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KRYLOV SUBSPACE RECYCLING FOR FASTER MODEL REDUCTION ALGORITHMS

Peter Benner

Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

Joint work with
Lihong Feng (MPI DCTS) and Jan Korvink (IMTEK/FRIAS, Freiburg)

Krylov Subspace Recycling

Problem

Solve sequence of linear systems

$$A^{(i)}x^{(i)} = b^{(i)}, \quad i = 1, \dots, k_{\max},$$

where $A^{(i)} \in \mathbb{C}^{n \times n}$ nonsingular, $x^{(i)}, b^{(i)} \in \mathbb{C}^n$ for all i, by Krylov subspace methods.

Question: can we re-use information from solving $A^{(i-1)}x^{(i-1)} = b^{(i-1)}$ when solving $A^{(i)}x^{(i)} = b^{(i)}$?

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Krylov subspace recycling

Store and use (part of)

$$\mathcal{K}(A^{(i-1)}, r_0^{(i-1)}, \ell) := \operatorname{span}\{r_0^{(i-1)}, A^{(i-1)}r_0^{(i-1)}, \dots, (A^{(i-1)})^{\ell-1}r_0^{(i-1)}\},$$

to accelerate convergence when solving $A^{(i)}x^{(i)} = b^{(i)}$.
(Typically, use harmonic Ritz vectors constructed as by-product.)

Krylov Subspace Recycling

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where $A^{(i)} \in \mathbb{C}^{n\times n}$ nonsingular, $x^{(i)}, b^{(i)} \in \mathbb{C}^n$ for all i, by Krylov subspace methods.

Special cases

- Fixed right-hand side $A^{(i)}x^{(i)} = b$ (e.g., frequency response analysis).
  - $A^{(i)} = A + \sigma_k I_n \quad \rightsquigarrow$ use shift invariance of Krylov subspace $(\mathcal{K}(A + \sigma I, r, \ell) \equiv \mathcal{K}(A, r, \ell))$ e.g., [Freund '90, Datta/Saad '91].
  - $A^{(i)} = A + bf^{(i)} \quad \rightsquigarrow$ use feedback invariance of Krylov subspace $(\mathcal{K}(A + bf^{(i)}, b, \ell) \equiv \mathcal{K}(A, b, \ell))$ e.g., [B./Beckermann '11].
- Fixed coefficient matrix $Ax^{(i)} = b^{(i)}$ (e.g., multiple right-hand sides, stationary iterative methods with $b^{(i)} = b^{(i)}(x^{(i-1)})$).

Overview

Introduction

Model Reduction

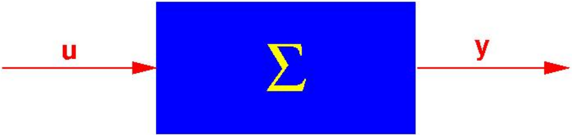
Dynamical Systems

$$\Sigma(p) : \begin{cases} E(p)\dot{x}(t;p) = f(t, x(t;p), u(t), p), & x(t_0) = x_0, \quad (a) \\ y(t;p) = g(t, x(t;p), u(t), p) & \quad (b) \end{cases}$$

with

- (generalized) states $x(t;p) \in \mathbb{R}^n$ ($E(p) \in \mathbb{R}^{n \times n}$),
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t;p) \in \mathbb{R}^q$, (b) is called output equation,
- $p \in \mathbb{R}^d$ is a parameter vector.

E singular $\Rightarrow$ (a) is system of differential-algebraic equations (DAEs)
otherwise $\Rightarrow$ (a) is system of ordinary differential equations (ODEs)



Model Reduction for Dynamical Systems

Original System

$$\Sigma(p): \begin{cases} E(p)\dot{x} = f(t,x,u,p), \\ y = g(t,x,u,p). \end{cases}$$

- states $x(t;p) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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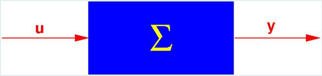
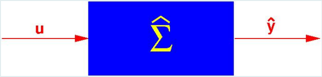
- parameters $p \in \mathbb{R}^d$.



