite or infinite)

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}, \\ \mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} + \mathbf{D}_{12}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} + \mathbf{D}_{22}\mathbf{u}, \end{cases}$$

where $\mathbf{A} : \mathrm{dom}(A) \subset \mathcal{X} \to \mathcal{X}$, etc.

- $\mathbf{x}$ – states of the system,
- $\mathbf{w}$ – exogenous inputs
- $\mathbf{u}$ – control inputs,
- $\mathbf{z}$ – performance outputs
- $\mathbf{y}$ – measured outputs



Robust Control

Transfer functions

Laplace transform $\Longrightarrow$ transfer function (in frequency domain)

$$\mathbf{G}(s) = \begin{bmatrix} \mathbf{G}_{11}(s) & \mathbf{G}_{12}(s) \\ \mathbf{G}_{21}(s) & \mathbf{G}_{22}(s) \end{bmatrix} \equiv \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{D}_{22} \end{array}\right].$$

where for $\mathbf{x}(0) = 0$, $\mathbf{G}_{ij}$ are the transfer functions

- $\mathbf{G}_{11}(s) = \mathbf{C}_1(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_1 + \mathbf{D}_{11}$,
- $\mathbf{G}_{12}(s) = \mathbf{C}_1(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_2 + \mathbf{D}_{12}$,
- $\mathbf{G}_{21}(s) = \mathbf{C}_2(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_1 + \mathbf{D}_{21}$,
- $\mathbf{G}_{22}(s) = \mathbf{C}_2(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_2 + \mathbf{D}_{22}$,

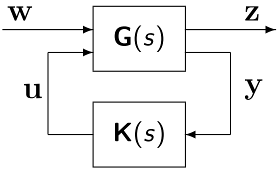
describing the transfer from inputs to outputs of Σ via

$$\begin{aligned} \mathbf{z}(s) &= \mathbf{G}_{11}(s)\mathbf{w}(s) + \mathbf{G}_{12}(s)\mathbf{u}(s), \\ \mathbf{y}(s) &= \mathbf{G}_{21}(s)\mathbf{w}(s) + \mathbf{G}_{22}(s)\mathbf{u}(s). \end{aligned}$$

Robust Control

The H_∞-Optimization Problem

Consider closed-loop system, where $\mathbf{K}(s)$ is an internally stabilizing controller, i.e., $\mathbf{K}$ stabilizes $\mathbf{G}$ for $\mathbf{w} \equiv 0$.



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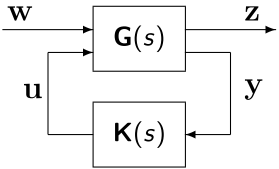

Goal:

find robust controller, i.e., $\mathbf{K}$ that minimizes error outputs

$$\mathbf{z} = \left(\mathbf{G}_{11} + \mathbf{G}_{12}\mathbf{K}(\mathbf{I} - \mathbf{G}_{22}\mathbf{K})^{-1}\mathbf{G}_{21}\right)\mathbf{w} =: \mathcal{F}(\mathbf{G},\mathbf{K})\mathbf{w},$$

where $\mathcal{F}(\mathbf{G},\mathbf{K})$ is the linear fractional transformation of $\mathbf{G}$, $\mathbf{K}$.

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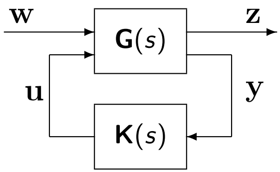

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H_∞-optimal control problem:

$$\min_{\mathbf{K}\text{ stabilizing}} \|\mathcal{F}(\mathbf{G},\mathbf{K})\|_{\mathcal{H}_\infty}.$$

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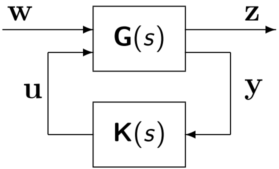

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H_∞-suboptimal control problem:

