Elgersburg Workshop 2013
February 11–14, 2013

On the Kalman–Yakubovich–Popov Lemma and its Application in Model Order Reduction

Peter Benner

Computational Methods in Systems and Control Theory
Max Planck Institute for Dynamics of Complex Technical Systems
Magdeburg, Germany

joint work (in parts) with Matthias Voigt and Xin Du, Guanghong Yang, and Dan Ye

Overview

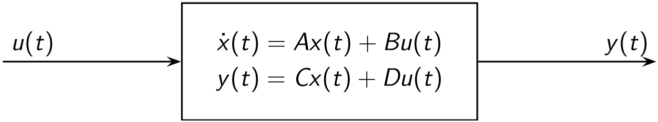
Linear Systems

LTI Systems

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) + Du(t), \end{cases}$$

with

- $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$,
- state vector $x(t) \in \mathbb{R}^n$,
- input vector $u(t) \in \mathbb{R}^m$,
- output vector $y(t) \in \mathbb{R}^p$.



Stability and Controllability

Definitions

The system Σ is called

- (asymptotically) stable if $\lim_{t\to\infty} x(t) = 0$ for $u \equiv 0$;
- controllable if for all $x_1 \in \mathbb{R}^n$ there exist $t_1 > 0$ and an input signal $u(t)$ such that $x(t_1) = x_1$.
- observable if $y(t) \equiv 0$ implies $x(t) \equiv 0$ (assuming $u(t) \equiv 0$).

Stability and Controllability

Definitions

The system Σ is called

- (asymptotically) stable if $\lim_{t\to\infty} x(t) = 0$ for $u \equiv 0$;
- controllable if for all $x_1 \in \mathbb{R}^n$ there exist $t_1 > 0$ and an input signal $u(t)$ such that $x(t_1) = x_1$.
- observable if $y(t) \equiv 0$ implies $x(t) \equiv 0$ (assuming $u(t) \equiv 0$).

Equivalent Conditions

The system Σ is

- (asymptotically) stable $\iff$ all eigenvalues of A are in the open left half-plane;
- controllable $\iff$ $\operatorname{rank} \begin{bmatrix} \lambda I_n - A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$.
- observable $\iff$ $\operatorname{rank} \begin{bmatrix} \lambda I_n - A^T & C^T \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$.
- minimal if it is controllable and observable.

Frequency Domain Analysis

Laplace transform

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

Frequency Domain Analysis

Laplace transform

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

Transfer function

Assume $x(0) = 0$. Then

$$\mathcal{L}(\Sigma) : \begin{cases} \mathcal{L}\{\dot{x}\}(s) = A\mathcal{L}\{x\}(s) + B\mathcal{L}\{u\}(s), \\ \mathcal{L}\{y\}(s) = C\mathcal{L}\{x\}(s) + D\mathcal{L}\{u\}(s), \end{cases}$$

Frequency Domain Analysis

Laplace transform

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

Transfer function

Assume $x(0) = 0$. Then

$$\mathcal{L}(\Sigma) : \begin{cases} s(\mathcal{L}\{x\}(s) - x(0)) = A\mathcal{L}\{x\}(s) + B\mathcal{L}\{u\}(s), \\ \mathcal{L}\{y\}(s) = C\mathcal{L}\{x\}(s) + D\mathcal{L}\{u\}(s), \end{cases}$$

Frequency Domain Analysis

Laplace transform

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

Transfer function

Assume $x(0) = 0$. Then

$$\mathcal{L}(\Sigma) : \begin{cases} s\mathcal{L}\{x\}(s) = A\mathcal{L}\{x\}(s) + B\mathcal{L}\{u\}(s), \\ \mathcal{L}\{y\}(s) = C\mathcal{L}\{x\}(s) + D\mathcal{L}\{u\}(s), \end{cases}$$

Frequency Domain Analysis

Laplace transform

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

Transfer function

Assume $x(0) = 0$. Then

$$\mathcal{L}(\Sigma) : \begin{cases} s\mathcal{L}\{x\}(s) = A\mathcal{L}\{x\}(s) + B\mathcal{L}\{u\}(s), \\ \mathcal{L}\{y\}(s) = C\mathcal{L}\{x\}(s) + D\mathcal{L}\{u\}(s), \end{cases}$$

Then

$$\mathcal{L}\{y\}(s) = \underbrace{C(sI_n - A)^{-1}B}_{=:G(s)}\mathcal{L}\{u\}(s).$$

Frequency Domain Analysis

Laplace transform

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

Transfer function

Assume $x(0) = 0$. Then

$$\mathcal{L}(\Sigma) : \begin{cases} s\mathcal{L}\{x\}(s) = A\mathcal{L}\{x\}(s) + B\mathcal{L}\{u\}(s), \\ \mathcal{L}\{y\}(s) = C\mathcal{L}\{x\}(s) + D\mathcal{L}\{u\}(s), \end{cases}$$

Then

$$\mathcal{L}\{y\}(s) = \underbrace{C(sI_n - A)^{-1}B}_{=:G(s)} \mathcal{L}\{u\}(s).$$

The transfer function $G(s)$ maps inputs to outputs in the frequency domain.