Reduced-Order System

$$\hat{\Sigma}(p): \begin{cases} \hat{E}(p)\dot{\hat{x}} = \hat{f}(t,\hat{x},u,p), \\ \hat{y} = \hat{g}(t,\hat{x},u,p). \end{cases}$$

- states $\hat{x}(t;p) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t;p) \in \mathbb{R}^q$,
- parameters $p \in \mathbb{R}^d$.



Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals and relevant parameter settings.

Model Reduction for Dynamical Systems

Original System

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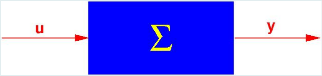

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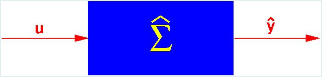


Goal:

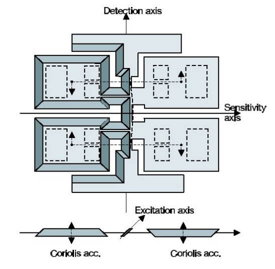
$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals and relevant parameter settings.

Motivation

Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

- Application: inertial navigation.



- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order: $N = 17.361 \rightsquigarrow n = 34.722,\ m = 1,\ p = 12.$
- Sensor for position control based on acceleration and rotation.

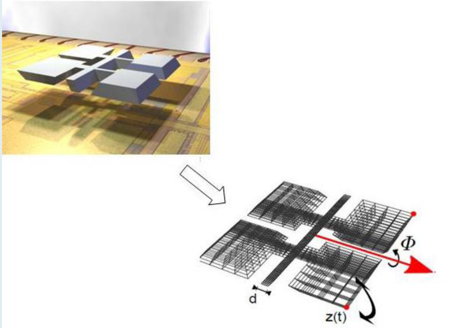
Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Motivation

Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: $M(w)\ddot{x}(t) + D(\theta, \alpha, \beta)\dot{x}(t) + T(w)x(t) = Bu(t).$



[Feng/B./Korvink '10]

Supported by DFG Projekt BE2174/7-1 *Automatic, Parameter-Preserving Model Reduction for Applications in Microsystems Technology* with IMTEK, Freiburg.

Motivation

Applications in Microsystems/MEMS Design

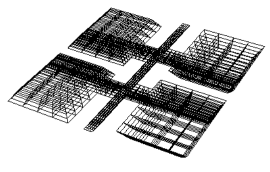
Microgyroscope (butterfly gyro)

Parametric FE model:

$$M(w)\ddot{x}(t) + D(\theta, \alpha, \beta)\dot{x}(t) + T(w)x(t) = Bu(t),$$

where

$$\begin{aligned} M(w) &= M_1 + wM_2, \\ D(\theta, \alpha, \beta) &= \theta(D_1 + wD_2) + \alpha M(w) + \beta T(w), \\ T(w) &= K_1 + \frac{1}{w}K_2 + dK_3, \end{aligned}$$



with

- width of bearing: w,
- angular velocity: θ,
- Rayleigh damping parameters: α, β,

[Feng/B./Korvink '10]

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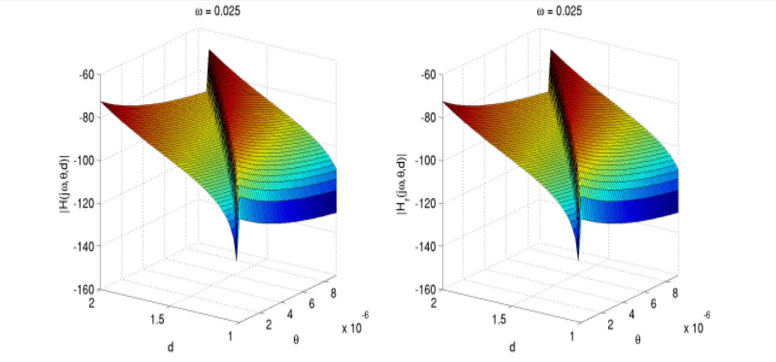
Motivation

Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Original. . . and reduced-order model.



