Mathematisches Kolloquium
Universität Bayreuth
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SYSTEM-THEORETIC MODEL REDUCTION FOR NONLINEAR SYSTEMS

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joint work with
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MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

Overview

Introduction

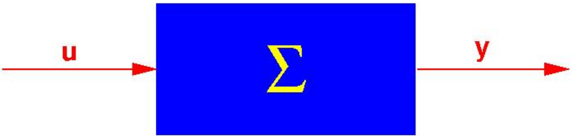
Model Reduction for Dynamical Systems

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.



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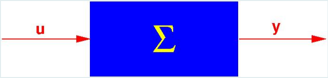

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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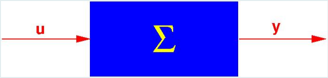

Reduced-Order Model (ROM)

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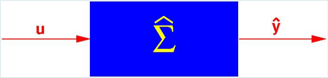

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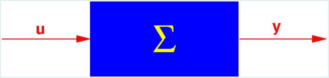

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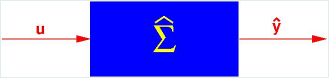

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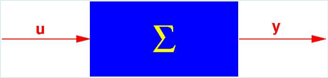

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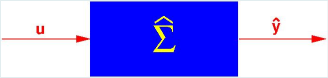


Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Model Reduction for Dynamical Systems

Linear Systems

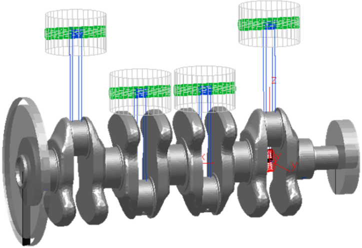
Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} E\dot{x} &= f(t,x,u) = Ax + Bu, \quad E, A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= g(t,x,u) = Cx + Du, \quad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

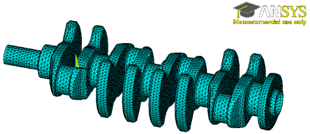
Application Areas

Structural Mechanics / Finite Element Modeling

since $\sim$1960ies



$\rightsquigarrow$



- Resolving complex 3D geometries $\Rightarrow$ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.

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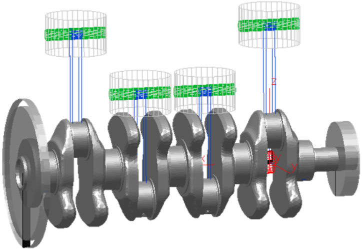

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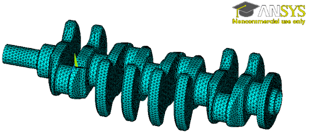


- Resolving complex 3D geometries ⇒ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.

Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) $\rightsquigarrow$ Craig-Bampton method.

Application Areas

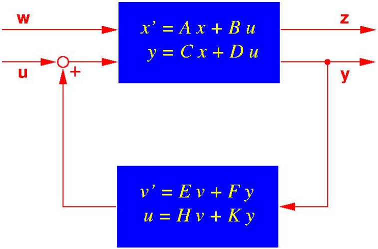
(Optimal) Control **since ∼1980ies**

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input = output of plant,
- output = input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



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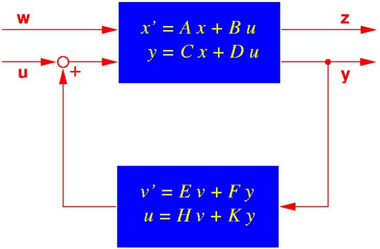

Practical controllers require small N ($N \sim 10$, say) due to

– real-time constraints,

– increasing fragility for larger N.

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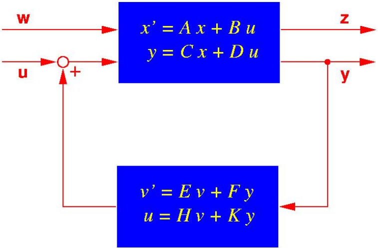

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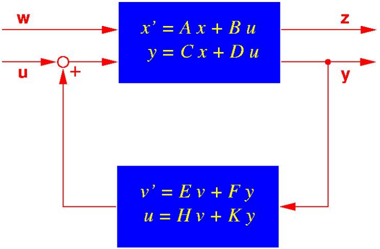

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Standard MOR techniques in systems and control: balanced truncation and related methods.

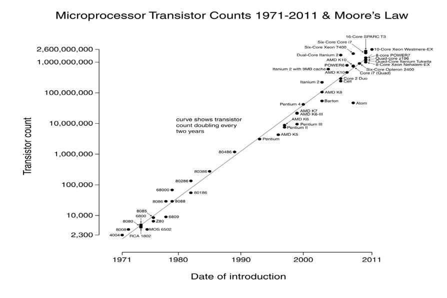
Application Areas

Micro Electronics/Circuit Simulation

since ∼1990ies

Progressive miniaturization

- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- **Moore's Law (1965/75)** states that the number of on-chip transistors doubles each 24 months.



Source: http://en.wikipedia.org/wiki/File:Transistor_Count_and_Moore'sLaw_-_2011.svg

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- Increase in packing density and multilayer technology requires modeling of interconncet to ensure that thermic/electro-magnetic effects do not disturb signal transmission.

Intel 4004 (1971)	Intel Core 2 Extreme (quad-core) (2007)
1 layer, 10μ technology	9 layers, 45*nm* technology
2,300 transistors	$>8,200,000$ transistors
64 kHz clock speed	>3 GHz clock speed.

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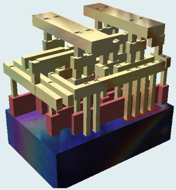

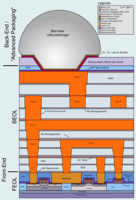

Source: http://en.wikipedia.org/wiki/Image:Silicon_chip_3d.png.

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- Here: mostly MOR for linear systems, they occur in micro electronics through modified nodal analysis (MNA) for RLC networks. e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

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Standard MOR techniques in circuit simulation:
Krylov subspace / Padé approximation / rational interpolation methods.

