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Computing Eigenvalues of $\mathcal{H}$(ackbusch) Matrices

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Max Planck Mathematicians...

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Courtesy of Joachim Heinze.

Eigenvalue Problem

Definition

The pair $(\lambda, v) \in \mathbb{R} \times \mathbb{R}^n$ is called an *eigenpair* of the symmetric matrix $M = M^T \in \mathbb{R}^{n \times n}$, if

$$Mv = v\lambda.$$

The set $\Lambda(M) = \{\lambda | \exists v : (\lambda, v) \text{ eigenpair of } M\}$ is the spectrum of M.

Similarity Transformation

$$\Lambda(M) = \Lambda(P^{-1}MP) \qquad \forall P \text{ invertible}$$

Classification of Eigenvalue Problems

[Golub, Van der Vorst '00]

- Is M real or complex?

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  $M \in \mathcal{H}(T_{\mathcal{I} \times \mathcal{I}}, k) \Rightarrow$ see next slide

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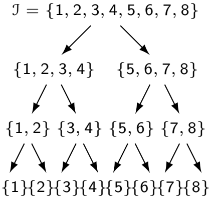
  $M \in \mathcal{H}(T_{\mathcal{I} \times \mathcal{I}}, k) \Rightarrow$ see next slide
- Which eigenvalues required?
  some (inner) or all eigenvalues

$\mathcal{H}$-Matrices

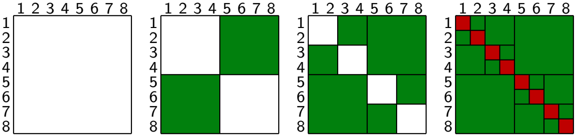
[Hackbusch 1998]

Some dense matrices, e.g. BEM or FEM, can be approximated by $\mathcal{H}$-matrices in a data-sparse manner.

hierarchical tree $T_{\mathcal{I}}$



block $\mathcal{H}$-tree $T_{\mathcal{I}\times\mathcal{I}}$



dense matrices, **rank-k-matrices**

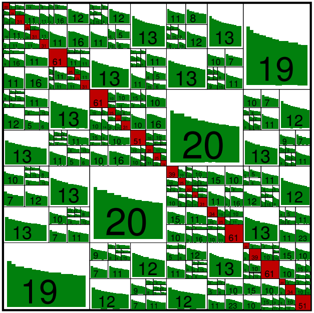
rank-k-matrix: $M_{a\times b} = AB^T$, $A \in \mathbb{R}^{n\times k}, B \in \mathbb{R}^{m\times k}$ $(k \ll n, m)$,

$\mathcal{H}$-Matrices

[Hackbusch 1998]

Hierarchical Matrices

$$\mathcal{H}(T_{\mathcal{I}\times\mathcal{I}}, k) = \left\{ M \in \mathbb{R}^{\mathcal{I}\times\mathcal{I}} \middle| \operatorname{rank}(M_{a\times b}) \leq k \ \forall a \times b \text{ admissible} \right\}$$



- adaptive rank $k(\varepsilon)$
- storage $N_{St,\mathcal{H}}(T, k) = \mathcal{O}(n \log n \; k(\varepsilon))$
- complexity of approximate arithmetic

$M_{\mathcal{H}}v$	$\mathcal{O}(n \log n \; k(\varepsilon))$
$+_{\mathcal{H}}, -_{\mathcal{H}}$	$\mathcal{O}(n \log n \; k(\varepsilon)^2)$
$*_{\mathcal{H}}, \mathcal{H}LU(\cdot), (\cdot)^{-1}_{\mathcal{H}}$	$\mathcal{O}(n (\log n)^2 \; k(\varepsilon)^2)$

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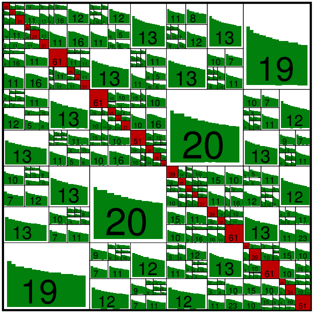

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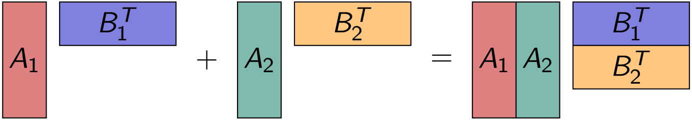
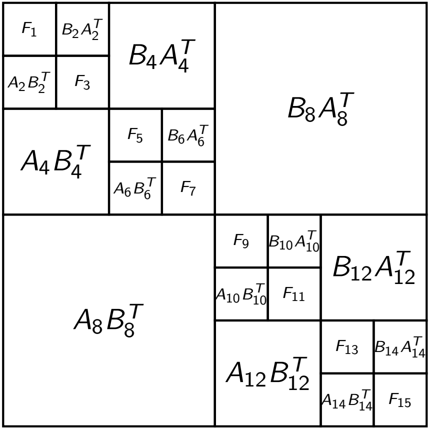


Special Case: $\mathcal{H}_\ell$-Matrices

[Hackbusch 1998]



Structure of a symmetric $\mathcal{H}_3(k)$-matrix.

$\mathcal{H}$lib

$\mathcal{H}$lib [Börm, Grasedyck, et al.]

We use the $\mathcal{H}$lib (`www.hlib.org`) for the $\mathcal{H}$-arithmetic operations and some examples out of the library for testing the eigenvalue algorithm.

Eigenvalues of Symmetric $\mathcal{H}$-Matrices

$$M = M^T \in \mathcal{H}(T, k)$$
$$\Downarrow$$

$$\Lambda_{\mathcal{H}}(M) = \{\lambda_1, \ldots, \lambda_n\} \text{ in } \mathcal{O}(n^2 (\log n)^\alpha k^\beta)$$

$$\{\lambda_i\} \in \Lambda_{\mathcal{H}}(M) \text{ in } \mathcal{O}(n (\log n)^\alpha k^\beta)?$$

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dense: $M + N$, Mv in $\mathcal{O}(n^2)$ and $\Lambda(M)$ in $\mathcal{O}(n^3)$

LR Cholesky Algorithm

QR-like Algorithm

LR Cholesky Algorithm

LR Cholesky Algorithm

LR Cholesky Algorithm

[RUTISHAUSER 1958]

LR-Cholesky Transformation

for $i = 1, \ldots$ **do**

$L_i L_i^T = M_i$

$M_{i+1} = L_i^T L_i$

end

LR Cholesky Algorithm

[Rutishauser 1958]

LR-Cholesky Transformation

for $i = 1, \ldots$ **do**

$$L_i L_i^T = M_i \Rightarrow L_i = M_i L_i^{-T}$$
$$M_{i+1} = L_i^T L_i = L_i^T M_i L_i^{-T}$$

end

$$\lim_{i\to\infty} M_i = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}(T, 0)$$

LR Cholesky Algorithm

[Rutishauser 1958]

LR-Cholesky Transformation

for $i = 1, \ldots$ **do**

$$L_i L_i^T = M_i - \mu_i \mathcal{I}$$

$$M_{i+1} = L_i^T L_i + \mu_i \mathcal{I}$$

end

$$\lim_{i \to \infty} M_i = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}(T, 0)$$

$\forall i$: $M_i - \mu_i \mathcal{I}$ symmetric positive definite

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$\mathcal{H}$-LR-Cholesky Transformation

for $i = 1, \ldots$ **do**

$\tilde{L}_i = \mathcal{H}$-`Cholesky factorization`$(\tilde{M}_i - \mu_i \mathcal{I})$