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Linear Parametric Systems

Linear, time-invariant systems depending on parameters

$$\begin{aligned} E(p)\dot{x}(t;p) &= A(p)x(t;p) + B(p)u(t), \quad & A(p), E(p) \in \mathbb{R}^{n\times n},\\ y(t;p) &= C(p)x(t;p), & B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}. \end{aligned}$$

Laplace Transformation / Frequency Domain

Application of Laplace transformation ($x(t;p) \mapsto x(s;p)$, $\dot{x}(t;p) \mapsto sx(s;p)$) to linear system with $x(0) = 0$:

$$sE(p)x(s;p) = A(p)x(s;p) + B(p)u(s), \quad y(s;p) = C(p)x(s;p),$$

yields I/O-relation in frequency domain:

$$y(s;p) = \Big(\underbrace{C(p)(sE(p) - A(p))^{-1}B(p)}_{=:G(s;p)} \Big) u(s)$$

$G(s;p)$ is the parameter-dependent transfer function of $\Sigma(p)$.

Linear Parametric Systems

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Model Reduction for Linear Parametric Systems

Problem

Approximate the dynamical system

$$\begin{array}{rcl} E(p)\dot{x} & = & A(p)x + B(p)u, \\ y & = & C(p)x, \end{array} \qquad \begin{array}{l} A(p), E(p) \in \mathbb{R}^{n\times n}, \\ B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}, \end{array}$$

by reduced-order system

$$\begin{array}{rcl} \hat{E}(p)\dot{\hat{x}} & = & \hat{A}(p)\hat{x} + \hat{B}(p)u, \\ \hat{y} & = & \hat{C}(p)\hat{x}, \end{array} \qquad \begin{array}{l} \hat{A}(p), \hat{E}(p) \in \mathbb{R}^{r\times r}, \\ \hat{B}(p) \in \mathbb{R}^{r\times m}, \hat{C}(p) \in \mathbb{R}^{q\times r}, \end{array}$$

of order $r \ll n$, such that for any feasible p,

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\|\|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G})\leq r} \|G - \hat{G}\|$.

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Model Reduction for Linear Parametric Systems

Parametric System

$$\Sigma(p): \begin{cases} E(p)\dot{x}(t;p) &= A(p)x(t;p) + B(p)u(t), \\ y(t;p) &= C(p)x(t;p). \end{cases}$$

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Appropriate representation:

$$\begin{aligned} E(p) &= E_0 + e_1(p)E_1 + \ldots + e_{q_E}(p)E_{q_E}, \\ A(p) &= A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \\ B(p) &= B_0 + b_1(p)B_1 + \ldots + b_{q_B}(p)B_{q_B}, \\ C(p) &= C_0 + c_1(p)C_1 + \ldots + c_{q_C}(p)C_{q_C}, \end{aligned}$$

allows easy parameter preservation for projection based model reduction.

Model Reduction for Linear Parametric Systems

Parametric System

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Applications:

- Repeated simulation for varying material or geometry parameters, boundary conditions,
- Optimization and design.

Model Reduction for Linear Parametric Systems

Parametric System

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Applications:

- Repeated simulation for varying material or geometry parameters, boundary conditions,
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Additional model reduction goal:

preserve parameters as symbolic quantities in reduced-order model:

$$\widehat{\Sigma}(p): \left\{ \begin{array}{rcl} \hat{E}(p)\dot{\hat{x}}(t;p) & = & \hat{A}(p)\hat{x}(t;p) + \hat{B}(p)u(t), \\ \hat{y}(t;p) & = & \hat{C}(p)\hat{x}(t;p) \end{array} \right.$$

with states $\hat{x}(t;p) \in \mathbb{R}^r$.