Application Areas

Many other disciplines in computational sciences and engineering like

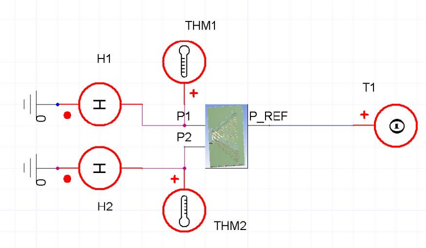
- computational fluid dynamics (CFD),
- computational electromagnetics,
- chemical process engineering,
- design of MEMS/NEMS (micro/nano-electrical-mechanical systems),
- computational acoustics,
- ...

Motivating Examples

Electro-Thermic Simulation of Integrated Circuit (IC)

[Source: Evgenii Rudnyi, CADFEM GmbH]

- SIMPLORER® test circuit with 2 transistors.



- Conservative thermic sub-system in SIMPLORER: voltage $\rightsquigarrow$ temperature, current $\rightsquigarrow$ heat flow.
- Original model: $n = 270.593$, $m = q = 2 \Rightarrow$ Computing time (on Intel Xeon dualcore 3GHz, 1 Thread):
  - Main computational cost for set-up data $\approx 22min$.
  - Computation of reduced models from set-up data: 44–49sec. ($r = 20$–70).
  - Bode plot (MATLAB on Intel Core i7, 2,67GHz, 12GB): 7.5h for original system, $<$ 1min for reduced system.
  - Speed-up factor: 18 including / $\geq$ 450 excluding reduced model generation!

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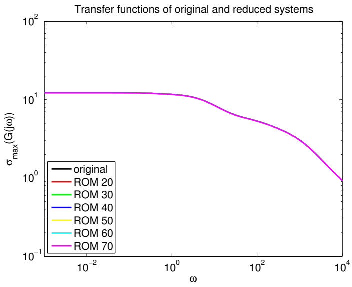
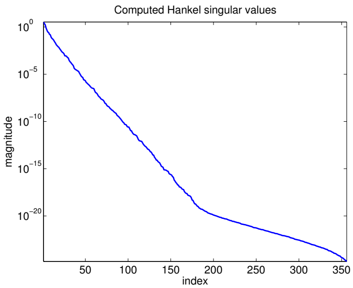
Bode Plot (Amplitude)



Hankel Singular Values



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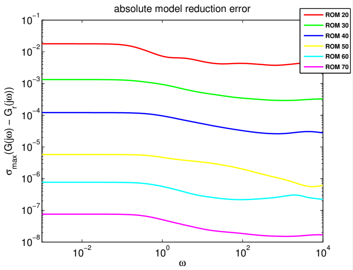
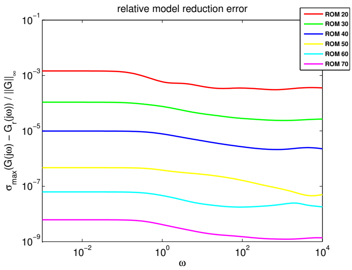
Absolute Error



Relative Error



Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

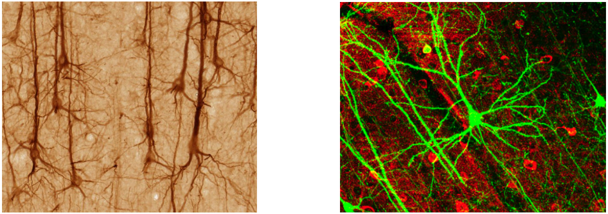
- Simple model for neuron (de-)activation [Chaturantabut/Sorensen 2009]

$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g,$$
$$w_t(x,t) = hv(x,t) - \gamma w(x,t) + g,$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$v(x,0) = 0, \qquad w(x,0) = 0, \qquad x \in [0,1]$$
$$v_x(0,t) = -i_0(t), \qquad v_x(1,t) = 0, \qquad t \geq 0,$$

where $\epsilon = 0.015, h = 0.5, \gamma = 2, g = 0.05, i_0(t) = 50000t^3 \exp(-15t)$.



Source: http://en.wikipedia.org/wiki/Neuron

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- Parameter g handled as an additional input.
- Original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$.

Motivating Examples

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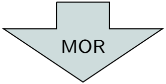
Introduction

Nonlinear Model Reduction

Given a large-scale control-affine nonlinear control system of the form

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t)) + bu(t), \\ y(t) = c^T x(t), \quad x(0) = x_0, \end{cases}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ nonlinear and $b, c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$.



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with $\hat{f} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}}$ and $\hat{b}, \hat{c} \in \mathbb{R}^{\hat{n}}$, $x \in \mathbb{R}^{\hat{n}}$, $u \in \mathbb{R}$ and

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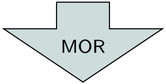

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Introduction

Common Reduction Techniques

Proper Orthogonal Decomposition (POD)

- Take computed or experimental 'snapshots' of full model: $[x(t_1), x(t_2), \ldots, x(t_N)] =: X$,
- perform SVD of snapshot matrix: $X = VSW^T \approx V_{\hat{n}} S_{\hat{n}} W_{\hat{n}}^T$.
- Reduction by POD-Galerkin projection: $\dot{\hat{x}} = V_{\hat{n}}^T f(V_{\hat{n}}\hat{x}) + V_{\hat{n}}^T Bu$.
- Requires evaluation of f
  $\rightsquigarrow$ discrete empirical interpolation [Sorensen/Chaturantabut '09].
- Input dependency due to 'snapshots'!

Introduction

Common Reduction Techniques

Proper Orthogonal Decomposition (POD)

- Take computed or experimental 'snapshots' of full model: $[x(t_1), x(t_2), \ldots, x(t_N)] =: X$,
- perform SVD of snapshot matrix: $X = VSW^T \approx V_{\hat{n}} S_{\hat{n}} W_{\hat{n}}^T$.
- Reduction by POD-Galerkin projection: $\dot{\hat{x}} = V_{\hat{n}}^T f(V_{\hat{n}} \hat{x}) + V_{\hat{n}}^T Bu$.
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  $\rightsquigarrow$ discrete empirical interpolation [Sorensen/Chaturantabut '09].
- Input dependency due to 'snapshots'!

Trajectory Piecewise Linear (TPWL)

- Linearize f along trajectory,
- reduce resulting linear systems,
- construct reduced model by weighted sum of linear systems.
- Requires simulation of original model and several linear reduction steps, many heuristics.

Introduction

Linear System Norms

Let us start with linear systems, i.e. $f(x) = Ax$.