Dissipative Systems

Definition [Scherer, Weiland '05]

A dynamical system Σ is called dissipative with respect to a supply function $s : \mathbb{R}^p \times \mathbb{R}^m \longrightarrow \mathbb{R}$ if there exists a storage function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the dissipation inequality

$$V(x(t_1)) \leq V(x(0)) + \int_0^{t_1} s(y(t), u(t))\mathrm{d}t$$

is fulfilled for all $0 \leq t_1$.

Dissipative Systems

Definition [Scherer, Weiland '05]

A dynamical system Σ is called dissipative with respect to a supply function $s : \mathbb{R}^p \times \mathbb{R}^m \longrightarrow \mathbb{R}$ if there exists a storage function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the dissipation inequality

$$V(x(t_1)) \leq V(x(0)) + \int_0^{t_1} s(y(t), u(t))\mathrm{d}t$$

is fulfilled for all $0 \leq t_1$.

Interpretation

- $\int_0^{t_1} s(y(t), u(t))\mathrm{d}t$ can be seen as the energy supplied to the system in the time interval $[0, t_1]$.
- $s(y(t), u(t))$ is a measure for the power at time t.
- $V(x(t))$ is the internal energy at time t.

Quadratic Supply Functions

Often, we consider

$$s(y(t), u(t)) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \quad \text{with } W = W^T,\ R = R^T$$

Quadratic Supply Functions

Often, we consider

$$
\begin{aligned}
s(y(t),u(t)) &= \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \quad \text{with } W = W^T,\ R = R^T \\
&= \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix}
\end{aligned}
$$

Quadratic Supply Functions

Often, we consider

$$\begin{aligned} s(y(t),u(t)) &= \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \quad \text{with } W = W^T,\ R = R^T \\ &= \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C^T W C & C^T W D + C^T S \\ D^T W C + S^T C & D^T W D + D^T S + S^T D + R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \end{aligned}$$

Quadratic Supply Functions

Often, we consider

$$
\begin{aligned}
s(y(t),u(t)) &= \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \quad \text{with } W = W^T,\ R = R^T \\
&= \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix} \\
&= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C^T WC & C^T WD + C^T S \\ D^T WC + S^T C & D^T WD + D^T S + S^T D + R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
&=: \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} \tilde{W} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}
\end{aligned}
$$

Quadratic Supply Functions

Often, we consider

$$
\begin{aligned}
s(y(t),u(t)) &= \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \quad \text{with } W = W^T,\ R = R^T \\
&= \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix} \\
&= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C^T WC & C^T WD + C^T S \\ D^T WC + S^T C & D^T WD + D^T S + S^T D + R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
&=: \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} \tilde{W} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
&=: \tilde{s}(x(t),u(t)).
\end{aligned}
$$

Special Cases

Passivity

$$s(y(t),u(t)) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix},$$

$$\tilde{s}(x(t),u(t)) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} 0 & C^T \\ C & D+D^T \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

Special Cases

Passivity

$$s(y(t),u(t)) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix},$$

$$\tilde{s}(x(t),u(t)) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} 0 & C^T \\ C & D+D^T \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

Contractivity

$$s(y(t),u(t)) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} -I_p & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix},$$

$$\tilde{s}(x(t),u(t)) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} -C^TC & -C^TD \\ -D^TC & I_m - D^TD \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

Dissipativity in the Frequency Domain

Definition: Popov function

$$\Phi(s) = \begin{bmatrix} (sI_n - A)^{-1}B \\ I_m \end{bmatrix}^H \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI_n - A)^{-1}B \\ I_m \end{bmatrix}$$

Dissipativity in the Frequency Domain

Definition: Popov function

$$\Phi(s) = \begin{bmatrix} (sI_n - A)^{-1}B \\ I_m \end{bmatrix}^H \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI_n - A)^{-1}B \\ I_m \end{bmatrix}$$

Theorem

Let Σ be controllable. Then, Σ is dissipative with respect to $\tilde{s}(x(t), u(t)) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ if and only if $\Phi(\mathrm{i}\omega) \succcurlyeq 0$ holds for all $\mathrm{i}\omega \in \mathrm{i}\mathbb{R} \backslash \Lambda(A)$.

Special Cases

Passivity and positive realness

A dynamical system is passive if and only its transfer function G is positive real, i.e.,

$$G(s) + G^H(s) \succcurlyeq 0 \quad \forall s \in \mathbb{C}^+.$$

Special Cases

Passivity and positive realness

A dynamical system is passive if and only its transfer function G is positive real, i.e.,

$$G(s) + G^H(s) \succcurlyeq 0 \quad \forall s \in \mathbb{C}^+.$$

Contractivity and bounded realness

A dynamical system is contractive if and only its transfer function G is bounded real, i.e.,

$$I_m - G^H(s)G(s) \succcurlyeq 0 \quad \forall s \in \mathbb{C}^+.$$

Special Cases

Passivity and positive realness

A dynamical system is passive if and only its transfer function G is positive real, i.e.,

$$G(s) + G^H(s) \succcurlyeq 0 \quad \forall s \in \mathbb{C}^+.$$

Contractivity and bounded realness

A dynamical system is contractive if and only its transfer function G is bounded real, i.e.,

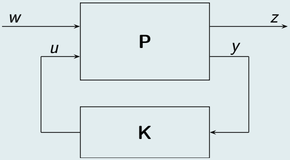
$$I_m - G^H(s)G(s) \succcurlyeq 0 \quad \forall s \in \mathbb{C}^+.$$

Remark

In contrast to general dissipativity, positive and bounded realness are properties of $\Phi(s)$ in the whole open right half-plane. It can be shown that for these cases $V(x(t)) = x(t)^T X x(t)$ for an $X = X^T \succcurlyeq 0$.