$\tilde{M}_{i+1} = \tilde{L}_i^T *_{\mathcal{H}} \tilde{L}_i + \mu_i \mathcal{I}$

end

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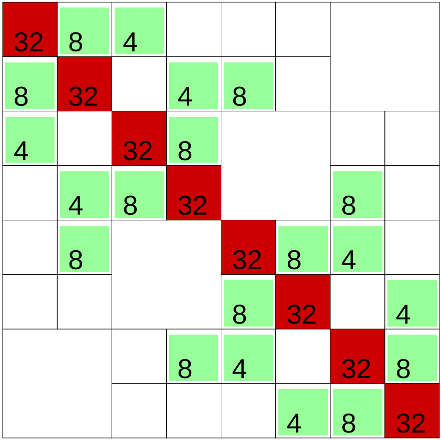
end

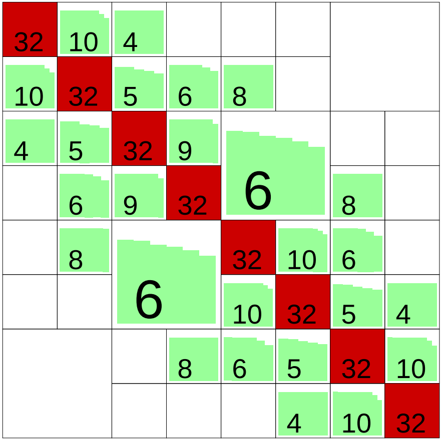
- shift strategy
- deflation

Example - $\mathcal{H}$-Fill-In



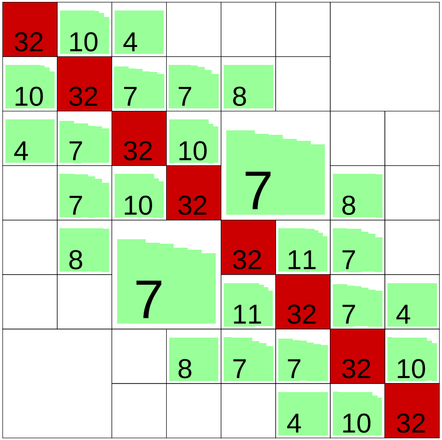
Matrix FEM16 ($\Delta_{2,h}$, 16 inner discr. points).

Example - $\mathcal{H}$-Fill-In



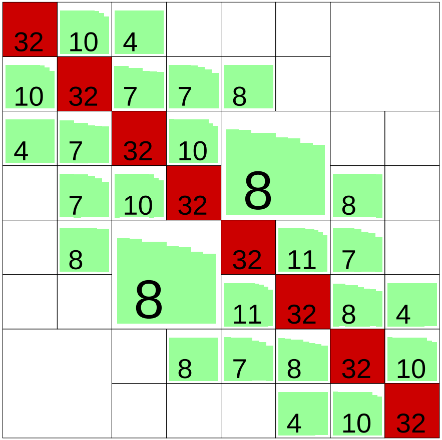
Matrix FEM16 ($\Delta_{2,h}$, 16 inner discr. points), after 1 step.

Example - $\mathcal{H}$-Fill-In



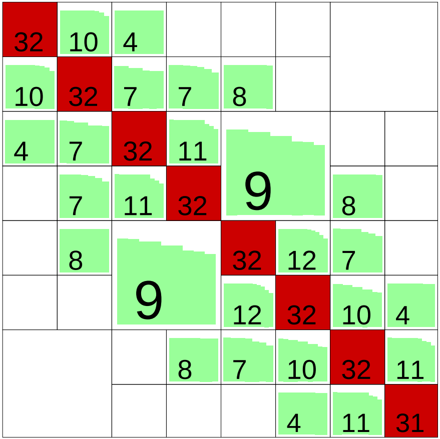
Matrix FEM16 ($\Delta_{2,h}$, 16 inner discr. points), after 2 steps.

Example - $\mathcal{H}$-Fill-In



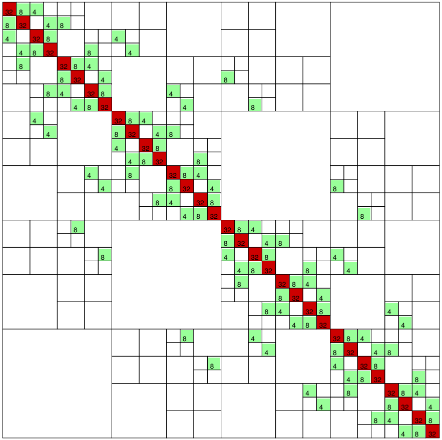
Matrix FEM16 ($\Delta_{2,h}$, 16 inner discr. points), after 3 steps.

Example - $\mathcal{H}$-Fill-In



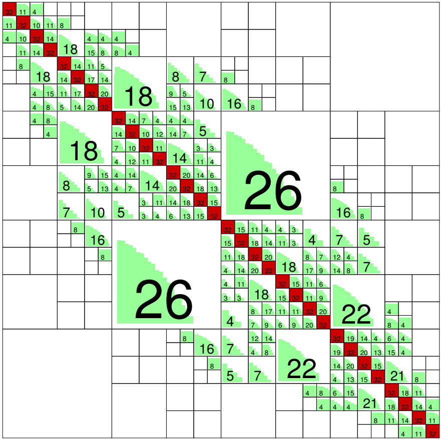
Matrix FEM16 ($\Delta_{2,h}$, 16 inner discr. points), after 4 steps.

Example - $\mathcal{H}$-Fill-In



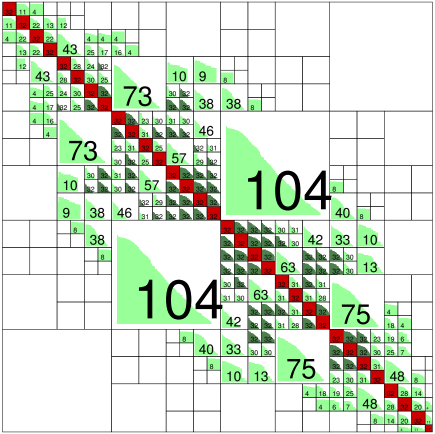
Matrix FEM32 ($\Delta_{2,h}$, 32 inner discr. points).

Example - $\mathcal{H}$-Fill-In



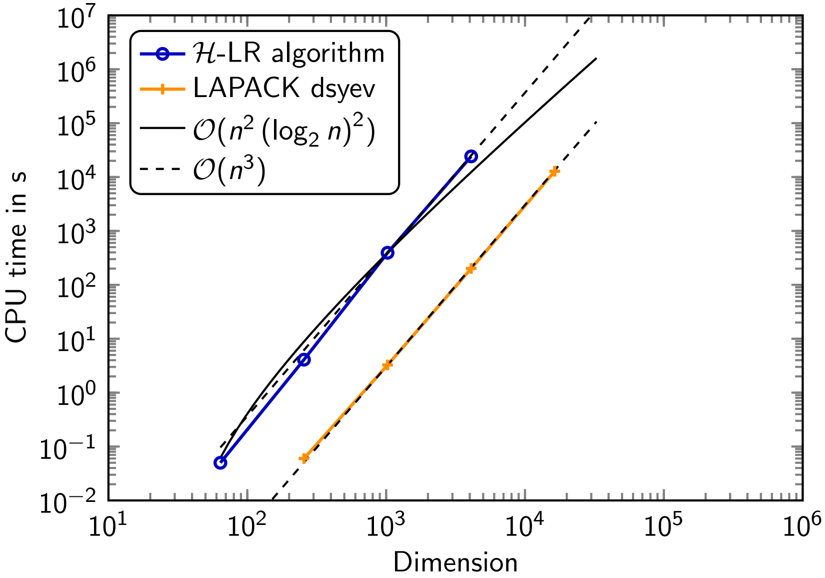
Matrix FEM32 ($\Delta_{2,h}$, 32 inner discr. points), after 1 step.