Two common system norms for measuring approximation quality:

- $\mathcal{H}_2$-norm, $||\Sigma||_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\left(H^*(-i\omega)H(i\omega)\right) d\omega\right)^{\frac{1}{2}}$,
- $\mathcal{H}_\infty$-norm, $||\Sigma||_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(H(i\omega)\right)$,

where

$$H(s) = C\left(sI - A\right)^{-1} B$$

denotes the corresponding transfer function of the linear system.

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$$H(s) = C\left(sI - A\right)^{-1} B$$

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We focus on the first one $\rightsquigarrow$ interpolation-based model reduction approaches.

Introduction

Error system and $\mathcal{H}_2$-Optimality [Meier/Luenberger '67]

In order to find an $\mathcal{H}_2$-optimal reduced system, consider the error system $H(s) - \hat{H}(s)$ which can be realized by

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

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$\rightsquigarrow$ first-order necessary $\mathcal{H}_2$-optimality conditions (SISO)

$$H(-\lambda_i) = \hat{H}(-\lambda_i),$$
$$H'(-\lambda_i) = \hat{H}'(-\lambda_i),$$

where λ_i are the poles of the reduced system $\hat{\Sigma}$.

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$\rightsquigarrow$ first-order necessary $\mathcal{H}_2$-optimality conditions (MIMO)

$$\begin{aligned} H(-\lambda_i)\tilde{B}_i &= \hat{H}(-\lambda_i)\tilde{B}_i, && \text{for } i = 1, \ldots, \hat{n}, \\ \tilde{C}_i^T H(-\lambda_i) &= \tilde{C}_i^T \hat{H}(-\lambda_i), && \text{for } i = 1, \ldots, \hat{n}, \\ \tilde{C}_i^T H'(-\lambda_i)\tilde{B}_i &= \tilde{C}_i^T \hat{H}'(-\lambda_i)\tilde{B}_i && \text{for } i = 1, \ldots, \hat{n}, \end{aligned}$$

where $\hat{A} = R\Lambda R^{-T}$ is the spectral decomposition of the reduced system and $\tilde{B} = \hat{B}^T R^{-T}$, $\tilde{C} = \hat{C}R$.

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$$\begin{aligned} &\operatorname{vec}(I_p)^T \left(e_j e_i^T \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A\right)^{-1} \left(\tilde{B}^T \otimes B\right) \operatorname{vec}(I_m) \\ &= \operatorname{vec}(I_p)^T \left(e_j e_i^T \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A}\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \operatorname{vec}(I_m), \end{aligned}$$

for $i = 1, \ldots, \hat{n}$ and $j = 1, \ldots, p$.

Introduction

Interpolation of the Transfer Function [Grimme '97]

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

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$$V = \left[(\sigma_1 I - A)^{-1}B, \ldots, (\sigma_r I - A)^{-1}B\right],$$
$$W = \left[(\sigma_1 I - A^T)^{-1}C^T, \ldots, (\sigma_r I - A^T)^{-1}C^T\right].$$

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Then

$$H(\sigma_i) = \hat{H}(\sigma_i) \quad \text{and} \quad H'(\sigma_i) = \hat{H}'(\sigma_i),$$

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$$H(\sigma_i) = \hat{H}(\sigma_i) \quad \text{and} \quad H'(\sigma_i) = \hat{H}'(\sigma_i),$$

for $i = 1, \ldots, r$.
$\rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. '08], [Bunse-Gerstner et al. '07], [Van Dooren et al. '08]

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Bilinear Control Systems

Now consider $\dot{x} = Ax + g(x, u)$ with

$$g(x, u) = Bu + \left[N_1, \dots, N_m\right] (I_m \otimes x)\, u,$$

i.e. bilinear control systems:

$$\Sigma : \quad \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} N_i x(t) u_i(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

where $A, N_i \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m},\ C \in \mathbb{R}^{p \times n}$.

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- Approximation of weakly nonlinear systems $\rightsquigarrow$ Carleman linearization.
- A lot of linear concepts can be extended, e.g. transfer functions, Gramians, Lyapunov equations, ...
- An equivalent structure arises for some stochastic control systems.

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Some Basic Facts

Output Characterization (SISO): Volterra series

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} K(t_1, \dots, t_k) u(t - t_1 - \dots - t_k) \cdots u(t - t_k) dt_k \cdots dt_1,$$

with kernels $K(t_1, \dots, t_k) = Ce^{At_k} N_1 \cdots e^{At_2} N_1 e^{At_1} B$.

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Multivariate Laplace-transform (SISO):

$$H_k(s_1, \ldots, s_k) = C(s_k I - A)^{-1} N_1 \cdots (s_2 I - A)^{-1} N_1 (s_1 I - A)^{-1} B.$$

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Bilinear $\mathcal{H}_2$-norm (MIMO):

$$||\Sigma||_{\mathcal{H}_2} := \left(\operatorname{tr} \left(\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} \overline{H_k(i\omega_1, \dots, i\omega_k)} H_k^T(i\omega_1, \dots, i\omega_k) \right) \right)^{\frac{1}{2}}.$$

[Zhang/Lam. '02]

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

$\mathcal{H}_2$-Norm Computation

Lemma [B./Breiten '11]

Let Σ denote a bilinear system. Then, the $\mathcal{H}_2$-norm is given as:

$$||\Sigma||_{\mathcal{H}_2}^2 = (\mathrm{vec}(I_p))^T (C \otimes C) \left(-A \otimes I - I \otimes A - \sum_{i=1}^{m} N_i \otimes N_i \right)^{-1} (B \otimes B) \, \mathrm{vec}(I_m).$$

Error System

In order to find an $\mathcal{H}_2$-optimal reduced system, define the error system $\Sigma^{err} := \Sigma - \hat{\Sigma}$ as follows:

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad N_i^{err} = \begin{bmatrix} N_i & 0 \\ 0 & \hat{N}_i \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

$\mathcal{H}_2$-Model Reduction

$\mathcal{H}_2$-Optimality Conditions

Let us assume $\hat{\Sigma}$ is given by its eigenvalue decomposition:

$$\hat{A} = R\Lambda R^{-1}, \quad \tilde{N}_i = R^{-1}\hat{N}_i R, \quad \tilde{B} = R^{-1}\hat{B}, \quad \tilde{C} = \hat{C}R.$$

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Using Λ, $\tilde{N}_i$, $\tilde{B}$, $\tilde{C}$ as optimization parameters, we can derive necessary conditions for $\mathcal{H}_2$-optimality, e.g.:

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Where is the connection to the interpolation of transfer functions?