Relations to $\mathcal{H}_\infty$ Optimal Control

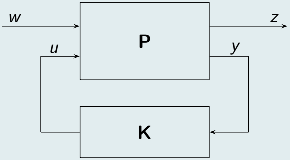
Problem setting [Green, Limebeer '95]



- Plant **P**, dynamic compensator **K**,
- noise w, estimation error z.
- **Goal:** Find **K** that stabilizes the system and minimizes the influence of w on z! ($=$ minimizing the $\mathcal{H}_\infty$-norm of closed-loop transfer function)

Relations to $\mathcal{H}_\infty$ Optimal Control

Problem setting [Green, Limebeer '95]



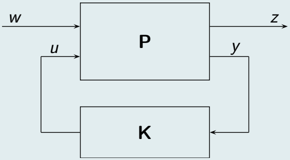
- Plant **P**, dynamic compensator **K**,
- noise w, estimation error z.
- **Goal:** Find **K** that stabilizes the system and minimizes the influence of w on z! (= minimizing the $\mathcal{H}_\infty$-norm of closed-loop transfer function)

$\mathcal{H}_\infty$-spaces

$\mathcal{H}_\infty^{p\times m}(\mathrm{i}\omega)$ = Banach space of $p \times m$ matrix-valued functions which are analytic and bounded in the open right half-plane.

Relations to $\mathcal{H}_\infty$ Optimal Control

Problem setting [Green, Limebeer '95]



- Plant **P**, dynamic compensator **K**,
- noise w, estimation error z.
- **Goal:** Find **K** that stabilizes the system and minimizes the influence of w on z! (= minimizing the $\mathcal{H}_\infty$-norm of closed-loop transfer function)

$\mathcal{H}_\infty$-spaces

$\mathcal{H}_\infty^{p\times m}(\mathrm{i}\omega)$ = Banach space of $p \times m$ matrix-valued functions which are analytic and bounded in the open right half-plane.

$\mathcal{H}_\infty$-norm (in this setting)

$$\|G\|_{\mathcal{H}_\infty} = \sup_{s\in\mathbb{C}^+} \sigma_{\max}(G(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega))$$
$$= \inf_{\gamma\geq 0} \left\{\gamma^2 I_m - G^H(\mathrm{i}\omega)G(\mathrm{i}\omega) \succcurlyeq 0 \; \forall \omega \in \mathbb{R}\right\}.$$

Algebraic Characterizations

Dissipativity can be characterized by properties of various algebraic structures such as

- linear matrix inequalities,
- quadratic matrix inequalities,
- algebraic matrix equations (Riccati equations, Lur'e equations),
- (structured matrices and matrix pencils).

Kalman-Yakubovich-Popov(-Anderson) Lemma

Consider again the dissipation inequality (in differential form):

$$\begin{aligned}\tilde{s}(x(t),u(t)) &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &\geq \dot{V}(x(t))\dot{x}(t)\end{aligned}$$

This obviously holds if there exists $X = X^T$ such that

$$\begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \geq \begin{bmatrix} A^T X + XA & XB \\ B^T X & 0 \end{bmatrix}.$$

Kalman-Yakubovich-Popov(-Anderson) Lemma

Consider again the dissipation inequality (in differential form):

$$\begin{aligned}\tilde{s}(x(t),u(t)) &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &\geq \dot{V}(x(t))\dot{x}(t) \qquad (\text{set } V(x(t)) = x(t)^T X x(t) \text{ with } X = X^T) \\ &= 2x(t)^T X(Ax(t) + Bu(t))\end{aligned}$$

This obviously holds if there exists $X = X^T$ such that

$$\begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \geq \begin{bmatrix} A^T X + XA & XB \\ B^T X & 0 \end{bmatrix}.$$

Kalman-Yakubovich-Popov(-Anderson) Lemma

Consider again the dissipation inequality (in differential form):

$$\begin{aligned}
\tilde{s}(x(t),u(t)) &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
&\geq \dot{V}(x(t))\dot{x}(t) \qquad \text{(set } V(x(t)) = x(t)^T X x(t) \text{ with } X = X^T\text{)} \\
&= 2x(t)^T X(Ax(t) + Bu(t)) \\
&= x(t)^T XAx(t) + x(t)^T XBu(t) + x(t)^T A^T Xx(t) + u(t)^T B^T Xx(t)
\end{aligned}$$

This obviously holds if there exists $X = X^T$ such that

$$\begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \geq \begin{bmatrix} A^T X + XA & XB \\ B^T X & 0 \end{bmatrix}.$$