For given constant $\gamma > 0$, find all internally stabilizing controllers satisfying

$$\|\mathcal{F}(\mathbf{G},\mathbf{K})\|_{\mathcal{H}_\infty} < \gamma.$$

Robust Control

Solution of the H_∞-(Sub-)Optimal Control Problem

Simplifying assumptions

1. $\mathbf{D}_{11} = \mathbf{0}$;
2. $\mathbf{D}_{22} = \mathbf{0}$;
3. $(\mathbf{A}, \mathbf{B}_1)$ stabilizable, $(\mathbf{C}_1, \mathbf{A})$ detectable;
4. $(\mathbf{A}, \mathbf{B}_2)$ stabilizable, $(\mathbf{C}_2, \mathbf{A})$ detectable ($\Longrightarrow \Sigma$ internally stabilizable);
5. $\mathbf{D}_{12}^* [\, \mathbf{C}_1 \;\; \mathbf{D}_{12} \,] = [\, \mathbf{0} \;\; \mathbf{I} \,]$;
6. $\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{D}_{21} \end{bmatrix} \mathbf{D}_{21}^* = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$.

Remark. 1.,2.,5.,6. only for notational convenience, 3. can be relaxed, but derivations get even more complicated.

Robust Control

Solution of the H_∞-(Sub-)Optimal Control Problem

Theorem [Doyle/Glover/Khargonekar/Francis '89, Van Keulen '93]

Given the Assumptions 1.–6., there exists an admissible controller $\mathbf{K}(s)$ solving the H_∞-suboptimal control problem $\iff$

(i) There exists a solution $\mathbf{X}_\infty = \mathbf{X}_\infty^* \geq 0$ to the operator Riccati equation

$$\mathbf{C}_1^*\mathbf{C}_1 + \mathbf{A}^*\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{X}(\gamma^{-2}\mathbf{B}_1\mathbf{B}_1^* - \mathbf{B}_2\mathbf{B}_2^*)\mathbf{X} = 0, \qquad (1)$$

such that $\mathbf{A}_\mathbf{X}$ generates an exponentially stable C_0 semigroup, where $\mathbf{A}_\mathbf{X} := \mathbf{A} + (\gamma^{-2}\mathbf{B}_1\mathbf{B}_1^* - \mathbf{B}_2\mathbf{B}_2^*)\mathbf{X}_\infty$.

(ii) There exists a solution $\mathbf{Y}_\infty = \mathbf{Y}_\infty^* \geq 0$ to the operator Riccati equation

$$\mathbf{B}_1\mathbf{B}_1^* + \mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^* + \mathbf{Y}(\gamma^{-2}\mathbf{C}_1^*\mathbf{C}_1 - \mathbf{C}_2^*\mathbf{C}_2)\mathbf{Y} = 0, \qquad (2)$$

such that $\mathbf{A}_\mathbf{Y}$ generates an exponentially stable C_0 semigroup, where $\mathbf{A}_\mathbf{Y} := \mathbf{A} + \mathbf{Y}_\infty(\gamma^{-2}\mathbf{C}_1^*\mathbf{C}_1 - \mathbf{C}_2^*\mathbf{C}_2)$.

(iii) $\gamma^2 > \rho(\mathbf{X}_\infty\mathbf{Y}_\infty)$.

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(iii) $\gamma^2 > \rho(\mathbf{X}_\infty\mathbf{Y}_\infty)$.

H_∞-optimal control

Find minimal γ for which (i)–(iii) are satisfied $\rightsquigarrow$ γ-iteration based on solving (1)–(2) repeatedly for different γ.

Robust Control

Solution of the H_∞-(Sub-)Optimal Control Problem

H_∞-(sub-)optimal controller

If (i)–(iii) hold, a suboptimal controller is given by

$$\hat{\mathbf{K}}(s) = \left[\begin{array}{c|c} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hline \hat{\mathbf{C}} & \mathbf{0} \end{array}\right] = \hat{\mathbf{C}}(s\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}},$$

where for

$$\mathbf{Z}_\infty := (\mathbf{I} - \gamma^{-2}\mathbf{Y}_\infty \mathbf{X}_\infty)^{-1},$$

$$\begin{aligned} \hat{\mathbf{A}} &:= \mathbf{A} + (\gamma^{-2}\mathbf{B}_1\mathbf{B}_1^* - \mathbf{B}_2\mathbf{B}_2^*)\mathbf{X}_\infty - \mathbf{Z}_\infty \mathbf{Y}_\infty \mathbf{C}_2^*\mathbf{C}_2, \\ \hat{\mathbf{B}} &:= \mathbf{Z}_\infty \mathbf{Y}_\infty \mathbf{C}_2^*, \\ \hat{\mathbf{C}} &:= -\mathbf{B}_2^*\mathbf{X}_\infty. \end{aligned}$$

$\hat{\mathbf{K}}(s)$ is the central or minimum entropy controller.