Interpolatory Model Reduction

Short Introduction

Computation of reduced-order model by projection

Given a linear (descriptor) system $E\dot{x} = Ax + Bu,\ y = Cx$ with transfer function $G(s) = C(sE - A)^{-1}B,$ a reduced-order model is obtained using truncation matrices $V, W \in \mathbb{R}^{n\times r}$ with $W^T V = I_r$ ($\rightsquigarrow (VW^T)^2 = VW^T$ is projector) by computing

$$\hat{E} = W^T EV,\ \hat{A} = W^T AV,\ \hat{B} = W^T B,\ \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

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$$\hat{E} = W^TEV,\ \hat{A} = W^TAV,\ \hat{B} = W^TB,\ \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

Rational Interpolation/Moment-Matching

Choose V, W such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i}G(s_j) = \frac{d^i}{ds^i}\hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$

Interpolatory Model Reduction

Short Introduction

Theorem (simplified) [Grimme '97, Villemagne/Skelton '87]

If

$$\begin{aligned}\operatorname{span}\left\{(s_1E-A)^{-1}B,\dots,(s_kE-A)^{-1}B\right\} &\subset \operatorname{Ran}(V),\\ \operatorname{span}\left\{(s_1E-A)^{-T}C^T,\dots,(s_kE-A)^{-T}C^T\right\} &\subset \operatorname{Ran}(W),\end{aligned}$$

then

$$G(s_j)=\hat{G}(s_j),\quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j),\quad \text{for } j=1,\dots,k.$$

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then

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Remarks:

computation of V, W from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme '97],
- Iterative Rational Krylov-Algo. [Antoulas/Beattie/Gugercin '07].

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then

$$G(s_j)=\hat{G}(s_j), \quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j), \quad \text{for } j=1,\dots,k.$$

Remarks:

using Galerkin/one-sided projection yields $G(s_j)=\hat{G}(s_j)$, but in general

$$\frac{d}{ds}G(s_j)\neq\frac{d}{ds}\hat{G}(s_j).$$

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then

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Remarks:

$k=1$, standard Krylov subspace(s) of dimension K $\rightsquigarrow$ moment-matching methods/Padé approximation,

$$\frac{d^i}{ds^i}G(s_1)=\frac{d^i}{ds^i}\hat{G}(s_1), \quad i=0,\ldots,K-1(+K).$$

Interpolatory Model Reduction

Notation

Parametric Systems

$$\Sigma(p) : \begin{cases} E(p)\dot{x}(t;p) &= A(p)x(t;p) + B(p)u(t)), \\ y(t;p) &= C(p)x(t;p). \end{cases}$$

Assume

$$\begin{aligned} E(p) &= E_0 + e_1(p)E_1 + \ldots + e_{q_E}(p)E_{q_E}, \\ A(p) &= A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \\ B(p) &= B_0 + b_1(p)B_1 + \ldots + b_{q_B}(p)B_{q_B}, \\ C(p) &= C_0 + c_1(p)C_1 + \ldots + c_{q_C}(p)C_{q_C}. \end{aligned}$$

Interpolatory Model Reduction

Structure-Preservation

Petrov-Galerkin-type projection

For given projection matrices $V, W \in \mathbb{R}^{n\times r}$ with $W^T V = I_r$
($\rightsquigarrow (VW^T)^2 = VW^T$ is projector), compute

$$
\begin{aligned}
\hat{E}(p) &= W^T E_0 V + e_1(p) W^T E_1 V + \ldots + e_{q_E}(p) W^T E_{q_E} V, \\
&= \hat{E}_0 + e_1(p)\hat{E}_1 + \ldots + e_{q_E}(p)\hat{E}_{q_E}, \\
\hat{A}(p) &= W^T A_0 V + a_1(p) W^T A_1 V + \ldots + a_{q_A}(p) W^T A_{q_A} V, \\
&= \hat{A}_0 + a_1(p)\hat{A}_1 + \ldots + a_{q_A}(p)\hat{A}_{q_A}, \\
\hat{B}(p) &= W^T B_0 + b_1(p) W^T B_1 + \ldots + b_{q_B}(p) W^T B_{q_B}, \\
&= \hat{B}_0 + b_1(p)\hat{B}_1 + \ldots + b_{q_B}(p)\hat{B}_{q_B}, \\
\hat{C}(p) &= C_0 V + c_1(p) C_1 V + \ldots + c_{q_C}(p) C_{q_C} V, \\
&= \hat{C}_0 + c_1(p)\hat{C}_1 + \ldots + c_{q_C}(p)\hat{C}_{q_C}.
\end{aligned}
$$