Kalman-Yakubovich-Popov(-Anderson) Lemma

Consider again the dissipation inequality (in differential form):

$$
\begin{aligned}
\tilde{s}(x(t),u(t)) &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
&\geq \dot{V}(x(t))\dot{x}(t) \qquad \text{(set } V(x(t)) = x(t)^T X x(t) \text{ with } X = X^T\text{)} \\
&= 2x(t)^T X(Ax(t) + Bu(t)) \\
&= x(t)^T XAx(t) + x(t)^T XBu(t) + x(t)^T A^T Xx(t) + u(t)^T B^T Xx(t) \\
&= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.
\end{aligned}
$$

This obviously holds if there exists $X = X^T$ such that

$$
\begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \geq \begin{bmatrix} A^T X + XA & XB \\ B^T X & 0 \end{bmatrix}.
$$

Kalman-Yakubovich-Popov(-Anderson) Lemma

Theorem [Willems '72]

Let Σ be controllable. Then Σ is dissipative with respect to $s(x(t), u(t))$ (or equivalently $\Phi(\mathrm{i}\omega) \succcurlyeq 0 \; \forall \mathrm{i}\omega \in \mathrm{i}\mathbb{R} \setminus \Lambda(A)$) if and only if there exists a symmetric matrix X such that the linear matrix inequality (LMI)

$$\begin{bmatrix} A^T X + XA - W & XB - S \\ B^T X - S^T & -R \end{bmatrix} \preccurlyeq 0$$

is fulfilled.

Kalman-Yakubovich-Popov(-Anderson) Lemma

Theorem [Willems '72]

Let Σ be controllable. Then Σ is dissipative with respect to $s(x(t), u(t))$ (or equivalently $\Phi(i\omega) \succcurlyeq 0 \ \forall i\omega \in i\mathbb{R} \backslash \Lambda(A)$) if and only if there exists a symmetric matrix X such that the linear matrix inequality (LMI)

$$\begin{bmatrix} A^T X + XA - W & XB - S \\ B^T X - S^T & -R \end{bmatrix} \preccurlyeq 0$$

is fulfilled.

History

- **'61:** Popov's criterion for stability of a feedback system with a memoryless nonlinearity.
- **'62/'63:** Original version of the lemma by Kalman and Yakubovich.
- **'67:** Anderson's positive real lemma for multivariate transfer functions.
- **until today:** Many generalizations and extensions.

Special Cases

Positive real lemma

Let Σ be controllable. Then Σ is passive (or equivalently $G(s)$ is positive real) if and only if there exists $X = X^T \succcurlyeq 0$ such that the LMI

$$\begin{bmatrix} A^T X + XA & XB - C^T \\ B^T X - C & -(D + D)^T \end{bmatrix} \preccurlyeq 0$$

is fulfilled.

Special Cases

Positive real lemma

Let Σ be controllable. Then Σ is passive (or equivalently $G(s)$ is positive real) if and only if there exists $X = X^T \succcurlyeq 0$ such that the LMI

$$\begin{bmatrix} A^TX + XA & XB - C^T \\ B^TX - C & -(D + D)^T \end{bmatrix} \preccurlyeq 0$$

is fulfilled.

Bounded real lemma

Let Σ be controllable. Then Σ is contractive (or equivalently $G(s)$ is bounded real) if and only if there exists $X = X^T \succcurlyeq 0$ such that the LMI

$$\begin{bmatrix} A^TX + XA + C^TC & XB + C^TD \\ B^TX + D^TC & D^TD - I_m \end{bmatrix} \preccurlyeq 0$$

is fulfilled.

Other Algebraic Characterizations — R nonsingular

Linear Matrix Inequality

$$\begin{bmatrix} A^T X + XA - W & XB - S \\ B^T X - S^T & -R \end{bmatrix} \preccurlyeq 0, \quad X = X^T \quad \text{solvable.}$$

Other Algebraic Characterizations — R nonsingular

Linear Matrix Inequality

$$\begin{bmatrix} A^T X + XA - W & XB - S \\ B^T X - S^T & -R \end{bmatrix} \preccurlyeq 0, \quad X = X^T \quad \text{solvable.}$$

$$\Updownarrow$$

Quadratic Matrix Inequality

$$A^T X + XA - W + (XB - S)\, R^{-1} \left(B^T X - S^T\right) \preccurlyeq 0, \quad X = X^T \quad \text{solvable.}$$

Other Algebraic Characterizations — R nonsingular

Linear Matrix Inequality

$$\begin{bmatrix} A^T X + XA - W & XB - S \\ B^T X - S^T & -R \end{bmatrix} \preccurlyeq 0, \quad X = X^T \quad \text{solvable.}$$

$$\Updownarrow$$

Quadratic Matrix Inequality

$$A^T X + XA - W + (XB - S)\, R^{-1} \left(B^T X - S^T\right) \preccurlyeq 0, \quad X = X^T \quad \text{solvable.}$$

$$\Updownarrow$$

Algebraic Riccati Equation

$$A^T X + XA - W + (XB - S)\, R^{-1} \left(B^T X - S^T\right) = 0, \quad X = X^T \quad \text{solvable.}$$

Other Algebraic Characterizations — R singular

Linear Matrix Inequality

$$\begin{bmatrix} A^T X + XA - W & XB - S \\ B^T X - S^T & -R \end{bmatrix} \preccurlyeq 0, \quad X = X^T \quad \text{solvable.}$$

Other Algebraic Characterizations — R singular

Linear Matrix Inequality

$$\begin{bmatrix} A^T X + XA - W & XB - S \\ B^T X - S^T & -R \end{bmatrix} \preccurlyeq 0, \quad X = X^T \quad \text{solvable.}$$

$$\Updownarrow$$

Quadratic Matrix Inequality

$$A^T X + XA - W + (XB - S)\, R^{-1} \left(B^T X - S^T\right) \preccurlyeq 0, \quad X = X^T$$

cannot be formulated!