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$\rightsquigarrow$ tangential interpolation at mirror images of reduced system poles

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$\rightsquigarrow$ tangential interpolation at mirror images of reduced system poles

Note: [Flagg 2011] shows equivalence to interpolating the Volterra series!

A First Iterative Approach

Algorithm 1 Bilinear IRKA

Input: A, N_i, B, C, $\hat{A}$, $\hat{N}_i$, $\hat{B}$, $\hat{C}$
Output: A^{opt}, N_i^{opt}, B^{opt}, C^{opt}

1: **while** (change in $\Lambda > \epsilon$) **do**

2: $R\Lambda R^{-1} = \hat{A}$, $\tilde{B} = R^{-1}\hat{B}$, $\tilde{C} = \hat{C}R$, $\tilde{N}_i = R^{-1}\hat{N}_i R$

3: $\mathrm{vec}(V) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^{m} \tilde{N}_i \otimes N_i\right)^{-1} \left(\tilde{B} \otimes B\right) \mathrm{vec}(I_m)$

4: $\mathrm{vec}(W) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A^T - \sum_{i=1}^{m} \tilde{N}_i^T \otimes N_i^T\right)^{-1} \left(\tilde{C}^T \otimes C^T\right) \mathrm{vec}(I_q)$

5: $V = \mathrm{orth}(V)$, $W = \mathrm{orth}(W)$

6: $\hat{A} = \left(W^T V\right)^{-1} W^T A V$, $\hat{N}_i = \left(W^T V\right)^{-1} W^T N_i V$,
$\hat{B} = \left(W^T V\right)^{-1} W^T B$, $\hat{C} = CV$

7: **end while**

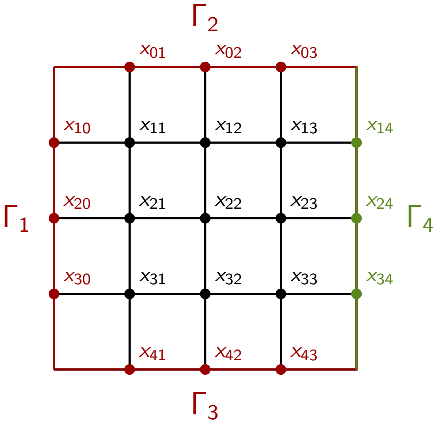
8: $A^{opt} = \hat{A}$, $N_i^{opt} = \hat{N}_i$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

A Heat Transfer Model

- 2-dimensional heat distribution [B./Saak '05]
- Boundary control by spraying intensities of a cooling fluid

$$\begin{aligned} \Omega &= (0,1) \times (0,1), & & \\ x_t &= \Delta x & & \text{in } \Omega, \\ n \cdot \nabla x &= c \cdot u_{1,2,3}(x-1) & & \text{on } \Gamma_1, \Gamma_2, \Gamma_3, \\ x &= u_4 & & \text{on } \Gamma_4. \end{aligned}$$



- Spatial discretization $k \times k$-grid

$$\Rightarrow \dot{x} \approx A_1 x + \sum_{i=1}^{3} N_i x u_i + Bu$$

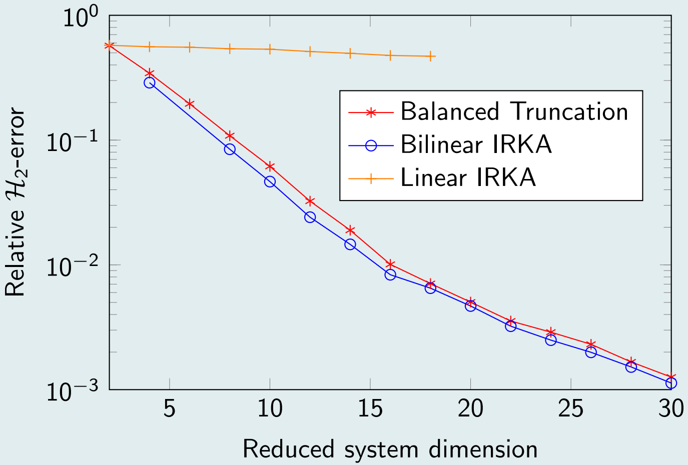
$$\Rightarrow A_2 = 0.$$

- Output: $y = \frac{1}{k^2}\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}.$

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

A Heat Transfer Model

Comparison of relative $\mathcal{H}_2$-error for $n = 10.000$



$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Fokker-Planck Equation

As a second example, we consider a dragged Brownian particle whose one-dimensional motion is given by

$$dX_t = -\nabla V(X_t, t)dt + \sqrt{2\sigma}dW_t,$$

with $\sigma = \frac{2}{3}$ and $V(x,u) = W(x,t) + \Phi(x,u_t) = (x^2-1)^2 - xu - x$.
Alternatively, one can consider ([Hartmann et al. '10]) ,

$$\rho(x,t)dx = \mathbf{P}\left[X_t \in [x, x+dx)\right]$$

which is described by the Fokker-Planck equation

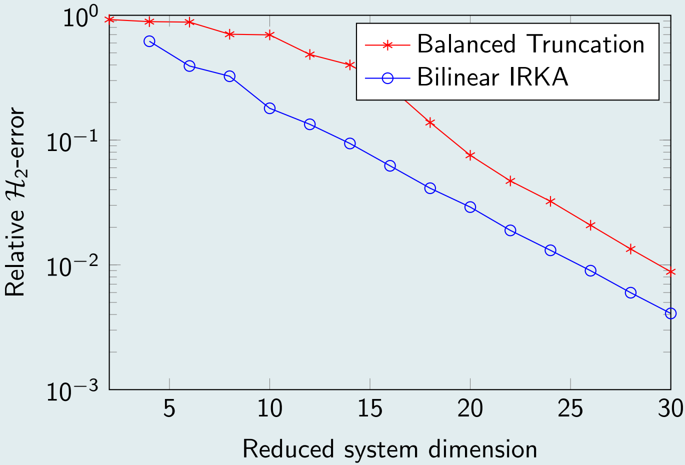
$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \sigma\Delta\rho + \nabla\cdot(\rho\nabla V), && (x,t) \in (-2,2)\times(0,T],\\
0 &= \sigma\nabla\rho + \rho\nabla B, && (x,t) \in \{-2,2\}\times[0,T],\\
\rho_0 &= \rho, && (x,t) \in (-2,2)\times 0.
\end{aligned}$$

Output C discrete characteristic function of the interval $[0.95, 1.05]$.