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\hat{A}(p) &= W^T A_0 V + a_1(p) W^T A_1 V + \ldots + a_{q_A}(p) W^T A_{q_A} V, \\
&= \hat{A}_0 + a_1(p)\hat{A}_1 + \ldots + a_{q_A}(p)\hat{A}_{q_A}, \\
\hat{B}(p) &= W^T B_0 + b_1(p) W^T B_1 + \ldots + b_{q_B}(p) W^T B_{q_B}, \\
&= \hat{B}_0 + b_1(p)\hat{B}_1 + \ldots + b_{q_B}(p)\hat{B}_{q_B}, \\
\hat{C}(p) &= C_0 V + c_1(p) C_1 V + \ldots + c_{q_C}(p) C_{q_C} V, \\
&= \hat{C}_0 + c_1(p)\hat{C}_1 + \ldots + c_{q_C}(p)\hat{C}_{q_C}.
\end{aligned}
$$

PMOR based on Multi-Moment Matching

Idea: choose appropriate frequency parameter $\hat{s}$ and parameter vector $\hat{p}$, expand into multivariate power series about $(\hat{s}, \hat{p})$ and compute reduced-order model, so that

$$G(s,p) = \hat{G}(s,p) + \mathcal{O}\left(|s-\hat{s}|^K + \|p-\hat{p}\|^L + |s-\hat{s}|^k\|p-\hat{p}\|^\ell\right),$$

i.e., first $K, L, k+\ell$ (mostly: $K = L = k+\ell$) coefficients (multi-moments) of Taylor/Laurent series coincide.

Algorithms:

- [Daniel et al. '04]: explicit computation of moments, numerically unstable.
- [Farle et al. '06/'07]: Krylov subspace approach, only polynomial parameter-dependance, numerical properties not clear, but appears to be robust.
- [Feng/B. '07–'10]: Arnoldi-MGS method, employ recursive dependance of multi-moments, numerically robust, r often larger as with [Farle et al.].

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PMOR based on Rational Interpolation

Theory: Interpolation of the Transfer Function

Theorem 1 [Baur/Beattie/B./Gugercin '07/'09]

Let
$$\hat{G}(s,p) := \hat{C}(p)(s\hat{E}(p) - \hat{A}(p))^{-1}\hat{B}(p) = C(p)V(sW^TE(p)V - W^TA(p)V)^{-1}W^TB(p)$$

and suppose $\hat{p} = [\hat{p}_1, \ldots, \hat{p}_d]^T$ and $\hat{s} \in \mathbb{C}$ are chosen such that both $\hat{s}\,E(\hat{p}) - A(\hat{p})$ and $\hat{s}\,\hat{E}(\hat{p}) - \hat{A}(\hat{p})$ are invertible.

If
$$(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}B(\hat{p}) \in \mathrm{Ran}(V)$$

or
$$\left(C(\hat{p})\,(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\right)^T \in \mathrm{Ran}(W),$$

then $G(\hat{s},\hat{p}) = \hat{G}(\hat{s},\hat{p})$.

Note: result extends to MIMO case using tangential interpolation:
Let $0 \neq b \in \mathbb{R}^m$, $0 \neq c \in \mathbb{R}^q$ be arbitrary.

a) If $(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}B(\hat{p})b \in \mathrm{Ran}(V)$, then $G(\hat{s},\hat{p})b = \hat{G}(\hat{s},\hat{p})b$;

b) If $\left(c^TC(\hat{p})\,(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\right)^T \in \mathrm{Ran}(W)$, then $c^TG(\hat{s},\hat{p}) = c^T\hat{G}(\hat{s},\hat{p})$.

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PMOR based on Rational Interpolation

Algorithm

Generic implementation of interpolatory PMOR

Define $\mathcal{A}(s,p) := sE(p) - A(p)$.