Numerical Computation of a Robust Controller

Discretization and Approximation

Numerical solution of H_∞ controller requires discretization by appropriate approximation scheme (dual convergence, etc., like in discretization of LQR problems [Banks/Kunisch '84, Burns/Ito/Probst '88]).

Theorem [Ito/Morris '98]

Under suitable assumptions and for N large enough, the operator Riccati equations and the resulting algebraic Riccati equations

$$
\begin{aligned}
(C_1^N)^T C_1^N + (A^N)^* X^N + X^N A^N + X^N(\gamma^{-2} B_1^N (B_1^N)^T - B_2^N (B_2^N)^T) X^N &= 0,\\
B_1^N (B_1^N)^T + A^N Y^N + Y^N (A^N)^* + Y^N(\gamma^{-2} (C_1^N)^T C_1^N - (C_2^N)^T C_2^N) Y^N &= 0
\end{aligned}
$$

have positive semidefinite stabilizing solutions for the same γ-levels, and the corresponding finite-dimensional controller $K^N(s)$ is a γ-sub-optimal (internally stabilizing) controller for the N- and infinite dimensional problem.

Numerical Computation of a Robust Controller

Solving Large-Scale AREs

Derive numerical algorithms for solving large-scale

(continuous-time) algebraic Riccati equation (ARE)

with indefinite Hessian,

$$\mathcal{R}(X) := C^T C + A^T X + XA + X(B_1 B_1^T - B_2 B_2^T)X = 0,$$

where

- $A \in \mathbb{R}^{n \times n}$ is large and sparse,
- $B_j \in \mathbb{R}^{n \times m_j}$ $(j = 1, 2)$,
- $C \in \mathbb{R}^{p \times n}$,
- $n \gg m_j, p$.

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Hessian of $\mathcal{R}(X)$

Frechét derivative of $\mathcal{R}(\,.\,)$ at X:

$$\mathcal{R}_X' : Z \to (A + GX)^T Z + Z(A + GX).$$

Hessian/2nd order Frechét derivative of $\mathcal{R}(\,.\,)$ at X:

$$\mathcal{H} : (Z, Y) \to ZGY + YGZ$$

is indefinite in general unless $B_1 = 0$ or $B_2 = 0$.

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Solving Large-Scale Standard AREs

General form for $A, G = G^T, W = W^T \in \mathbb{R}^{n\times n}$ given and $X \in \mathbb{R}^{n\times n}$ unknown:

$$0 \quad = \quad \mathcal{R}(X) := A^T X + XA - XGX + W.$$

Large-scale AREs from semi-discretized PDE control problems:

- $n = 10^3 - 10^6$ ($\Longrightarrow 10^6 - 10^{12}$ unknowns!),
- A has sparse representation ($A = -M^{-1}K$ for FEM),
- G, W low-rank with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n\times m}$, $m \ll n$, $\quad C \in \mathbb{R}^{p\times n}$, $p \ll n$.
- Standard (eigenproblem-based) $\mathcal{O}(n^3)$ methods are not applicable!

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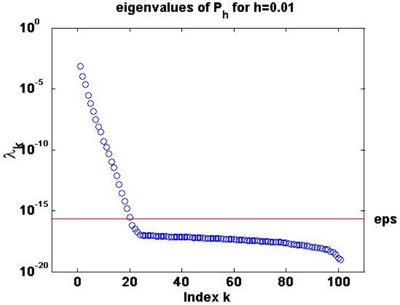
Low-Rank Approximation

ARE $0 = A^T X + XA - XBB^T X + CC^T$

Consider spectrum of ARE solution (analogous for Lyapunov equations).

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \implies n = 101$.