Example - $\mathcal{H}$-Fill-In

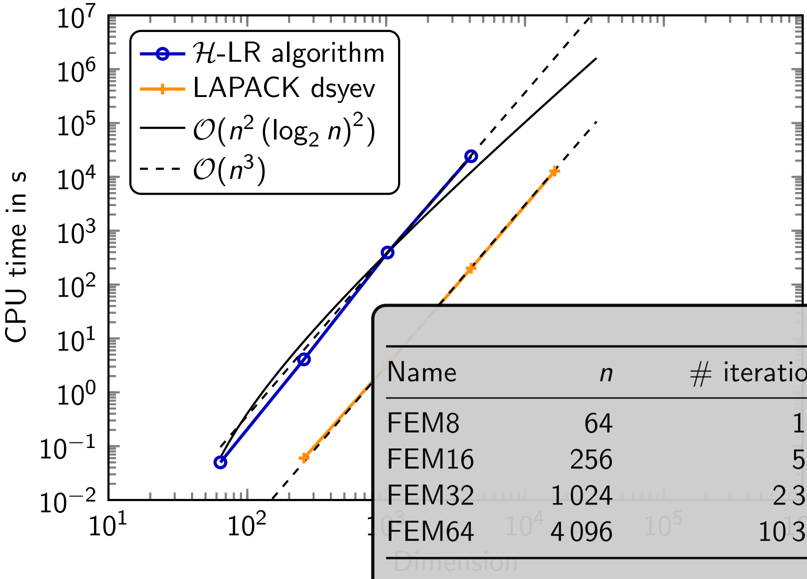


Matrix FEM32 ($\Delta_{2,h}$, 32 inner discr. points), after 50 steps.

Computation Time



Computation Time



Name	n	# iterations
FEM8	64	101
FEM16	256	556
FEM32	1 024	2 333
FEM64	4 096	10 320

Theorem

Adaption of [Fasino '05/Plestenjak, Van Barel, Van Camp '08]

$$M = \mathrm{diag}\left(d\right) + \sum_{i=1}^{r} \left(\mathrm{tril}\left(u_i v_i^T\right) + \mathrm{triu}\left(v_i u_i^T\right)\right)$$

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Structure Preservation of dpss Matrices

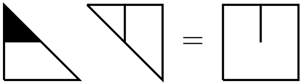
Let M be a symmetric positive definite diagonal plus semiseparable matrix, with a decomposition as in the definition. The Cholesky factor L of $M = LL^T$ can be written in the form

$$L = \operatorname{diag}\left(\tilde{d}\right) + \sum_{i=1}^{r} \operatorname{tril}\left(u_i \tilde{v}_i^T\right).$$

Multiplying the Cholesky factors in reverse order gives the next iterate $N = L^T L$ of the LR Cholesky algorithm. The matrix N has the same form as M,

$$N = \operatorname{diag}\left(\hat{d}\right) + \sum_{i=1}^{r} \left(\operatorname{tril}\left(\hat{u}_i \tilde{v}_i^T\right) + \operatorname{triu}\left(\tilde{v}_i \hat{u}_i^T\right)\right).$$

Proof Idea



$$L_{1:p-1,1:p-1} L_{p,1:p-1}^T = M_{1:p-1,p} = \sum_i v_i u_i^T$$

$$\Rightarrow L_{1:p-1,1:p-1}\ \tilde{v}_i\big|_{1:p-1} = v_i\big|_{1:p-1} \quad \text{and} \quad L_{p,1:p-1} = \sum_i u_i\big|_p\ \tilde{v}_i^T\Big|_{1:p-1}$$

$$\tilde{d}_p + \sum_i u_i\big|_p\ \tilde{v}_i\big|_p = L_{pp} = \sqrt{M_{pp} - L_{p,1:p-1} L_{p,1:p-1}^T}$$

L is a dpss matrix.

Proof Idea

$$N = L^T L = \left(\operatorname{diag}\left(\tilde{d}\right) + \sum_i \operatorname{tril}\left(u_i \tilde{v}_i^T\right)\right)^T \left(\operatorname{diag}\left(\tilde{d}\right) + \sum_i \operatorname{tril}\left(u_i \tilde{v}_i^T\right)\right)$$

$$\hat{u}^i = \left(Z + \operatorname{diag}\left(\tilde{d}\right)\right) u_i, \quad \text{with}$$

$$Z_{p,:} = \sum_j \tilde{v}_j\big|_p \begin{bmatrix} 0 & \cdots & 0 & u_j\big|_p & u_j\big|_{p+1} & \cdots & u_j\big|_n \end{bmatrix}$$

$$\operatorname{tril}(N, -1) = \sum_i \operatorname{tril}\left(\left(\operatorname{diag}(\tilde{d}) u_i + Z u_i\right) \tilde{v}_i^T, -1\right)$$

$$= \sum_i \operatorname{tril}\left(\hat{u}_i \tilde{v}_i^T, -1\right)$$

N is a dpss matrix.

Structure of $\hat{u}$ and $\tilde{v}$

$$M = \operatorname{diag}(d) + \sum_{i=1}^{r} \operatorname{tril}\left(u_i v_i^T\right) + \dots$$

$$N = \operatorname{diag}(d) + \sum_{i=1}^{r} \operatorname{tril}\left(\hat{u}_i \tilde{v}_i^T\right) + \dots$$

$$v_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightsquigarrow \tilde{v}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \\ \color{blue}{*} \\ \color{blue}{\vdots} \\ \color{blue}{*} \end{bmatrix} \qquad u_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightsquigarrow \hat{u}_i = \begin{bmatrix} \color{blue}{*} \\ \color{blue}{\vdots} \\ \color{blue}{*} \\ * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Hierarchical Matrices

$$\operatorname{rank}\left(M_{a:b,c:d}\right) = k \qquad M_{a:b,c:d} = AB^T$$

$$u_{i_r}^T = \begin{bmatrix} 0 & \cdots & 0 & A_{j,r}^T & 0 & \cdots & 0 \end{bmatrix}$$

$$v_{i_r}^T = \begin{bmatrix} 0 & \cdots & 0 & B_{j,r}^T & 0 & \cdots & 0 \end{bmatrix}, \quad r = 1, \ldots, k$$

$$\rightsquigarrow \hat{u}_{i_r}^T = \begin{bmatrix} * & \cdots & * & * & 0 & \cdots & 0 \end{bmatrix}$$

$$\tilde{v}_{i_r}^T = \begin{bmatrix} 0 & \cdots & 0 & * & * & \cdots & * \end{bmatrix}$$

$$\operatorname{tril}\left(u_i v_i^T\right) = \begin{bmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & \\ 0 & * & * & 0 & 0 & & \\ 0 & * & * & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \operatorname{tril}\left(\hat{u}_i \tilde{v}_i^T\right) = \begin{bmatrix} 0 & & & & & & \\ 0 & * & & & & & \\ 0 & * & * & & & & \\ 0 & * & * & * & & & \\ 0 & * & * & * & * & & \\ 0 & * & * & * & * & * & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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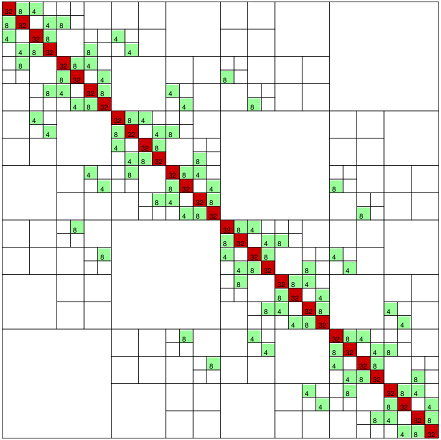
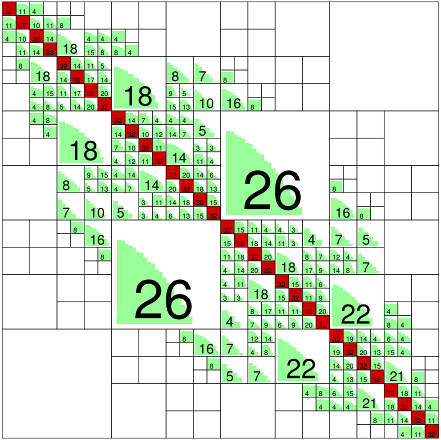
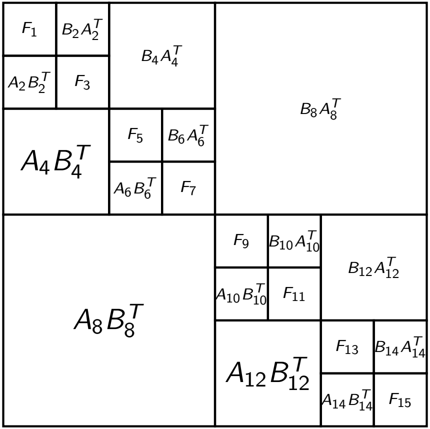
The structure of hierarchical matrices is not preserved under LR Cholesky transformations.