1. Select "frequencies" $s_1, \ldots, s_k \in \mathbb{C}$ and parameters $p^{(1)}, \ldots, p^{(\ell)} \in \mathbb{R}^d$.
2. Compute (orthonormal) basis of
$$\begin{aligned}\mathcal{V} \quad = \quad & \operatorname{span}\{\mathcal{A}(s_1,p^{(1)})^{-1}B(p^{(1)}), \ldots, \mathcal{A}(s_1,p^{(1)})^{-K_1}B(p^{(1)}), \ldots \\ & \ldots, \mathcal{A}(s_k,p^{(\ell)})^{-1}B(p^{(\ell)}), \ldots, \mathcal{A}(s_k,p^{(\ell)})^{-K_\ell}B(p^{(\ell)})\}.\end{aligned}$$
3. For two-sided approach: compute (orthonormal) basis of
$$\begin{aligned}\mathcal{W} \quad = \quad & \operatorname{span}\{\mathcal{A}^T(s_1,p^{(1)})^{-1}C(p^{(1)}), \ldots, \mathcal{A}^T(s_1,p^{(1)})^{-K_1}C(p^{(1)}), \ldots, \\ & \ldots, \mathcal{A}^T(s_k,p^{(\ell)})^{-1}C(p^{(\ell)})^T, \ldots, \mathcal{A}^T(s_k,p^{(\ell)})^{-K_\ell}C(p^{(\ell)})^T\}.\end{aligned}$$
4. Set $V := [v_1, \ldots, v_{k\ell}]$, $\tilde{W} := [w_1, \ldots, w_{k\ell}]$, and $W := \tilde{W}(\tilde{W}^T V)^{-1}$.
(Note: $r \leq k\ell$ after eliminating potential linear dependencies by appropriate orthogonalization).
5. Compute $\begin{cases} \hat{A}(p) := W^T A(p) V, & \hat{B}(p) := W^T B(p) V, \\ \hat{C}(p) := W^T C(p) V, & \hat{E}(p) := W^T E(p) V. \end{cases}$

Recycling for PMOR

Problem setting

Let $A^{(i)} := \mathcal{A}(s_i, p^{(i)})$, $b_k^{(i)} :=$ kth column of $B(p^{(i)})$, etc.

- For two-sided projection, need to solve "dual" linear systems,

$$A^{(i)} v^{(i)} = b^{(i)}, \qquad \left(A^{(i)}\right)^T w^{(i)} = c^{(i)}, \quad i = 1, 2, \ldots$$

  Use variants of unsymmetric Lanczos process $\rightsquigarrow$ Kapil Ahuja's talk this session.

- Here: consider one-sided method following robust implementation based on repeated modified Gram-Schmidt [Feng/B. '07/'08].
- Need to solve for $i = 1, \ldots, \ell =$ number of interpolation points:

$$\left.\begin{array}{rcll} A^{(i)} v_k^{(i)} & = & b_k^{(i)}, & k = 1, \ldots, m, \\ A^{(i)} v_k^{(i)} & = & v_{k-m}^{(i)}, & k = m+1, \ldots, 2m, \\ & \vdots & & \\ A^{(i)} v_k^{(i)} & = & v_{k-m}^{(i)}, & k = (K_i - 1)m + 1, \ldots, K_i m. \end{array}\right\}$$

No change in coefficient matrix!

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Recycling for PMOR

The Method

GCRO-DR [Parks/de Sturler/Mackey/Johnson/Maiti '06]

- Acronym: Generalized Conjugate Residual method with inner Orthogonalization and Deflated Restarting.
- Based on GCR(OT) [de Sturler '96], and GMRES-DR [Morgan '02], which tries to recover some of the GMRES convergence rate in restarted GMRES (GMRES(m)) by "recycling" k harmonic Ritz values.
- GCRO-DR is equivalent to GMRES-DR if applied to a single linear system.
- If applied to a sequence $A^{(i)}x^{(i)} = b^{(i)}$ of linear systems, it recycles the harmonic Ritz vectors from $A^{(i-1)}$ and updates them in the ith solve.
- In the PMOR settings, $A^{(i)}$ only changes a few times $\rightsquigarrow$ no need to update the harmonic Ritz vectors in system solves associated to $A^{(i)}$ $\rightsquigarrow$ significant savings in number of matvecs and orthogonalization steps possible.
- When switching from $A^{(i-1)}$ to $A^{(i)}$, we "turn on" the harmonic Ritz vector updating.
- $\Rightarrow$ variant G-DRvar of GCRO-DR, with restart length/size of Krylov subspace m and recycle space dimension k.