Idea: $X = X^T \geq 0 \implies$

$$X = YY^T = \sum_{k=1}^{n} \lambda_k y_k y_k^T \approx Y^{(r)}(Y^{(r)})^T = \sum_{k=1}^{r} \lambda_k y_k y_k^T.$$

Newton's Method for AREs

[Kleinman '68, Mehrmann '91, Lancaster/Rodman '95, B./Byers '94/'98, B. '97, Guo/Laub '99]

- Consider $0 = \mathcal{R}(X) = C^T C + A^T X + XA - XBB^T X.$
- Frechét derivative of $\mathcal{R}(X)$ at X:

$$\mathcal{R}'_X : Z \to (A - BB^T X)^T Z + Z(A - BB^T X).$$

- Newton-Kantorovich method:

$$X_{j+1} = X_j - \left(\mathcal{R}'_{X_j}\right)^{-1} \mathcal{R}(X_j), \quad j = 0, 1, 2, \dots$$

Newton's method (with line search) for AREs

FOR $j = 0, 1, \dots$

1. $A_j \leftarrow A - BB^T X_j =: A - BK_j.$
2. Solve the Lyapunov equation $A_j^T N_j + N_j A_j = -\mathcal{R}(X_j).$
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END FOR j

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Newton's Method for AREs

Properties and Implementation

- Convergence for K_0 stabilizing:
  - $A_j = A - BK_j = A - BB^T X_j$ is stable $\forall\ j \geq 0$.
  - $\lim_{j\to\infty} \|\mathcal{R}(X_j)\|_F = 0$ (monotonically).
  - $\lim_{j\to\infty} X_j = X_* \geq 0$ (locally quadratic).
- Need large-scale Lyapunov solver; here, ADI iteration:
  linear systems with dense, but "sparse+low rank" coefficient matrix

$$A_j = A - B \cdot K_j$$

- $m \ll n \implies$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j)^{-1} = (I_n + A^{-1}B(I_m - K_jA^{-1}B)^{-1}K_j)A^{-1}.$$

- BUT: $X = X^T \in \mathbb{R}^{n\times n} \implies n(n+1)/2$ unknowns!

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ADI Method for Lyapunov Equations

- For $A \in \mathbb{R}^{n\times n}$ stable, $B \in \mathbb{R}^{n\times m}$ $(w \ll n)$, consider Lyapunov equation
$$AX + XA^T = -BB^T.$$
- ADI Iteration: [Wachspress 1988]
$$\begin{aligned}(A + p_k I)X_{(k-1)/2} &= -BB^T - X_{k-1}(A^T - p_k I)\\ (A + \overline{p_k} I)X_k{}^T &= -BB^T - X_{(k-1)/2}(A^T - \overline{p_k} I)\end{aligned}$$
with parameters $p_k \in \mathbb{C}^-$ and $p_{k+1} = \overline{p_k}$ if $p_k \notin \mathbb{R}$.
- For $X_0 = 0$ and proper choice of p_k: $\lim_{k\to\infty} X_k = X$ superlinear.
- Re-formulation using $X_k = Y_k Y_k^T$ yields iteration for Y_k...

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Factored ADI Iteration

Lyapunov equation $0 = AX + XA^T = -BB^T$.

Setting $X_k = Y_k Y_k^T$, some algebraic manipulations $\Longrightarrow$

Algorithm [Penzl '97, Li/White '02, B./Li/Penzl '99/'08]

$$V_1 \leftarrow \sqrt{-2\Re p_1}(A + p_1 I)^{-1}B, \qquad Y_1 \leftarrow V_1$$

FOR $j = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\Re p_k}{\Re p_{k-1}}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right),$$

$$Y_k \leftarrow \begin{bmatrix} Y_{k-1} & V_k \end{bmatrix}$$

$$Y_k \leftarrow \mathrm{rrqr}(Y_k, \tau) \qquad \text{\% column compression}$$

At convergence, $Y_{k_{\max}} Y_{k_{\max}}^T \approx X$, where

$$\mathrm{range}\left(Y_{k_{\max}}\right) = \mathrm{range}\left(\begin{bmatrix} V_1 & \ldots & V_{k_{\max}} \end{bmatrix}\right), \quad V_k = \Big[\,\Big] \in \mathbb{C}^{n \times m}.$$

Note: Implementation in real arithmetic possible, saves even one solve for complex conjugate pair of shifts [B./Kürschner/Saak '11].

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Low-Rank Newton-ADI for AREs

Re-write Newton's method for AREs

$$A_j^T N_j + N_j A_j = -\mathcal{R}(X_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + \underbrace{(X_j + N_j)}_{=X_{j+1}} A_j = \underbrace{-C^T C - X_j B B^T X_j}_{=:-W_j W_j^T}$$

Set $X_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

Factored Newton Iteration [B./Li/Penzl '99/'08]

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and use 'sparse + low-rank' structure of A_j.