Other Algebraic Characterizations — R singular

Linear Matrix Inequality

$$\begin{bmatrix} A^T X + XA - W & XB - S \\ B^T X - S^T & -R \end{bmatrix} \preccurlyeq 0, \quad X = X^T \quad \text{solvable.}$$

$$\Updownarrow$$

Lur'e Equation

$$\begin{aligned} A^T X + XA - W &= -K^T K, \\ XB - S &= -K^T L, \\ -R &= -L^T L, \\ X &= X^T \end{aligned}$$

solvable for $(X, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ and p as small as possible.

first formulated in [Lur'e '57]

More on KYP and Lur'e equations in M. Voigt's talk on Wednesday!

Some Remarks on Numerical Aspects

- In the control literature, one often finds statements:

  We have reduced the problem to an LMI $\Longrightarrow$ *problem solved!*

  Good reference for LMI formulaion of control problems:

  V. Balakrishnan, L. Vandenberghe, "Semidefinite programming duality and linear time-invariant systems", IEEE TAC, 2003.

Some Remarks on Numerical Aspects

- In the control literature, one often finds statements:

  We have reduced the problem to an LMI $\Longrightarrow$ problem solved!

  Good reference for LMI formulaion of control problems:

  V. Balakrishnan, L. Vandenberghe, "Semidefinite programming duality and linear time-invariant systems", IEEE TAC, 2003.

- True for small dimensions, say $n < 10$.

Some Remarks on Numerical Aspects

- In the control literature, one often finds statements:

  We have reduced the problem to an LMI $\Longrightarrow$ *problem solved!*

  Good reference for LMI formulaion of control problems:

  V. Balakrishnan, L. Vandenberghe, "Semidefinite programming duality and linear time-invariant systems", IEEE TAC, 2003.
- True for small dimensions, say $n < 10$.
- But: numerical solution of LMIs requires Semidefinite Programming (SDP) methods, this requires generically $\mathcal{O}(n^6)$ floating point operations (flops), with some tricks and exploiting structures $\mathcal{O}(n^{4.5})$.

Some Remarks on Numerical Aspects

- In the control literature, one often finds statements:

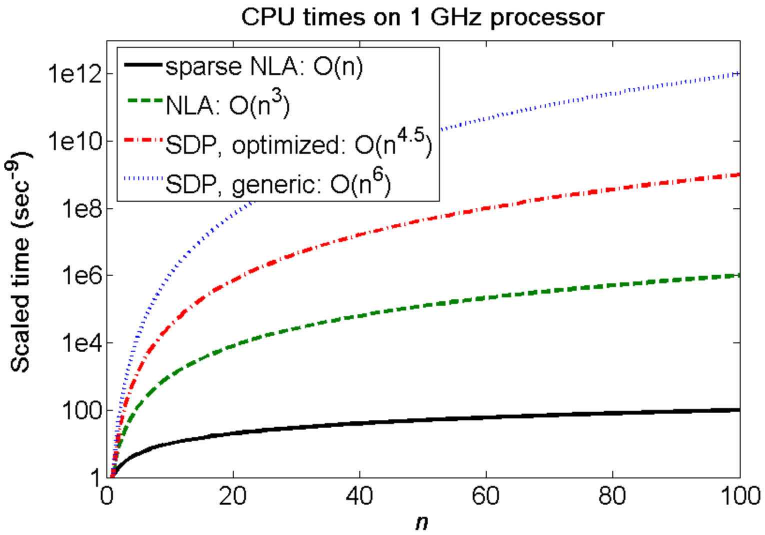
  We have reduced the problem to an LMI $\Longrightarrow$ problem solved!

  Good reference for LMI formulaion of control problems:

  V. Balakrishnan, L. Vandenberghe, "Semidefinite programming duality and linear time-invariant systems", IEEE TAC, 2003.
- True for small dimensions, say $n < 10$.
- But: numerical solution of LMIs requires Semidefinite Programming (SDP) methods, this requires generically $\mathcal{O}(n^6)$ floating point operations (flops), with some tricks and exploiting structures $\mathcal{O}(n^{4.5})$.
- Methods based on Lyapunov or Riccati equations, invariant subspaces of Hamiltonian matrices or even pencils generically require only $\mathcal{O}(n^3)$ flops, and can be implemented in $\mathcal{O}(nmp)$ flops for some large-scale problems with sparse state matrix A.