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Numerical Examples

Setting

- Use MATLAB® implementation `gmres` for comparison.
- All solvers are preconditioned with incomplete LU, drop tolerance 10^{-3}.
- Tested different versions of simplified GCRO-DR, here only results for most successful method "G-DRvar".
- Focus on applications in MEMS design.
- For all tests, see Max Planck Institute Magdeburg Preprint MPIMD/12-08, 2012, http://www.mpi-magdeburg.mpg.de/preprints/2012/08/.

Numerical Examples

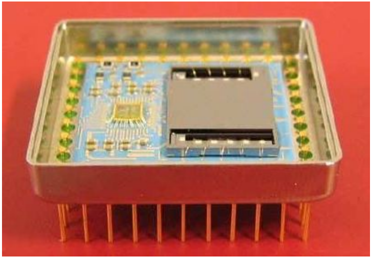
Butterfly microgyroscope [Billger '05]

FEM model ($n = 17,361,\ m = 1,\ q = 12$) of microgyroscope:

$$\begin{aligned}(M_1 + wM_2)\ddot{x}(t) + \big(\theta(D_1 + wD_2) + \alpha(M_1 + wM_2) + \\ \beta(K_1 + \tfrac{1}{w}K_2 + wK_3)\big)\dot{x}(t) + \ (K_1 + \tfrac{1}{w}K_2 + wK_3)x(t) \quad &= \quad Bu(t), \quad y = Cx.\end{aligned}$$

Parameters:

- width of bearing w,
- angular velocity θ,
- Rayleigh damping coefficients α, β.



For model with linear parameter dependency, introduce $d = 11$ artificial parameters, e.g. $p_8 = \frac{s}{w}\beta$.

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Computation of reduced-order model:

- 4 interpolation points $[s^{(i)}, p^{(i)}] = [\jmath\omega_i, w_i, \theta_i]$ $(\alpha_i = 0 = \beta_i)$,
- $K_i = 43$ for $i = 1, \ldots, 4$ $\Rightarrow$ 172 linear system solves with 4 different coefficient matrices

$$A^{(i)} := -\omega_i^2(M_1 + wM_2) + \jmath\omega_i\theta_i(D_1 + w_iD_2) + K_1 + \frac{1}{w_i}K_2 + w_iK_3,$$

- stopping criterion: $\|\mathrm{res}(x_j^{(i)})\| \leq 10^{-9}\|\mathrm{res}(x_0^{(i)})\|$,
- reduced second-order model with $r = 289$.

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	k	m	matvecs (average)	matvecs (total)	CPU time [s]
G-DRvar	30	90	633	101,340	21,931
GCRO-DR	30	90	638	109,800	23,192
GMRES(m)		90	no convergence		
GMRES			out of memory		

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	k	m	matvecs (average)	matvecs (total)	CPU time [s]
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GCRO-DR	30	85	689	118,440	23,525
G-DRvar	30	90	633	108,900	21,106
GCRO-DR	30	90	638	109,800	23,192
G-DRvar	30	95	597	102,645	19,812
GCRO-DR	30	95	601	103,425	23,038
G-DRvar	40	120	663	114,000	23,004
GCRO-DR	40	120	682	117,360	27,123

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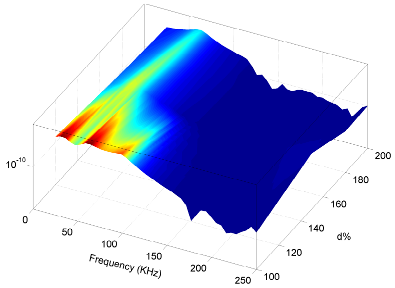
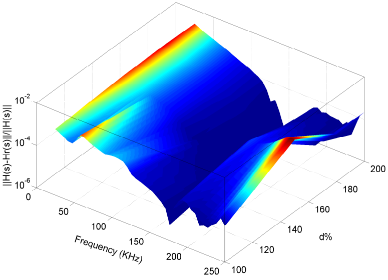
Model reduction performance: (for $G_{11}(\jmath\omega, p)$ for ...