Example - $\mathcal{H}$-Fill-In





$\mathcal{H}_\ell$-Matrices



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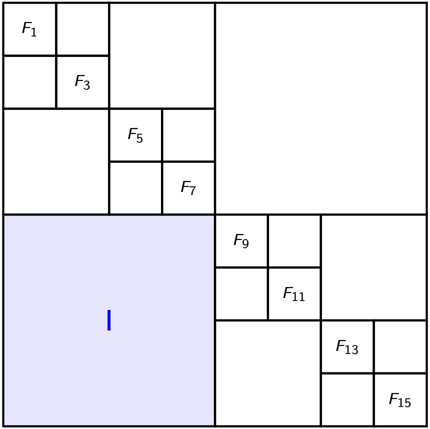

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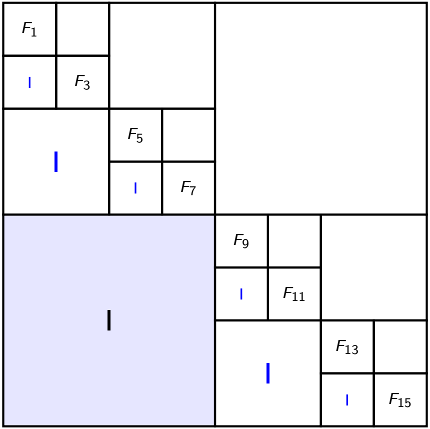

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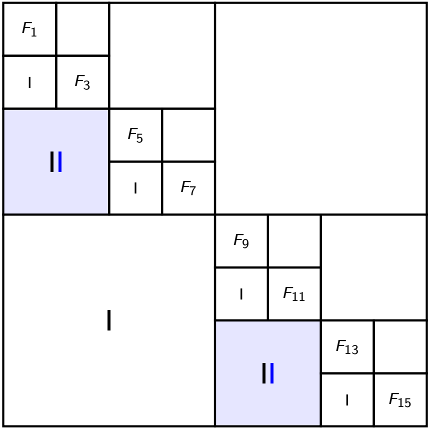

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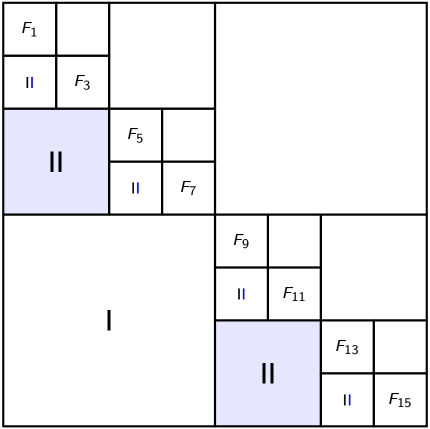
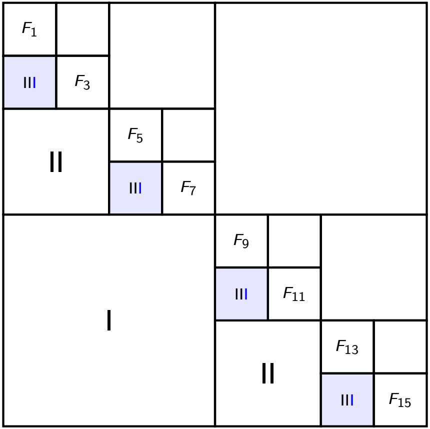


$\mathcal{H}_\ell$-Matrices



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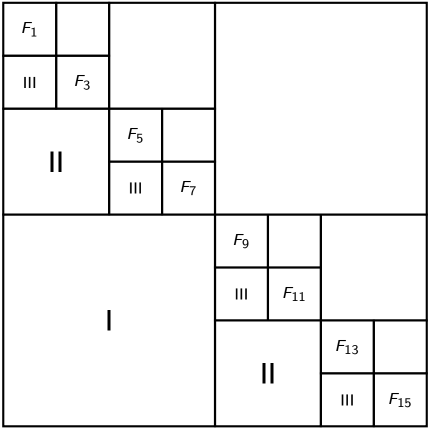
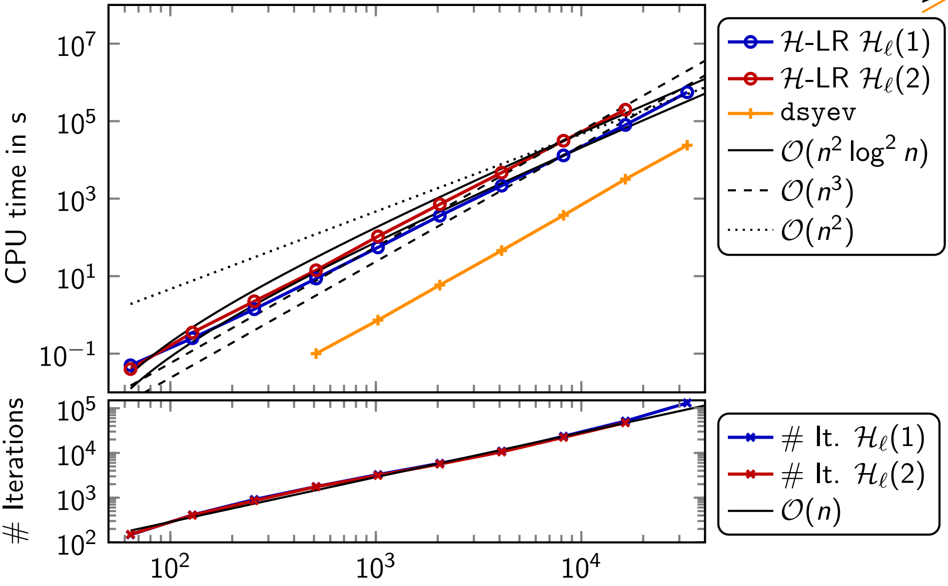


$\Rightarrow$ rank bounded by ℓk instead of k

$\Rightarrow$ total storage required by the low-rank parts of M is increased only from $2nk\ell$ to $2nk\frac{\ell(\ell-1)}{2}$

Computation Time $\mathcal{H}_\ell$-Matrices

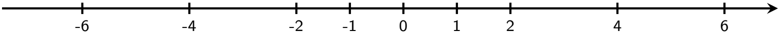


Slicing the Spectrum

Slicing the Spectrum

Bisectioning

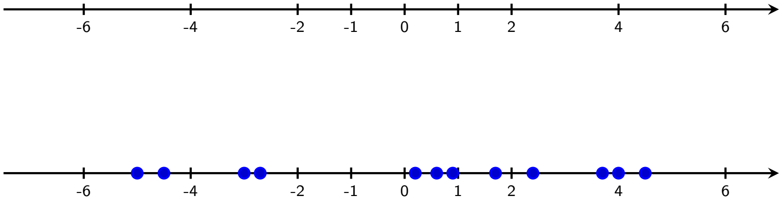
[Parlett '80]



Bisectioning

[Parlett '80]

$\lambda_3 = ?$



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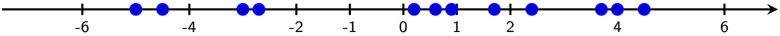