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Fokker-Planck Equation

Comparison of relative $\mathcal{H}_2$-error for $n = 500$



Nonlinear Model Reduction

Quadratic-Bilinear Differential Algebraic Equations (QBDAEs)

Coming back to the more general case with nonlinear $f(x)$, we consider the class of quadratic-bilinear differential algebraic equations

$$\Sigma : \quad \begin{cases} E\dot{x}(t) = A_1 x(t) + A_2 x(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \\ \quad y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

where $E, A_1, N \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times n^2}$ (Hessian tensor), $B, C^T \in \mathbb{R}^n$ are quite helpful.

- A large class of smooth nonlinear control-affine systems can be transformed into the above type of control system.
- The transformation is exact, but a slight increase of the state dimension has to be accepted.
- Input-output behavior can be characterized by generalized transfer functions $\rightsquigarrow$ enables us to use Krylov-based reduction techniques.

Nonlinear Model Reduction

Transformation via McCormick Relaxation

Theorem [Gu'09]

Assume that the state equation of a nonlinear system Σ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking derivatives and adding algebraic equations, respectively, Σ can be transformed into a system of QBDAEs.

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Example

- $\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \quad \dot{x}_2 = -x_2 + u.$

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  $\dot{z}_2 = \frac{2 \cdot x_1 \cdot z_1 \cdot z_2}{2 \cdot z_2} = x_1 \cdot z_1.$

Nonlinear Model Reduction

Variational Analysis and Linear Subsystems

Analysis of nonlinear systems by variational equation approach:

Nonlinear Model Reduction

Variational Analysis and Linear Subsystems

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- consider input of the form $\alpha u(t)$,

Nonlinear Model Reduction

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- consider input of the form $\alpha u(t)$,
- nonlinear system is assumed to be a series of homogeneous nonlinear subsystems, i.e. response should be of the form

$$x(t) = \alpha x_1(t) + \alpha^2 x_2(t) + \alpha^3 x_3(t) + \dots.$$

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- comparison of terms $\alpha^i, i = 1, 2, \ldots$ leads to series of systems

$$\begin{aligned} E\dot{x}_1 &= A_1 x_1 + Bu, \\ E\dot{x}_2 &= A_1 x_2 + A_2 x_1 \otimes x_1 + N x_1 u, \\ E\dot{x}_3 &= A_1 x_3 + A_2 \left(x_1 \otimes x_2 + x_2 \otimes x_1 \right) + N x_2 u \\ &\vdots \end{aligned}$$

Nonlinear Model Reduction

Variational Analysis and Linear Subsystems

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- although i-th subsystem is coupled nonlinearly to preceding systems, linear systems are obtained if terms x_j, $j < i$, are interpreted as pseudo-inputs.

Nonlinear Model Reduction

Generalized Transfer Functions

In a similar way, a series of generalized symmetric transfer functions can be obtained via the growing exponential approach:

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$$H_2(s_1, s_2) = \frac{1}{2!} C\left((s_1 + s_2)E - A_1\right)^{-1} \left[N\left(G_1(s_1) + G_1(s_2)\right)\right.$$
$$\left. + A_2\left(G_1(s_1) \otimes G_1(s_2) + G_1(s_2) \otimes G_1(s_1)\right)\right],$$

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$$\begin{aligned} H_3(s_1, s_2, s_3) = \frac{1}{3!} C\left((s_1 + s_2 + s_3)E - A_1\right)^{-1} & \\ \Big[N(G_2(s_1, s_2) &+ G_2(s_2, s_3) + G_2(s_1, s_3)) \\ + A_2\big(G_1(s_1) &\otimes G_2(s_2, s_3) + G_1(s_2) \otimes G_2(s_1, s_3) \\ &+ G_1(s_3) \otimes G_2(s_1, s_3) + G_2(s_2, s_3) \otimes G_1(s_1) \\ &+ G_2(s_1, s_3) \otimes G_1(s_2) + G_2(s_1, s_2) \otimes G_1(s_3)\big)\Big]. \end{aligned}$$

Nonlinear Model Reduction

Characterization via Multimoments

For simplicity, focus on the first two transfer functions. For $H_1(s_1)$, choosing σ and making use of the Neumann lemma leads to

$$H_1(s_1) = \sum_{i=0}^{\infty} C \underbrace{\left((A_1 - \sigma E)^{-1} E\right)^i (A_1 - \sigma E)^{-1} B \, (s_1 - \sigma)^i}_{m^i_{s_1,\sigma}}.$$

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Similarly, specifying an expansion point (τ, ξ) yields

$$H_2(s_1, s_2) = \frac{1}{2} \sum_{i=0}^{\infty} C \left((A_1 - (\tau + \xi)E)^{-1} E\right)^i (A_1 - (\tau + \xi)E)^{-1} \, (s_1 + s_2 - \tau - \xi)^i \cdot$$

$$\left[A_2 \left(\sum_{j=0}^{\infty} m^j_{s_1,\tau} \otimes \sum_{k=0}^{\infty} m^k_{s_2,\xi} + \sum_{k=0}^{\infty} m^k_{s_2,\xi} \otimes \sum_{j=0}^{\infty} m^j_{s_1,\tau}\right) + N \left(\sum_{p=0}^{\infty} m^p_{s_1,\tau} + \sum_{p=0}^{\infty} m^q_{s_2,\xi}\right)\right]$$

Nonlinear Model Reduction

Constructing the Projection Matrix

Goal: $\frac{\partial}{\partial s_1^{q-1}} H_1(\sigma) = \frac{\partial}{\partial s_1^{q-1}} \hat{H}_1(\sigma), \quad \frac{\partial}{\partial s_1^l s_2^m} H_2(\sigma,\sigma) = \frac{\partial}{\partial s_1^l s_2^m} \hat{H}_2(\sigma,\sigma), \ l+m \leq q-1.$

Construct the following sequence of nested Krylov subspaces

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for $i = 1:q$

$$V_2^i = \mathcal{K}_{q-i+1}\left((A_1-2\sigma E)^{-1}E, (A_1-2\sigma E)^{-1}NV_1(:,i)\right),$$

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& \qquad\quad \textbf{for } j = 1:\min(q-i+1, i) \\
& \qquad\qquad V_3^{i,j} = \mathcal{K}_{q-i-j+2}\left((A_1-2\sigma E)^{-1}E, (A_1-2\sigma E)^{-1}A_2V_1(:,i)\otimes V_1(:,j)\right),
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$V_1(:,i)$ denoting the i-*th* column of V_1.