Some Remarks on Numerical Aspects

Complexity of Numerical Linear Algebra (NLA) and SDP Solutions to Control Problems



Model Reduction for LTI Systems

Balanced truncation for linear systems

Idea

- $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

Model Reduction for LTI Systems

Balanced truncation for linear systems

Idea

- $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.
- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Model Reduction for LTI Systems

Balanced truncation for linear systems

Idea

- $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$, is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right). \end{aligned}$$

Model Reduction for LTI Systems

Balanced truncation for linear systems

Idea

- $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,
  is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.
- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right). \end{aligned}$$

- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (A_{11}, B_1, C_1, D)$.

Model Reduction for LTI Systems

Balanced truncation for linear systems

Motivation:

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

"functional analyst's point of view"

Model Reduction for LTI Systems

Balanced truncation for linear systems

Motivation:

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

"functional analyst's point of view"

In balanced coordinates, energy transfer from u_- to y_+ is

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\, dt}{\int_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{||x_0||_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2.$$

"engineer's point of view"

Model Reduction for LTI Systems

Balanced truncation for linear systems

Motivation:

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+ .$$

"functional analyst's point of view"

In balanced coordinates, energy transfer from u_- to y_+ is

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int\limits_0^\infty y(t)^T y(t)\, dt}{\int\limits_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{||x_0||_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2 .$$

"engineer's point of view" $\Longrightarrow$ Truncate states corresponding to "small" HSVs

$\Longrightarrow$ analogy to best approximation via SVD, therefore balancing-related methods are sometimes called SVD methods.

Model Reduction for LTI Systems

Balanced truncation for linear systems

Implementation: SR Method

1. Compute (Cholesky) factors of the solutions of the Lyapunov equations,

$$P = S^T S, \quad Q = R^T R.$$

Model Reduction for LTI Systems

Balanced truncation for linear systems

Implementation: SR Method

1. Compute (Cholesky) factors of the solutions of the Lyapunov equations,
$$P = S^T S, \quad Q = R^T R.$$
2. Compute SVD
$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

Model Reduction for LTI Systems

Balanced truncation for linear systems

Implementation: SR Method

1. Compute (Cholesky) factors of the solutions of the Lyapunov equations,
$$P = S^T S, \quad Q = R^T R.$$
2. Compute SVD
$$SR^T = [\, U_1, \; U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$
3. Set
$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$
4. Reduced model is $(W^T A V, W^T B, CV)$.

Model Reduction for LTI Systems

Balanced truncation for linear systems

Implementation: SR Method

1. Compute (Cholesky) factors of the solutions of the Lyapunov equations,
$$P = S^T S, \quad Q = R^T R.$$
2. Compute SVD
$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$
3. Set
$$W = R^T V_1 \Sigma_1^{-1/2}, \qquad V = S^T U_1 \Sigma_1^{-1/2}.$$
4. Reduced model is $(W^T AV, W^T B, CV)$.

Note: $T := \Sigma^{-\frac{1}{2}} V^T R$ yields balancing state-space transformation with $T^{-1} = S^T U \Sigma^{-\frac{1}{2}}$, so that $T = \begin{bmatrix} W^T \\ * \end{bmatrix}$ and $T^{-1} = \begin{bmatrix} V & * \end{bmatrix}$.

Model Reduction for LTI Systems

Balanced truncation for linear systems

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

Model Reduction for LTI Systems

Balanced truncation for linear systems

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$||y - \hat{y}||_2 \leq \underbrace{\left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right)}_{=:\delta} ||u||_2 \,.$$

Model Reduction for LTI Systems

Balanced truncation for linear systems

Relation to KYP

- Structural properties of reduced-order models can be proved using KYP.
- Error bound can be proved using KYP as follows:

$$E(s) = \begin{bmatrix} C & -\hat{C} \end{bmatrix} \left(sI_{n+r} - \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ \hat{B} \end{bmatrix} =: \tilde{C} \left(sI_{n+r} - \tilde{A} \right)^{-1} \tilde{B}.$$

is a stable transfer function, i.e., $E \in \mathcal{H}_\infty$.

Model Reduction for LTI Systems

Balanced truncation for linear systems

Relation to KYP

- Structural properties of reduced-order models can be proved using KYP.
- Error bound can be proved using KYP as follows:

$$E(s) = \begin{bmatrix} C & -\hat{C} \end{bmatrix} \left(sI_{n+r} - \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ \hat{B} \end{bmatrix} =: \tilde{C} \left(sI_{n+r} - \tilde{A} \right)^{-1} \tilde{B}.$$

is a stable transfer function, i.e., $E \in \mathcal{H}_\infty$. Hence,

$$\|E\|_{\mathcal{H}_\infty} < \delta \quad \Longleftrightarrow \quad \Phi_\delta(\mathrm{i}\omega) \succcurlyeq 0 \ \forall \omega$$

for Popov function

$$\Phi_\delta(s) = \begin{bmatrix} (sI_{n+r} - \tilde{A})^{-1}\tilde{B} \\ I_m \end{bmatrix}^H \begin{bmatrix} -\tilde{C}^T\tilde{C} & 0 \\ 0 & \delta^2 I_m \end{bmatrix} \begin{bmatrix} (sI_{n+r} - \tilde{A})^{-1}\tilde{B} \\ I_m \end{bmatrix}.$$