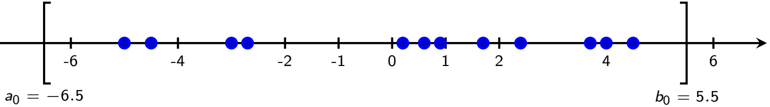

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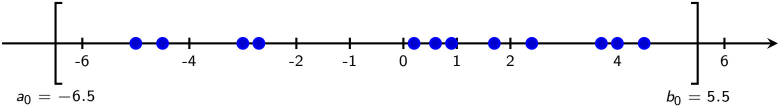

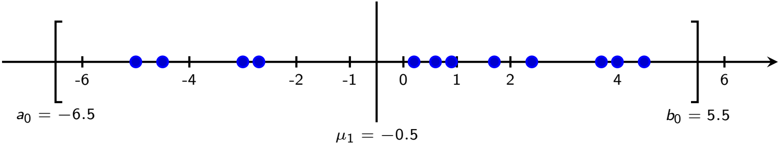

Bisectioning

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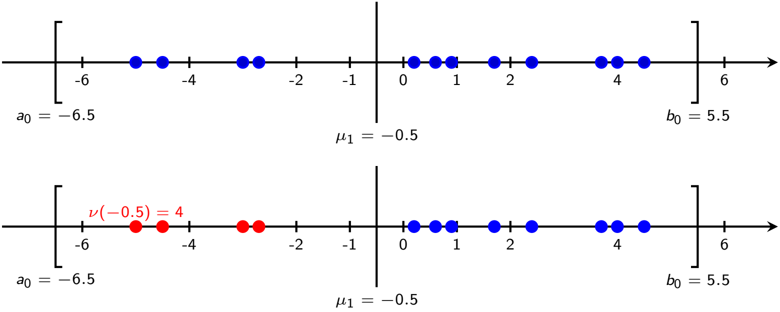

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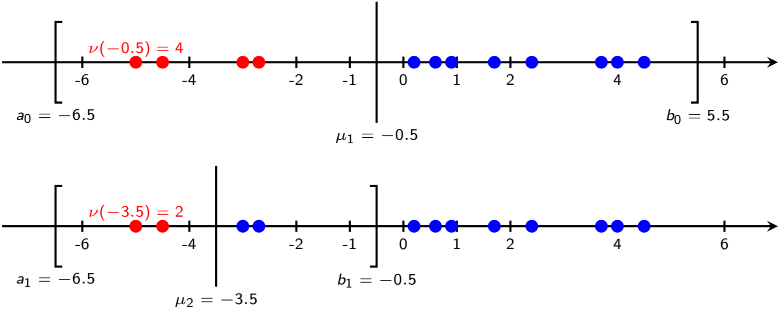

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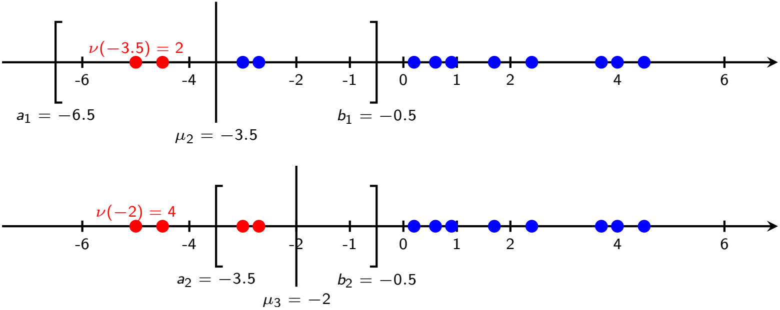
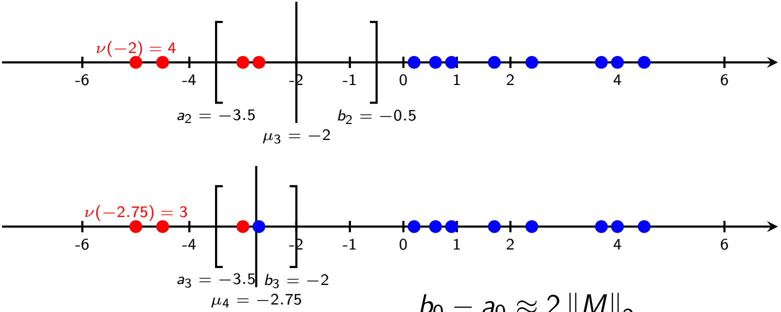


Bisectioning

[PARLETT '80]

$\lambda_3 \in [-3.5, -2.75]$, $\hat{\lambda}_3 = -3.125$



$$b_0 - a_0 \approx 2\,\|M\|_2$$
$$\left|\lambda_i - \hat{\lambda}_i\right| < \epsilon \Leftrightarrow b_n - a_n < 2\epsilon$$
$$b_{i+1} - a_{i+1} = \tfrac{1}{2}\,(b_i - a_i)$$
$$\Rightarrow \mathcal{O}\left(\log_2(\|M\|_2\,/\epsilon)\right)$$

Evaluation of $\nu(\mu)$

Sylvester's Law of Inertia

Each matrix M is congruent to a matrix

$$\operatorname{diag}\left(-I_\nu, I_{\operatorname{rank}(M)-\nu}, 0_{n-\operatorname{rank}(M)}\right),$$

where ν is the number of negative eigenvalues. The triple

$$(\nu, \operatorname{rank}(M) - \nu, n - \operatorname{rank}(M))$$

is called the *inertia* of M.

$$\begin{aligned} M = LDL^T \quad &\Rightarrow \quad \nu(M) = \nu(D) \\ M - \mu I = L_\mu D_\mu L_\mu^T \quad &\Rightarrow \quad \nu(\mu) = \nu(M - \mu I) = \nu(D_\mu) \end{aligned}$$

Complexity

Flops per factorization (for $\mathcal{H}_\ell$-matrices):

$$\mathcal{O}\left(nk^2 (\log n)^4\right).$$

Factorization per eigenvalue:

$$\mathcal{O}\left(\log(\|M\|_2 / \epsilon)\right).$$

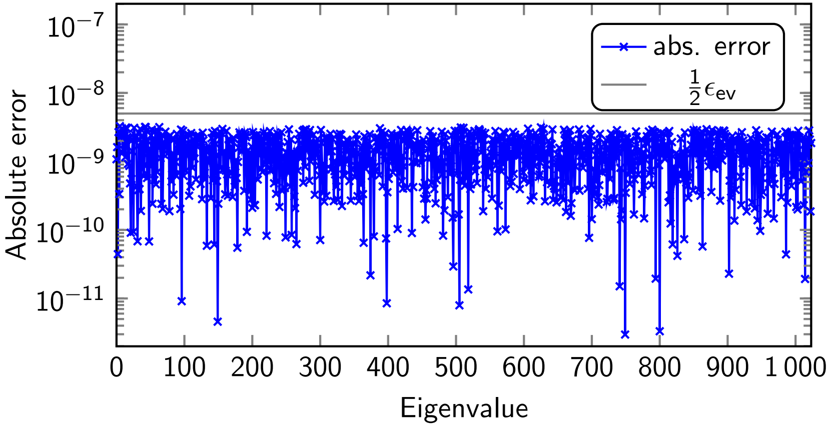
Flops per eigenvalue:

$$\mathcal{O}\left(nk^2 (\log n)^4 \log \|M\|_2 / \epsilon\right).$$

Slicing the whole spectrum:

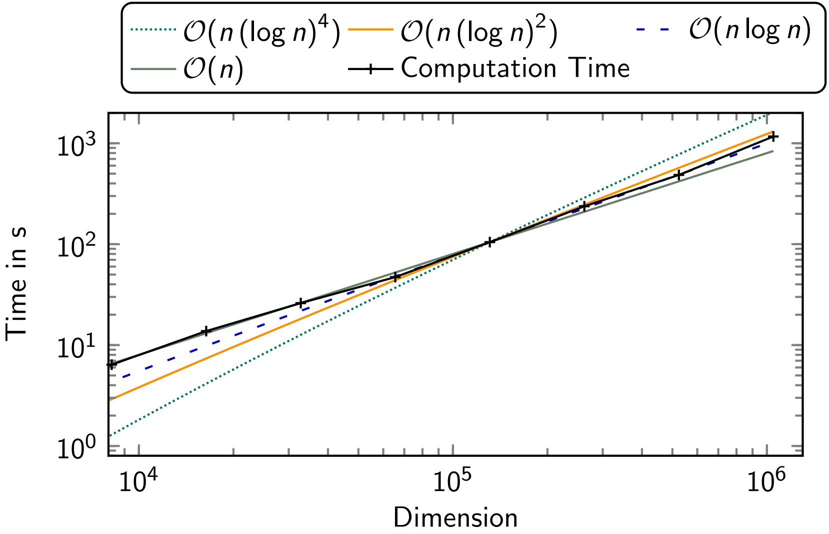
$$\mathcal{O}\left(n^2 k^2 (\log n)^4 \log(\|M\|_2 / \epsilon)\right).$$

Absolute Error for $\mathcal{H}_5(1)$-Matrix



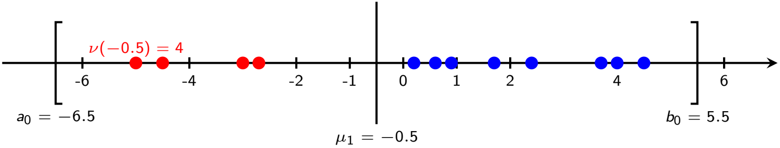
Absolute error $|\lambda_i - \hat{\lambda}_i|$ for the $1\,024 \times 1\,024$ $\mathcal{H}_5(1)$-matrix, $\epsilon_{\text{ev}} = 10^{-8}$.

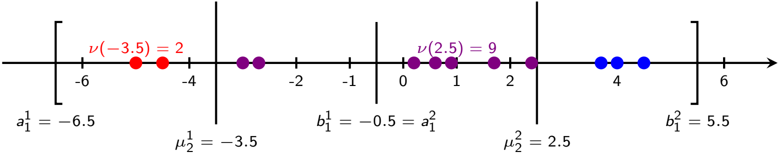
CPU Time for 10 Eigenvalues



Computation times for 10 eigenvalues of $\mathcal{H}_\ell(1)$-matrices ($\ell = 8, \ldots, 15$).

Parallelization





Parallelization Speedup

OpenMP:

Name	n	$t_{1\text{ core}}$ in s	t_{1c}/t_{2c}	t_{1c}/t_{4c}	$t_{8\text{ core}}$	t_{1c}/t_{8c}
H2 r1	128	0.33	1.83	3.30	0.06	5.50
H4 r1	512	9.44	1.94	3.67	1.43	6.60
H6 r1	2 048	219.28	1.91	3.64	33.88	6.47
H8 r1	8 192	4 022.80	1.87	3.44	676.57	5.95
H10 r1	32 768	49 012.24	1.93	3.18	10 006.60	4.90

Parallelization Speedup

Open MPI: $(\mathcal{H}_9(1) \in \mathbb{R}^{16\,384 \times 16\,384})$

No. of Processes	t in s	Speedup	Efficiency
1	16 564.58	1.00	1.00
2+1	8 340.22	1.99	0.66
4+1	4 044.13	4.10	0.82
6+1	2 678.95	6.18	0.88
11+1	1 494.33	11.08	0.92
23+1	713.80	23.21	0.96
35+1	476.44	34.77	0.96
47+1	364.30	45.47	0.95
95+1	188.92	87.68	0.91
191+1	100.61	164.64	0.86
287+1	71.86	230.50	0.80
383+1	61.91	267.56	0.70

$\mathcal{H}$-Matrices

- Shifting affects the structure.
- The LDL^T factorization is of almost linear complexity only for the original $\mathcal{H}$-matrix and not necessarily for the shifted ones.
- For $\mathcal{H}_\ell$-matrices we use the exact LDL^T factorization.
  For general $\mathcal{H}$-matrices: The truncation introduces errors in the LDL^T factorization — the computed D may have another inertia
  $\Rightarrow$ some eigenvalues may lie outside the computed interval
  $\Rightarrow$ larger errors

Preconditioned Inverse Iteration

Preconditioned Inverse Iteration for Hierarchical Matrices

Preconditioned Inverse Iteration

[Knyazev, Neymeyr, et al.]

Definition

The function

$$\mu(x) = \mu(x, M) = \frac{x^T M x}{x^T x}$$

is called the *Rayleigh quotient*.

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Minimize the Rayleigh quotient by a gradient method:

$$x_{i+1} := x_i - \alpha \nabla \mu(x_i), \quad \nabla \mu(x) = \frac{2}{x^T x} \left(Mx - x\mu(x) \right),$$

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$+$ preconditioning $\Rightarrow$ update equation:

$$x_{i+1} := x_i - B^{-1} \left(M x_i - x_i \mu(x_i) \right).$$

Preconditioned Inverse Iteration

[Knyazev, Neymeyr 2009]

$$x_{i+1} := x_i - B^{-1}\left(Mx_i - x_i\mu(x_i)\right)$$

If

- $M \in \mathbb{R}^{n\times n}$ symmetric positive definite and
- B^{-1} approximates the inverse of M, such that

$$\left\|\mathcal{I} - B^{-1}M\right\|_M \leq c < 1,$$

then Preconditioned INVerse ITeration (PINVIT) converges and the number of iterations is independent of n.

Algorithm and Complexity

The number of iterations is independent of matrix size n.

$\mathcal{H}$-PINVIT

Input: $M \in \mathbb{R}^{n \times n}$, $X_0 \in \mathbb{R}^{n \times d}$ ($X_0^T X_0 = I$, e.g. randomly chosen)
Output: $X_p \in \mathbb{R}^{n \times d}$, $\mu \in \mathbb{R}^{d \times d}$, with $\|MX_p - X_p\mu\| \leq \epsilon$

$B^{-1} = (M)_{\mathcal{H}}^{-1}$ or $B^{-1} = L_{\mathcal{H}}^{-T} L_{\mathcal{H}}^{-1}$
$R := MX_0 - X_0\mu, \quad \mu = X_0^T MX_0$
for $(i := 1;\ \|R\|_F > \epsilon;\ i++)$ **do**
  $X_i := \text{Orthogonalize}\left(X_{i-1} - B^{-1}R\right)$
  $R := MX_i - X_i\mu, \quad \mu = X_i^T MX_i$
end

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$B^{-1} = (M)_{\mathcal{H}}^{-1}$ or $B^{-1} = L_{\mathcal{H}}^{-T} L_{\mathcal{H}}^{-1}$ $\qquad \mathcal{O}(n\,(\log n)^2\,k\,(c)^2)$
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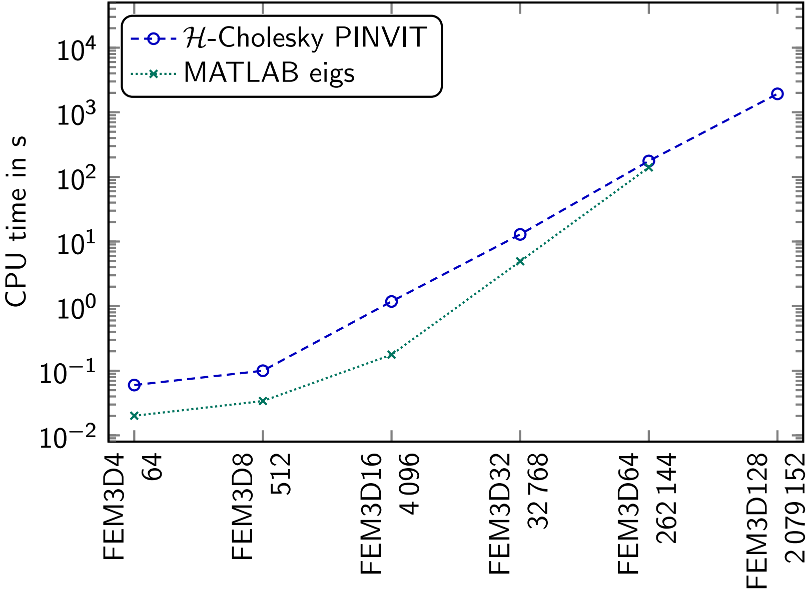
  $R := MX_i - X_i\mu, \quad \mu = X_i^T MX_i$ $\qquad \mathcal{O}(n\,(\log n)\,k\,(c))$
end