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$V_1(:,i)$ denoting the i-*th* column of V_1. Set $\mathcal{V} = \operatorname{orth}[V_1, V_2^i, V_3^{i,j}]$ and construct $\hat{\Sigma}$ by the Galerkin-Projection $\mathcal{P} = \mathcal{V}\mathcal{V}^T$:

$$\hat{A}_1 = \mathcal{V}^T A_1 \mathcal{V} \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{A}_2 = \mathcal{V}^T A_2(\mathcal{V}\otimes\mathcal{V}) \in \mathbb{R}^{\hat{n}\times\hat{n}^2},$$

$$\hat{N} = \mathcal{V}^T N \mathcal{V} \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{b} = \mathcal{V}^T b \in \mathbb{R}^{\hat{n}}, \quad \hat{c}^T = c^T \mathcal{V} \in \mathbb{R}^{\hat{n}}.$$

Nonlinear Model Reduction

Tensors and Matricizations: A Short Excursion [Kolda/Bader '09, Grasedyck '10]

A tensor is a vector

$$(A_i)_{i \in \mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$$

indexed by a product index set

$$\mathcal{I} = \mathcal{I}_1 \times \cdots \times \mathcal{I}_d, \quad \#\mathcal{I}_j = n_j.$$

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For a given tensor A, the t-matricization $A^{(t)}$ is defined as

$$A^{(t)} \in \mathbb{R}^{\mathcal{I}_t \times \mathcal{I}_{t'}}, \quad A^{(t)}_{(i_\mu)_{\mu\in t},\,(i_\mu)_{\mu\in t'}} := A_{(i_1,\ldots,i_d)}, \quad t' := \{1,\ldots,d\}\backslash t.$$

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Example: For a given 3-tensor $A_{(i_1,i_2,i_3)}$ with $i_1, i_2, i_3 \in \{1,2\}$, we have:

$$A^{(1)} = \begin{bmatrix} A_{(1,1,1)} & A_{(1,2,1)} & A_{(1,1,2)} & A_{(1,2,2)} \\ A_{(2,1,1)} & A_{(2,2,1)} & A_{(2,1,2)} & A_{(2,2,2)} \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} A_{(1,1,1)} & A_{(2,1,1)} & A_{(1,1,2)} & A_{(2,1,2)} \\ A_{(1,2,1)} & A_{(2,2,1)} & A_{(1,2,2)} & A_{(2,2,2)} \end{bmatrix}.$$

Nonlinear Model Reduction

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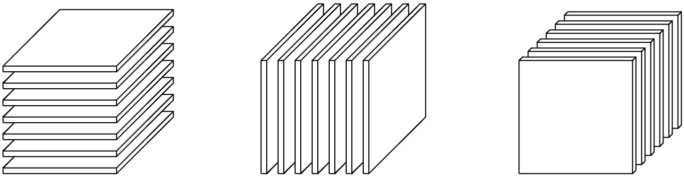

(a) Horizontal slices (b) Lateral slices (c) Frontal slices

Figure : Slices of a 3rd-order tensor. [Courtesy of Tammy Kolda]

Nonlinear Model Reduction

Tensors and Matricizations: A Short Excursion [Kolda/Bader '09, Grasedyck '10]

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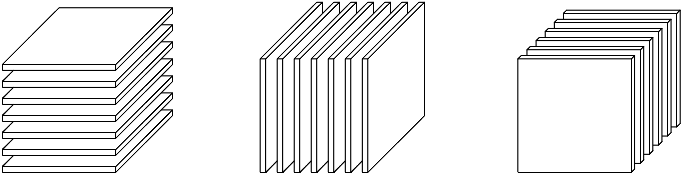

(a) Horizontal slices (b) Lateral slices (c) Frontal slices

Figure : Slices of a 3rd-order tensor. [Courtesy of Tammy Kolda]

$\rightsquigarrow$ Allows to compute matrix products more efficiently.

Nonlinear Model Reduction

Two-Sided Projection Methods

Similarly to the linear case, one can exploit duality concepts, in order to construct two-sided projection methods.

Nonlinear Model Reduction

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Interpreting $\mathcal{A}^{(2)}$ now as the 2-matricization of the Hessian 3-tensor corresponding to A_2, one can show that the dual Krylov spaces have to be constructed as follows

$$
\begin{aligned}
W_1 =& \mathcal{K}_q\left((A_1 - 2\sigma E)^{-T} E^T, (A_1 - 2\sigma E)^{-T} c\right) \\
&\textbf{for } i = 1 : q \\
&\quad W_2^i = \mathcal{K}_{q-i+1}\left((A_1 - \sigma E)^{-T} E^T, (A_1 - \sigma E)^{-T} N^T W_1(:,i)\right), \\
&\quad\quad \textbf{for } j = 1 : \min(q-i+1, i) \\
&\quad\quad\quad W_3^{i,j} = \mathcal{K}_{q-i-j+2}\left((A_1 - \sigma E)^{-T} E^T, (A_1 - \sigma E)^{-T} \mathcal{A}^{(2)} V_1(:,i) \otimes W_1(:,j)\right),
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Note: Due to the symmetry of the Hessian tensor, the 3-matricization $\mathcal{A}^{(3)}$ coincides with $\mathcal{A}^{(2)}$.