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AREs with Indefinite Hessian

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$$\mathcal{R}(X) := C^T C + A^T X + XA + X(B_1 B_1^T - B_2 B_2^T)X = 0.$$

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Problems

- For large-scale problems, resulting, e.g., from H_∞ control, standard methods based on Hamiltonian/even eigenvalue problem can not be used due to $\mathcal{O}(n^3)$ complexity/dense matrix algebra.
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Problems

Quick-and-dirty solution: consider $X^{-1}\mathcal{R}(X)X^{-1} = 0$ [Damm '02]

$\rightsquigarrow$ standard ARE for $\tilde{X} \equiv X^{-1}$

$$\tilde{\mathcal{R}}(\tilde{X}) := (B_1 B_1^T - B_2 B_2^T) + \tilde{X}A^T + A\tilde{X} + \tilde{X}C^T C\tilde{X} = 0.$$

Newton's method will converge to stabilizing solution, Newton-ADI can be employed (with modification for indefinite constant term).

But: low-rank approximation of $\tilde{X}$ will not yield good approximation of X
$\Rightarrow$ not feasible for large-scale problems!

Lyapunov Iterations/Perturbed Hessian Approach

[Cherfi/Abou-Kandil/Bourles '05 (Proc. ACSE 2005)]

Idea

Perturb Hessian to enforce semi-definiteness: write

$$0 = A^T X + XA + Q - XGX = A^T X + XA + Q - XDX + X(D - G)X,$$

where $D = G + \alpha I \geq 0$ with $\alpha \geq \min\{0, -\lambda_{\max}(G)\}$.

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Here: $G = B_2 B_2^T - B_1 B_1^T$

$\Rightarrow$ use $\alpha = \|B_1\|^2$ for spectral/Frobenius norm or

$$\alpha = \|B_1\|_1 \cdot \|B_1\|_\infty.$$

Remark

$W \geq -G$ can be used instead of αI, e.g., $W = \beta B_1 B_1^T$ with $\beta \geq 1$.

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Lyapunov iteration

Based on

$$(A - DX)^T X + X(A - DX) = -Q - XDX - \alpha X^2,$$

iterate

FOR $k = 0, 1, \ldots,$ solve Lyapunov equation

$$(A - DX_k)^T X_{k+1} + X_{k+1}(A - DX_k) = -Q - X_k D X_k - \alpha X_k^2.$$

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Easy to convert to low-rank iteration employing low-rank ADI for Lyapunov equations, e.g. with $W = B_1 B_1^T$ instead of αI: the Lyapunov equation becomes

$$\begin{aligned}(A - B_2 B_2^T Y_k Y_k)^T Y_{k+1} Y_{k+1}^T + Y_{k+1} Y_{k+1}^T (A - B_2 B_2^T Y_k Y_k) \\ = -CC^T - Y_k Y_k^T B_1 B_1^T Y_k Y_k^T - Y_k Y_k^T B_2 B_2^T Y_k Y_k^T \\ = -[\, C,\ Y_k Y_k^T B_1,\ Y_k Y_k^T B_2 \,] \begin{bmatrix} C^T \\ B_1^T Y_k Y_k^T \\ B_2^T Y_k Y_k^T \end{bmatrix}.\end{aligned}$$

Lyapunov Iterations/Perturbed Hessian Approach

Convergence

Theorem [Cherfi/Abou-Kandil/Bourles '05]

If

- $\exists\ \hat{X}$ such that $\mathcal{R}(\hat{X}) \geq 0$,
- $\exists\ X_0 = X_0^T \geq \hat{X}$ such that $\mathcal{R}(X_0) \leq 0$ and $A - DX_0$ is Hurwitz,

then

a) $X_0 \geq \ldots \geq X_k \geq X_{k+1} \geq \ldots \geq \hat{X}$,

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d) $\exists \quad \lim_{k\to\infty} X_k =: \underline{X} \geq \hat{X}$,

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Main problems

- Conditions for initial guess make its computation difficult.
- Observed convergence is linear.

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d) $\exists\ \lim_{k\to\infty} X_k =: \underline{X} \geq \hat{X}$,

e) $\underline{X}$ is semi-stabilizing.

Main problems

- Conditions for initial guess make its computation difficult.
- Observed convergence is linear.