Model Reduction for LTI Systems

Balanced truncation for linear systems

Relation to KYP

- Structural properties of reduced-order models can be proved using KYP.
- Error bound can be proved using KYP as follows:

$$E(s) = \begin{bmatrix} C & -\hat{C} \end{bmatrix} \left(sI_{n+r} - \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ \hat{B} \end{bmatrix} =: \tilde{C} \left(sI_{n+r} - \tilde{A} \right)^{-1} \tilde{B}.$$

is a stable transfer function, i.e., $E \in \mathcal{H}_\infty$. Hence,

$$\|E\|_{\mathcal{H}_\infty} < \delta \quad \Longleftrightarrow \quad \Phi_\delta(\mathrm{i}\omega) \succcurlyeq 0 \;\forall \omega$$

for Popov function

$$\Phi_\delta(s) = \begin{bmatrix} (sI_{n+r} - \tilde{A})^{-1}\tilde{B} \\ I_m \end{bmatrix}^H \begin{bmatrix} -\tilde{C}^T\tilde{C} & 0 \\ 0 & \delta^2 I_m \end{bmatrix} \begin{bmatrix} (sI_{n+r} - \tilde{A})^{-1}\tilde{B} \\ I_m \end{bmatrix}.$$

Using KYP and properties of balanced realizations, one can prove existence of symmetric solution of corresponding LMI.

Motivation

Disadvantages of Balanced Truncation

Global error bound can be pessimistic in relevant frequency bands, e.g., in mechanical systems, often only frequencies $0 \leq 2\pi\omega \leq 1000$ (in Hz) are relevant, in VLSI design only an operating frequency, e.g., 2.6 GHz, may be of interest.

Remedies

1. Frequency-weighted BT (FWBT): aim at minimizing $\|G_o(G - \hat{G})G_i\|_{\mathcal{H}_\infty}$, where G_i, G_o are rational transfer functions, e.g., lowpass/highpass filters.

Motivation

Disadvantages of Balanced Truncation

Global error bound can be pessimistic in relevant frequency bands, e.g., in mechanical systems, often only frequencies $0 \leq 2\pi\omega \leq 1000$ (in Hz) are relevant, in VLSI design only an operating frequency, e.g., 2.6 GHz, may be of interest.

Remedies

1. Frequency-weighted BT (FWBT): aim at minimizing $\|G_o(G - \hat{G})G_i\|_{\mathcal{H}_\infty}$, where G_i, G_o are rational transfer functions, e.g., lowpass/highpass filters.
2. Use Frequency-limited Gramians: recall that the Gramians of stable systems satisfy

$$AP + PA^T + BB^T = 0$$
$$A^T Q + QA + C^T C = 0$$

Motivation

Disadvantages of Balanced Truncation

Global error bound can be pessimistic in relevant frequency bands, e.g., in mechanical systems, often only frequencies $0 \leq 2\pi\omega \leq 1000$ (in Hz) are relevant, in VLSI design only an operating frequency, e.g., 2.6 GHz, may be of interest.

Remedies

1. Frequency-weighted BT (FWBT): aim at minimizing $\|G_o(G - \hat{G})G_i\|_{\mathcal{H}_\infty}$, where G_i, G_o are rational transfer functions, e.g., lowpass/highpass filters.
2. Use Frequency-limited Gramians: recall that the Gramians of stable systems satisfy

$$AP + PA^T + BB^T = 0 \;\Leftrightarrow\; P = \int_0^\infty e^{At} BB^T e^{A^T t}\, dt$$
$$A^T Q + QA + C^T C = 0 \;\Leftrightarrow\; P = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt$$

Motivation

Disadvantages of Balanced Truncation

Global error bound can be pessimistic in relevant frequency bands, e.g., in mechanical systems, often only frequencies $0 \leq 2\pi\omega \leq 1000$ (in Hz) are relevant, in VLSI design only an operating frequency, e.g., 2.6 GHz, may be of interest.

Remedies

1. Frequency-weighted BT (FWBT): aim at minimizing $\|G_o(G - \hat{G})G_i\|_{\mathcal{H}_\infty}$, where G_i, G_o are rational transfer functions, e.g., lowpass/highpass filters.
2. Use Frequency-limited Gramians: recall that the Gramians of stable systems satisfy

$$AP + PA^T + BB^T = 0 \;\Leftrightarrow\; P = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\jmath\omega I - A)^{-1} BB^T (\jmath\omega I - A)^{-H} dt$$

$$A^T Q + QA + C^T C = 0 \;\Leftrightarrow\; Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\jmath\omega I - A)^{-H} C^T C (\jmath\omega I - A)^{-1} dt$$

Motivation

Disadvantages of Balanced Truncation

Global error bound can be pessimistic in relevant frequency bands, e.g., in mechanical systems, often only frequencies $0 \leq 2\pi\omega \leq 1000$ (in Hz) are relevant, in VLSI design only an operating frequency, e.g., 2.6 GHz, may be of interest.