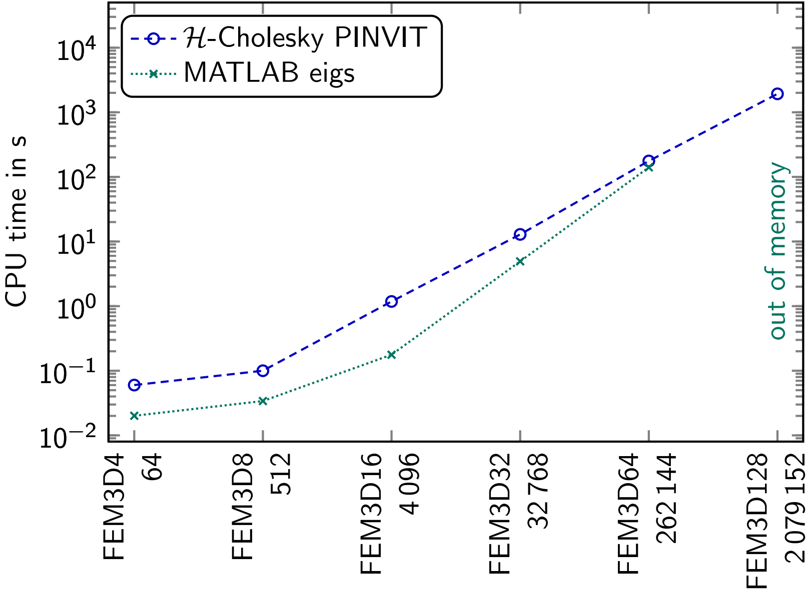
The complexity of the algorithm is determined by the $\mathcal{H}$-matrix inversion/Cholesky decomposition: $\Rightarrow \mathcal{O}(n\,(\log n)^2\,k\,(c)^2)$.

CPU Time



$\mathcal{H}$-Cholesky PINVIT
MATLAB eigs
CPU time in s
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
FEM3D4 64
FEM3D8 512
FEM3D16 4 096
FEM3D32 32 768
FEM3D64 262 144
FEM3D128 2 079 152

CPU Time


$\mathcal{H}$-Cholesky PINVIT
MATLAB eigs
CPU time in s
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
out of memory
FEM3D4 64
FEM3D8 512
FEM3D16 4 096
FEM3D32 32 768
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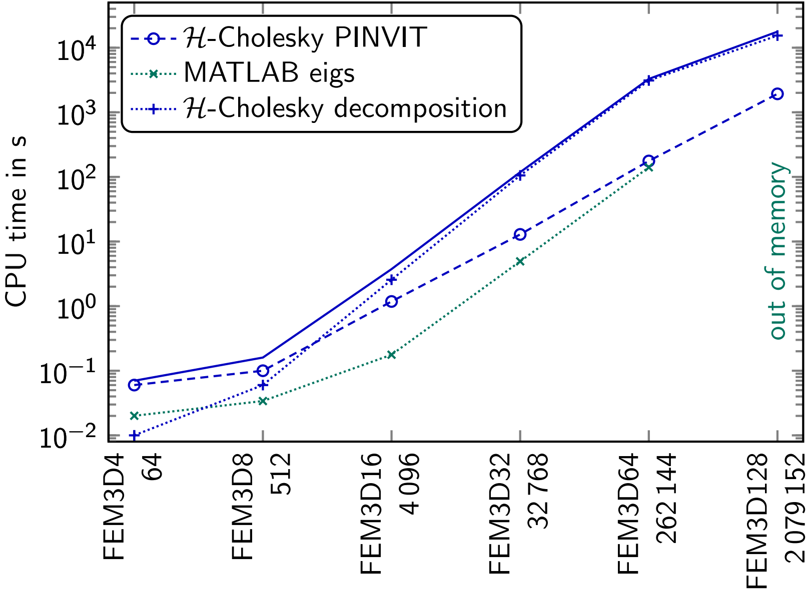
CPU Time


$\mathcal{H}$-Cholesky PINVIT
MATLAB eigs
$\mathcal{H}$-Cholesky decomposition
CPU time in s
10^{-2}
10^{-1}
10^{0}
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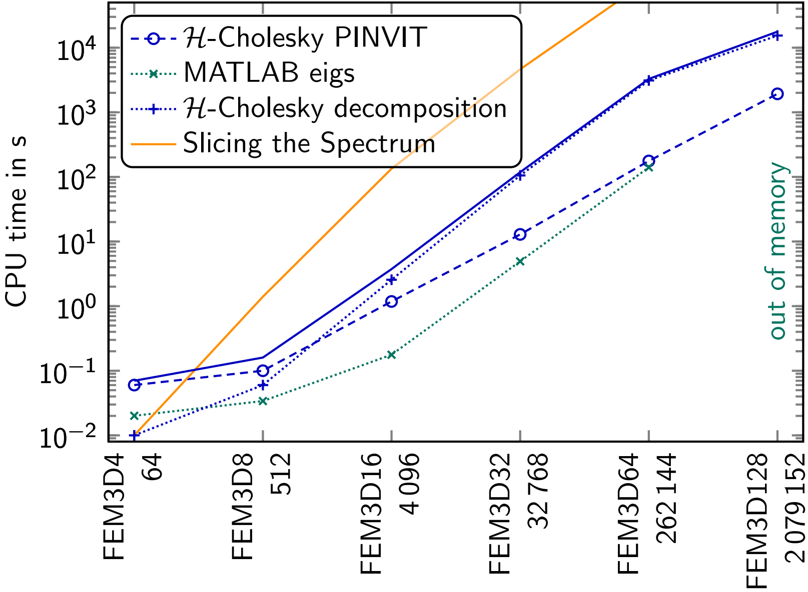

$\mathcal{H}$-Cholesky PINVIT
MATLAB eigs
$\mathcal{H}$-Cholesky decomposition
Slicing the Spectrum
CPU time in s
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
out of memory
FEM3D4 64
FEM3D8 512
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FEM3D32 32 768
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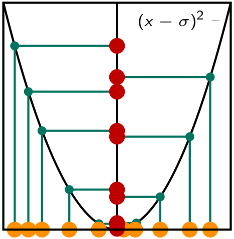
Folded Spectrum Method

Folded Spectrum Method [Wang, Zunger 1994]

$$M_\sigma = (M - \sigma\mathcal{I})^2$$

M_σ is s.p.d., if M is s.p.d. and $\sigma \neq \lambda_i$.