Lyapunov Iterations/Perturbed Hessian Approach

Convergence

Theorem [Cherfi/Abou-Kandil/Bourles '05]

If

- $\exists\ \hat{X}$ such that $\mathcal{R}(\hat{X}) \geq 0$,
- $\exists\ X_0 = X_0^T \geq \hat{X}$ such that $\mathcal{R}(X_0) \leq 0$ and $A - DX_0$ is Hurwitz,

then

a) $X_0 \geq \ldots \geq X_k \geq X_{k+1} \geq \ldots \geq \hat{X}$,

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Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson '07 (Proc. ECC 2007)]

Idea

Consider

$$A^T X + XA + C^T C + X(B_1 B_1^T - B_2 B_2^T)X =: \mathcal{R}(X).$$

Then

$$\mathcal{R}(X+Z) = \mathcal{R}(X) + (\underbrace{A + (B_1 B_1^T - B_2 B_2^T)X}_{=:\widehat{A}})^T Z + Z\widehat{A} + Z(B_1 B_1^T - B_2 B_2^T)Z.$$

Furthermore, if $X = X^T$, $Z = Z^T$ solve the standard ARE

$$0 = \mathcal{R}(X) + \widehat{A}^T Z + Z\widehat{A} - ZB_2 B_2^T Z,$$

then

$$\mathcal{R}(X+Z) = ZB_1 B_1^T Z,$$
$$\|\mathcal{R}(X)\|_2 = \|B_1^T Z\|_2.$$

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Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson '07 (Proc. ECC 2007)]

Riccati iteration

1. Set $X_0 = 0$.
2. FOR $k = 1, 2, \ldots,$
   (i) Set $A_k := A + B_1(B_1^T X_k) - B_2(B_2^T X_k)$.
   (ii) Solve the ARE
   $$\mathcal{R}(X_k) + A_k^T Z_k + Z_k A_k - Z_k B_2 B_2^T Z_k = 0.$$
   (iii) Set $X_{k+1} := X_k + Z_k$.
   (iv) IF $\|B_1^T Z_k\|_2 < \text{tol}$ THEN Stop.

Remark. ARE for $k = 1$ is the standard LQR/H_2 ARE.

Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson '07 (Proc. ECC 2007)]

Theorem [Lanzon/Feng/B.D.O. Anderson 2007]

If

- (A, B_2) stabilizable,
- (A, C) has no unobservable purely imaginary modes, and
- $\exists$ stabilizing solution X_-,

then

a) $(A + B_1 B_1^T X_k, B_2)$ stabilizable for all $k = 0, 1, \ldots$,

b) $Z_k \geq 0$ for all $k = 0, 1, \ldots$,

c) $A + B_1 B_1^T X_k - B_2 B_2^T X_{k+1}$ is Hurwitz for all $k = 0, 1, \ldots$,

d) $\mathcal{R}(X_{k+1}) = Z_k B_1 B_1^T Z_k$ for all $k = 0, 1, \ldots$,

e) $X_- \geq \ldots \geq X_{k+1} \geq X_k \geq \ldots \geq 0$.

f) If $\exists \lim_{k\to\infty} X_k =: \underline{X}$, then $\underline{X} = X_-$, and

g) convergence is locally quadratic.

Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson '07 (Proc. ECC 2007)]

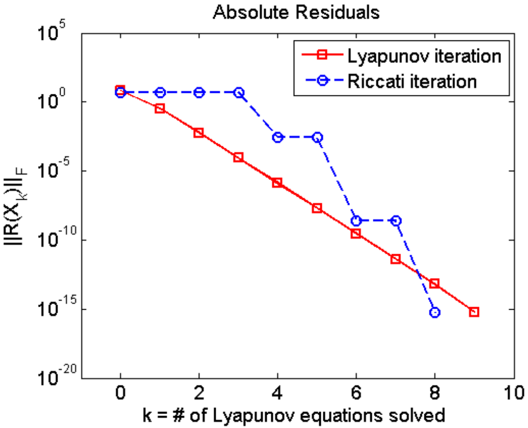
Riccati iteration – low-rank version [B. '08/'12]