Absolute Error



Relative Error

Numerical Examples

Microhotplate gas sensor [Bechtold et al '10]

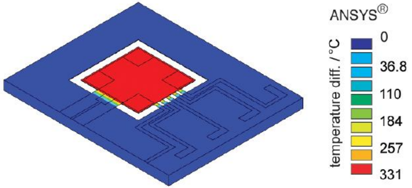
FEM model $(n = 60,020,\ m = 1,\ q = 2)$ of a gas sensor with four parameters:

$$(E_0 + \rho c_p E_1)\dot{x} + (K_0 + \kappa K_1 + h K_2)x = Bu(t), \quad y = Cx.$$

$$\leadsto \quad A^{(i)} := s_i(E_0 + \rho c_{p,i} E_1) + (K_0 + \kappa_i K_1 + h_i K_2)$$

Parameters:

- mass density
  ρ [kg/m^3],
- specific heat capacity
  c_p [J/kg/K],
- thermal conductivity
  κ [W/m/K],
- heat transfer coefficient
  h [W/m^2/K].



Numerical Examples

Microhotplate gas sensor [Bechtold et al '10]

FEM model $(n = 60{,}020,\ m = 1,\ q = 2)$ of a gas sensor with four parameters:

$$(E_0 + \rho c_p E_1)\dot{x} + (K_0 + \kappa K_1 + hK_2)x = Bu(t), \quad y = Cx.$$

$$\rightsquigarrow \quad A^{(i)} := s_i(E_0 + \rho c_{p,i} E_1) + (K_0 + \kappa_i K_1 + h_i K_2)$$

Computation of reduced-order model:

- Interpolation points $[s^{(1)}, p^{(1)}] = [0, 4, 700, 3100, 11]$, $[s^{(2)}, p^{(2)}] = [0, 3, 500, 3100, 10.5]$, $[s^{(3)}, p^{(3)}] = [0, 2.5, 439, 3100, 10]$,
- $K_i = 21$ for $i = 1, 2, 3 \Rightarrow r = 63$ linear system solves with 3 different coefficient matrices, with stopping criterion: $\|\mathrm{res}(x_j^{(i)})\| \leq 10^{-9} \|\mathrm{res}(x_0^{(i)})\|$.

	k	m	matvecs (total)	CPU time [s]
G-DRvar	50	60	5,220	1,777
GCRO-DR	50	60	13,650	2,602
GMRES(m)		60	?	$>$ 24h
GMRES			out of memory	

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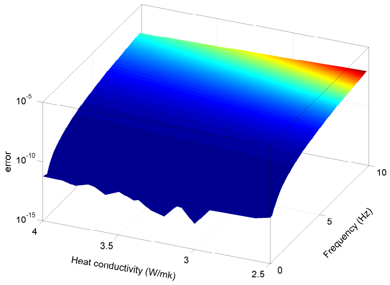
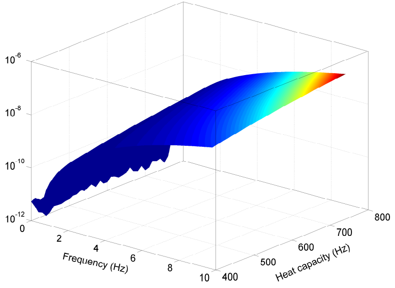
Model reduction performance: $|G_{11}(\jmath\omega, p) - \hat{G}_{11}(\jmath\omega, p)|/|G_{11}(\jmath\omega, p)|$ for …



… varying κ,



… varying c_p.

Conclusions and Outlook

Conclusions:

- Offline phase of PMOR algorithms can significantly be accelerated using Krylov subspace recycling.
- Due to special requirements in PMOR applications, usual recycling methods can be simplified.
- Application to real-world MEMS examples demonstrates significant savings over standard Krylov solvers.

Future Work:

- Use more advanced recycling techniques based on BiCG(Stab), TFQMR, IDR, ...
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Muchas gracias por su atención!

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