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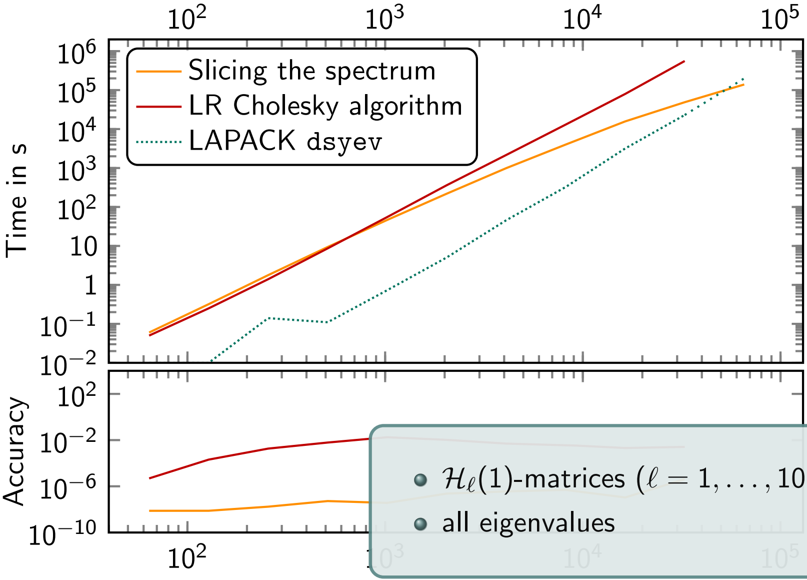
- Multiple eigenvalues of M_σ may lead to incomplete subspace information.

$\Rightarrow$ $v^T M v / v^T v$ does not approximate λ.

Conclusions

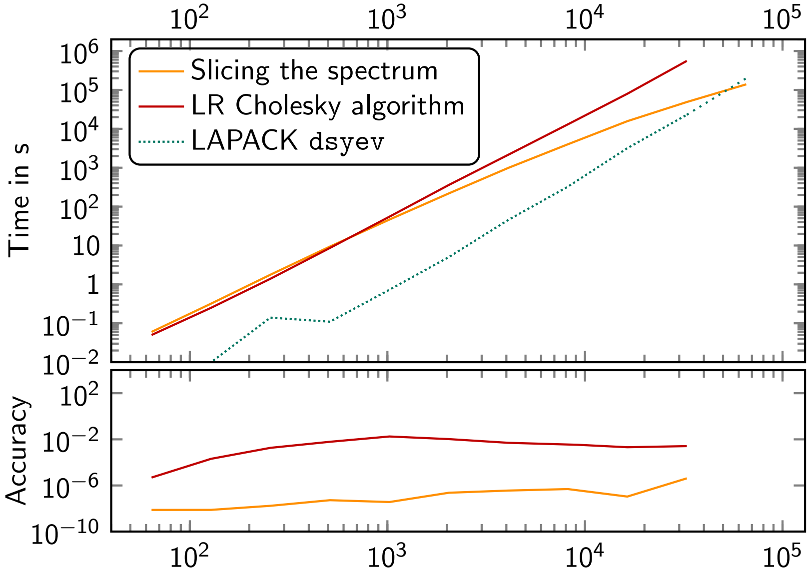
Conclusions

Slicing the Spectrum vs. LR Cholesky

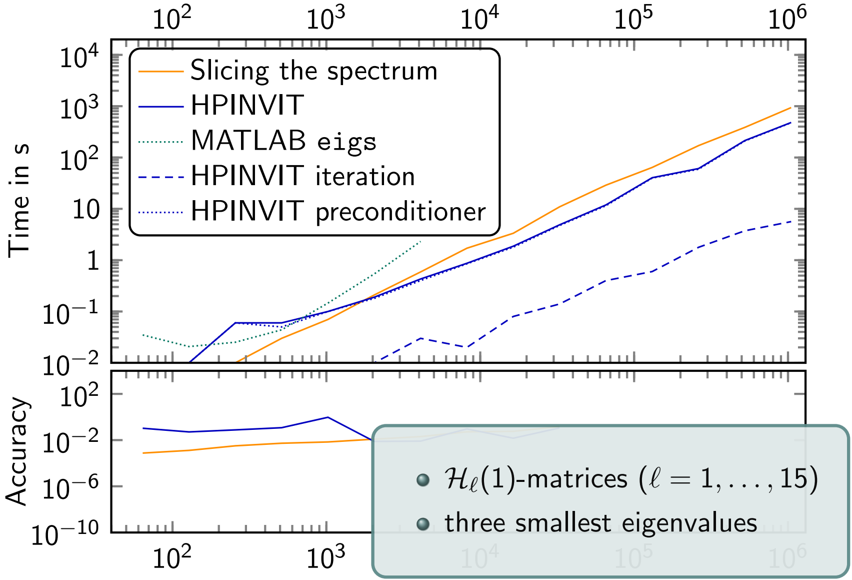


- $\mathcal{H}_\ell(1)$-matrices ($\ell = 1, \ldots, 10$)
- all eigenvalues

Slicing the Spectrum vs. LR Cholesky

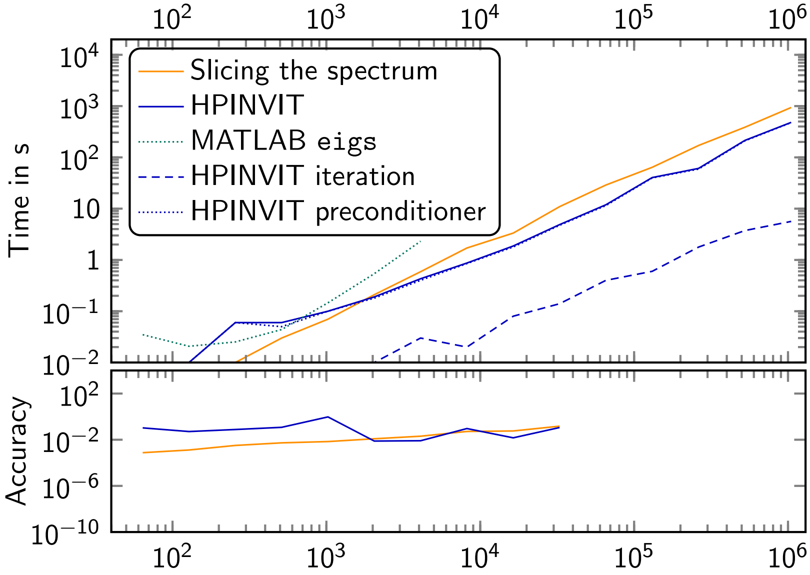


Slicing the Spectrum vs. $\mathcal{H}$-PINVIT



- $\mathcal{H}_\ell(1)$-matrices ($\ell = 1, \ldots, 15$)
- three smallest eigenvalues

Slicing the Spectrum vs. $\mathcal{H}$-PINVIT



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- Other ideas ot compute eigenvalues of $\mathcal{H}$-matrices:

J. Gördes.
Eigenwertproblem von hierarchischen Matrizen mit lokalem Rang 1.
Diplomarbeit, Mathematisch-Naturwissenschaftlichen Fakultät, CAU Kiel, May 2009.

W. Hackbusch and W. Kress.
A projection method for the computation of inner eigenvalues using high degree rational operators.
Computing 81:259–268, 2007.

S. Delvaux, K. Frederix, and M. Van Barel.
Transforming a hierarchical into a unitary-weight representation.
Electr. Trans. Num. Anal. 33:163–188, 2009.

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- Probably similar conclusions for unsymmetric case.
- Possible improvements employing PDE theory?
- Concepts carry over to TT format, but so far not competitive with tensor-specific methods.

Details to be found in...

Peter Benner and Thomas Mach.
The LR Cholesky algorithm for symmetric hierarchical matrices.
Linear Algebra and its Applications 439(4):1150–1166, 2013.

Peter Benner and Thomas Mach.
Computing all or some eigenvalues of symmetric $\mathcal{H}_\ell$-matrices.
SIAM Journal on Scientific Computing 34(1):A485–A496, 2012.

Peter Benner and Thomas Mach.
The preconditioned inverse iteration for hierarchical matrices.
Numerical Linear Algebra with Applications 20(1):150–166, 2013.

Peter Benner and Thomas Mach.
Locally optimal block preconditioned conjugate gradient method for hierarchical matrices.
Proceedings in Applied Mathematics and Mechanics 11:741-742, 2011.

Peter Benner and Thomas Mach.
On the QR decomposition of $\mathcal{H}$-matrices.
Computing 88(3–4):111–129, 2010.

All about $\mathcal{H}$ackbusch matrices to be found in. . .

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HAPPY BIRTHDAYS!