1. Solve the ARE
$$C^TC + A^TZ_0 + Z_0A - Z_0B_2B_2^TZ_0 = 0$$
using Newton-ADI, yielding Y_0 with $Z_0 \approx Y_0Y_0^T$.
2. Set $R_1 := Y_0$. $\{\% \; R_1R_1^T \approx X_1.\}$
3. FOR $k = 1, 2, \ldots,$
   (i) Set $A_k := A + B_1(B_1^TR_k)R_k^T - B_2(B_2^TR_k)R_k^T$.
   (ii) Solve the ARE
   $$Y_{k-1}(Y_{k-1}^TB_1)(B_1^TY_{k-1})Y_{k-1}^T + A_k^TZ_k + Z_kA_k - Z_kB_2B_2^TZ_k = 0$$
   using Newton-ADI, yielding Y_k with $Z_k \approx Y_kY_k^T$.
   (iii) Set $R_{k+1} := \mathtt{rrqr}([R_k, \; Y_k], \tau)$. $\{\% \; R_{k+1}R_{k+1}^T \approx X_{k+1}\}$
   (iv) IF $\|(B_1^TY_k)Y_k^T\|_2 <$ tol THEN Stop.

Numerical Examples

Artificial Example

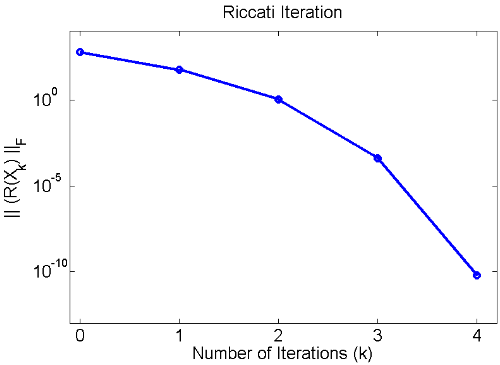
- Trivial example ($n = 2$) from [Cherfi/Abou-Kandil/Bourles '05].
- Compare convergence of Lyapunov and Riccati iterations.
- Solution of standard AREs with Newton's method.



Numerical Examples

Heat Equation

- Heat equation on $[0,1]^2$, heating/cooling in a vertical strip, random noise injection operator, temperature measurement in a strip at other side of the region ($\rightsquigarrow$ single-input, single-output system).
- FDM discretization, $n = 900$.
- Numerical ranks of Riccati iterates: 15 (for all iterations).



Conclusions and Open Problems

- Numerical computation of robust (H_∞-) controller for parabolic systems requires the solution of large scale AREs with indefinite Hessian.
- Low-rank Riccati iteration yields (hopefully) a reliable and efficient method for large-scale AREs with indefinite Hessian.
- Low-rank Lyapunov iteration is an extremely simple variant for large-scale problems, but exhibits slower convergence and requires difficult-to-compute initial value.
- To-Do list:
  - Implement Riccati iteration in LyaPack/M.E.S.S. style.
  - Practically relevant numerical tests.
  - Re-write Riccati iteration as feedback iteration.
  - Apply to practical robust control problem of parabolic systems (and to robust stabilization of flow problems, cf. [Dharmatti/Raymond/Thevenet SICON 49:2318–2348, 2011]).
  - Efficient computation of initial value for Lyapunov iterations?
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Fin.

Assumptions for Approximation Schemes

Let P^N be the canonical orthogonal projection

$$P^N : \mathcal{H} \to \mathcal{H}^N,$$

(i) For all $\varphi \in \mathcal{H}$ it holds that $T^N(t)P^N\varphi \to \mathbf{T}(t)\varphi$ uniformly on any bounded t-interval.

(ii) For all $\phi \in \mathcal{H}$ it holds that $T^N(t)^*P^N\phi \to \mathbf{T}(t)^*\phi$ uniformly on any bounded t-interval.

(iii) For all $v \in \mathcal{U}$, $w \in \mathcal{W}$ it holds $B_2^N v \to \mathbf{B}_2 v$, $B_1^N w \to \mathbf{B}_1 w$ and for all $\varphi \in \mathcal{H}$ it holds that $(B_j^N)^* P^N \varphi \to \mathbf{B}_j^* \varphi, \quad j = 1, 2.$

(iv) The family of pairs (A^N, B^N) is uniformly exponentially stabilizable, i.e., there exists a uniformly bounded sequence $F^N : \mathcal{H}^N \mapsto \mathcal{U}$ such that $A^N - B^N F^N$ generates an exponentially stable C_0-semigroup.

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