Nonlinear Model Reduction

Multimoment matching

Theorem

- $\Sigma = (E, A_1, A_2, N, b, c)$ original QBDAE system.
- Reduced system by Petrov-Galerkin projection $\mathcal{P} = \mathcal{V}\mathcal{W}^T$ with

$$V_1 = \mathcal{K}_{q_1}\left(E, A_1, b, \sigma\right), \quad W_1 = \mathcal{K}_{q_1}\left(E^T, A_1^T, c, 2\sigma\right)$$

$$\begin{aligned}
&\textbf{for } i = 1 : q_2 \\
&\quad V_2 = \mathcal{K}_{q_2-i+1}\left(E, A_1, NV_1(:,i), 2\sigma\right) \\
&\quad W_2 = \mathcal{K}_{q_2-i+1}\left(E^T, A_1^T, N^T W_1(:,i), \sigma\right) \\
&\qquad \textbf{for } j = 1 : \min(q_2 - i + 1, i) \\
&\qquad\quad V_3 = \mathcal{K}_{q_2-i-j+2}\left(E, A_1, A_2 V_1(:,i) \otimes V_1(:,j), 2\sigma\right) \\
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\end{aligned}$$

Then, it holds:

$$\frac{\partial^i H_1}{\partial s_1^i}(\sigma) = \frac{\partial^i \hat{H}_1}{\partial s_1^i}(\sigma), \quad \frac{\partial^i H_1}{\partial s_1^i}(2\sigma) = \frac{\partial^i \hat{H}_1}{\partial s_1^i}(2\sigma), \qquad i = 0, \ldots, q_1 - 1,$$

$$\frac{\partial^{i+j}}{\partial s_1^i s_2^j} H_2(\sigma, \sigma) = \frac{\partial^{i+j}}{\partial s_1^i s_2^j} \hat{H}_2(\sigma, \sigma), \qquad i + j \leq 2q_2 - 1.$$

Numerical Examples

Two-Dimensional Burgers Equation

- 2D-Burgers equation on $\underbrace{(0,1)\times(0,1)}_{:=\Omega}\times[0,T]$

$$u_t = -\left(u\cdot\nabla\right)u + \nu\Delta u$$

with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.

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- Consider initial and boundary conditions

$$\begin{aligned}
&u_x(x,y,0)=\frac{\sqrt{2}}{2}, && u_y(x,y,0)=\frac{\sqrt{2}}{2}, && \text{for } (x,y)\in\Omega_1:=(0,0.5],\\
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- Spatial discretization $\rightsquigarrow$ QBDAE system with nonzero I.C. and $N=0$ $\rightsquigarrow$ reformulate as system with zero I.C. and constant input.

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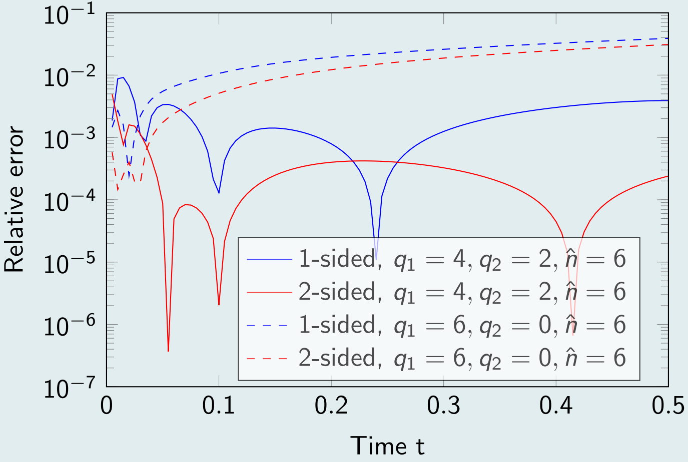
- Output C chosen to be average x-velocity.

Numerical Examples

Two-Dimensional Burgers Equation

Comparison of relative time-domain error for $n = 1600$



Numerical Examples

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  with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.
- Now consider initial and boundary conditions

$$\begin{aligned}
&u_x(x,y,0)=0, && u_y(x,y,0)=0, && \text{for } \ x,y\in\Omega,\\
&u_x=\cos(\pi t), && u_y=\cos(2\pi t), && \text{for } \ (x,y)\in\{0,1\}\times(0,1),\\
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- Spatial discretization $\rightsquigarrow$ QBDAE system with zero I.C. and 4 inputs $B\in\mathbb{R}^{n\times 4}$, N_1,N_2,N_3,N_4, ROM with $q_1=5, q_2=2, \sigma=0, \hat{n}=52$.

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- State reconstruction by reduced model $x\approx V\hat{x}$, max. rel. err $<3\%$.

Numerical Examples

The Chafee-Infante equation

- Consider PDE with a cubic nonlinearity:

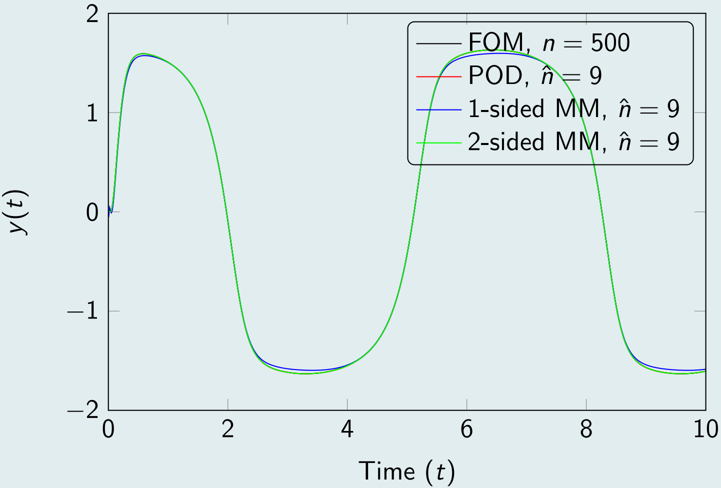
$$\begin{aligned} v_t + v^3 &= v_{xx} + v, && \text{in } (0,1)\times(0,T),\\ v(0,\cdot) &= u(t), && \text{in } (0,T),\\ v_x(1,\cdot) &= 0, && \text{in } (0,T),\\ v(x,0) &= v_0(x), && \text{in } (0,1) \end{aligned}$$

- original state dimension $n = 500$, QBDAE dimension $N = 2\cdot 500$, reduced QBDAE dimension $r = 9$

Numerical Examples

The Chafee-Infante equation

Comparison between moment-matching and POD ($u(t) = 5\cos(t)$)



Numerical Examples

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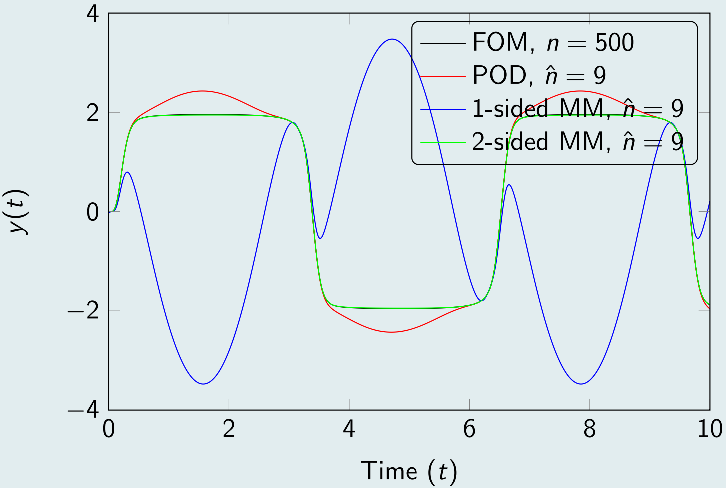
Comparison between moment-matching and POD ($u(t) = 50\sin(t)$)



Numerical Examples

The FitzHugh-Nagumo System

- FitzHugh-Nagumo system modeling a neuron

[Chaturantabut, Sorensen '09]

$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1], \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

where

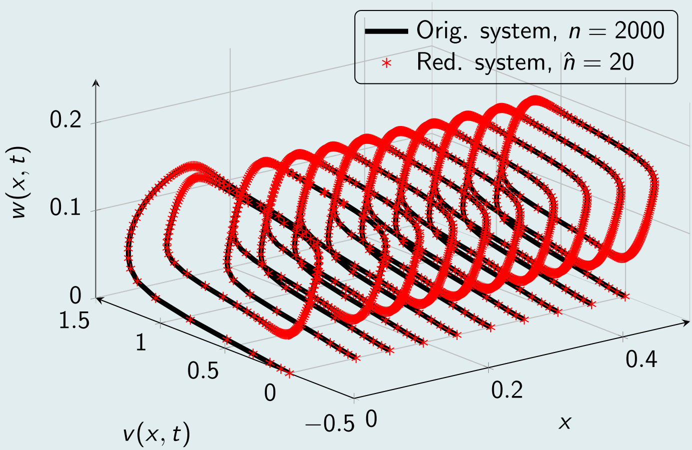
$\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ g = 0.05,\ i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)$

- original state dimension $n = 2 \cdot 1000$, QBDAE dimension $N = 3 \cdot 1000$, reduced QBDAE dimension $r = 20$

Numerical Examples

The FitzHugh-Nagumo System

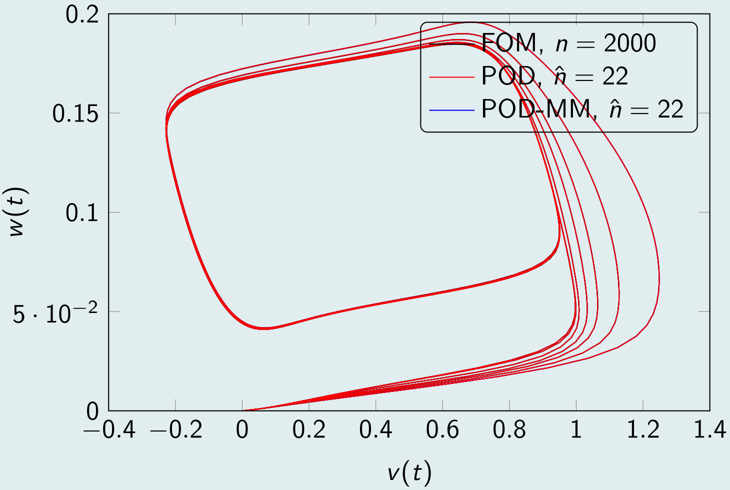
Limit cycle behavior for 1-sided proj. (ROM, $\hat{n} = 20, \sigma = 4$)



Numerical Examples

The FitzHugh-Nagumo System

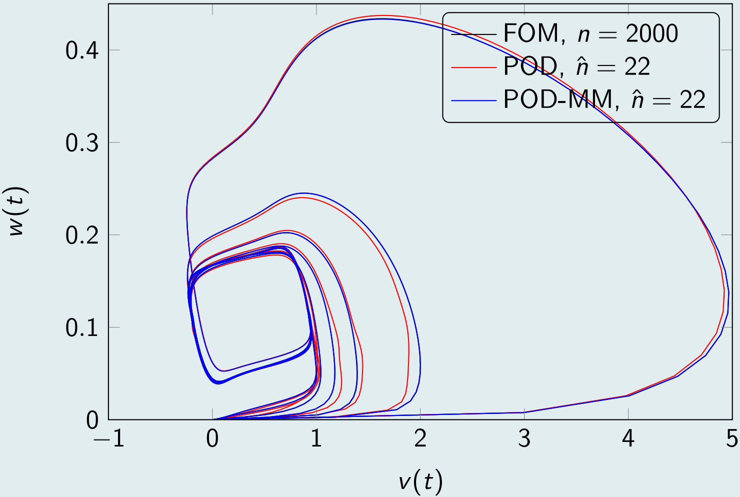
POD via moment-matching (training input)



Numerical Examples

The FitzHugh-Nagumo System

POD via moment-matching (varying input)



Conclusions and Outlook

- Many nonlinear dynamics can be expressed by a system of quadratic-bilinear differential algebraic equations.
- For this type of systems, a frequency domain analysis leads to certain generalized transfer functions.
- There exist Krylov subspace methods that extend the concept of moment-matching $\rightsquigarrow$ using basic tools from tensor theory allows for better approximations.
- In contrast to other methods like TPWL and POD, the reduction process is independent of the control input.

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- In contrast to other methods like TPWL and POD, the reduction process is independent of the control input.
- Optimal choice of interpolation points?
- Stability/index-preserving reduction possible?

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