Remedies

1. Frequency-weighted BT (FWBT): aim at minimizing $\|G_o(G - \hat{G})G_i\|_{\mathcal{H}_\infty}$, where G_i, G_o are rational transfer functions, e.g., lowpass/highpass filters.
2. Use Frequency-limited Gramians:

$$P(\varpi) := \frac{1}{2\pi}\int_{-\varpi}^{\varpi} (\jmath\omega I - A)^{-1}BB^T(\jmath\omega I - A)^{-H}dt$$

$$Q(\varpi) := \frac{1}{2\pi}\int_{-\varpi}^{\varpi} (\jmath\omega I - A)^{-H}C^T C(\jmath\omega I - A)^{-1}dt$$

Motivation

Disadvantages of Balanced Truncation

Global error bound can be pessimistic in relevant frequency bands, e.g., in mechanical systems, often only frequencies $0 \leq 2\pi\omega \leq 1000$ (in Hz) are relevant, in VLSI design only an operating frequency, e.g., 2.6 GHz, may be of interest.

Remedies

1. Frequency-weighted BT (FWBT): aim at minimizing $\|G_o(G - \hat{G})G_i\|_{\mathcal{H}_\infty}$, where G_i, G_o are rational transfer functions, e.g., lowpass/highpass filters.
2. Use Frequency-limited Gramians:

$$P(\varpi) := \frac{1}{2\pi}\int_{-\varpi}^{\varpi} (\jmath\omega I - A)^{-1} BB^T (\jmath\omega I - A)^{-H} dt$$

$$Q(\varpi) := \frac{1}{2\pi}\int_{-\varpi}^{\varpi} (\jmath\omega I - A)^{-H} C^T C (\jmath\omega I - A)^{-1} dt$$

Both approaches yield good local approximation properties, but error bounds are still global and stability preservation often requires some modifications!

The Frequency-dependent KYP Lemma

Theorem [Iwasaki/Hara '05]

Consider $G(\jmath\omega) = C(\jmath\omega I - A)^{-1}B + D$, $\varpi \in \mathbb{R}$ such that $\jmath\varpi$ is not a pole of G, and let $\Pi = \Pi^T \in \mathbb{R}^{n\times n}$. Then TFAE:

a) $\begin{bmatrix} G(\jmath\varpi) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(\jmath\varpi) \\ I \end{bmatrix} \preccurlyeq 0.$

b) There exist symmetric matrices P and $Q \succ 0$ of appropriate dimensions, satisfying

$$\begin{bmatrix} A & I \\ C & 0 \end{bmatrix} \begin{bmatrix} -Q & P + \jmath\varpi Q \\ P - \jmath\varpi Q & -\jmath\varpi^2 Q \end{bmatrix} \begin{bmatrix} A & I \\ C & 0 \end{bmatrix}^T + \begin{bmatrix} B & 0 \\ D & I \end{bmatrix} \Pi \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}^T \preccurlyeq 0.$$

Note: in standard KYP, we used $-\Pi = \begin{bmatrix} W & S \\ S^T & R \end{bmatrix}$.

The Frequency-dependent KYP Lemma

A family of frequency-dependent systems

Given $\epsilon, \varpi \in \mathbb{R}$, we define

$$\begin{aligned} \dot{x}(t) &= A_{\varpi} x(t) + B_{\varpi} u(t), \\ y(t) &= C_{\varpi} x(t) + D_{\varpi} u(t), \end{aligned}$$

by

$$\begin{aligned} A_{\varpi} &:= \jmath\varpi I - \epsilon((\epsilon + \jmath\varpi)I - A)^{-1}(\jmath\varpi I - A), \\ B_{\varpi} &:= \epsilon((\epsilon + \jmath\varpi)I - A)^{-1}B, \\ C_{\varpi} &:= \epsilon C((\epsilon + \jmath\varpi)I - A)^{-1}, \\ D_{\varpi} &:= D + C((\epsilon + \jmath\varpi I) - A)^{-1}B. \end{aligned}$$

The associated transfer function is

$$G_{\varpi}(\jmath\omega) = C_{\varpi}(\jmath\omega I - A_{\varpi})^{-1}B_{\varpi} + D_{\varpi}.$$

The Frequency-dependent KYP Lemma

Properties of the frequency-dependent systems

Theorem 1

a) G stable $\Longrightarrow$ G_{ϖ} is stable for all $\epsilon > 0$.

The Frequency-dependent KYP Lemma

Properties of the frequency-dependent systems

Theorem 1

a) G stable $\Longrightarrow$ G_{ϖ} is stable for all $\epsilon > 0$.

b) If G is unstable, then G_{ϖ} is stable for $0 < \epsilon < \hat{\epsilon}_{\varpi}$, where

$$\hat{\epsilon}_{\varpi} = \min_{\lambda_u \in \Lambda(A) \cap \mathbb{C}_0^+} \left\{ \frac{(\varpi - \Im(\lambda_u))^2}{\Re(\lambda_u)} + \Re(\lambda_u) \right\}.$$

The Frequency-dependent KYP Lemma

Properties of the frequency-dependent systems

Theorem 1

a) G stable $\Longrightarrow$ G_ϖ is stable for all $\epsilon > 0$.

b) If G is unstable, then G_ϖ is stable for $0 < \epsilon < \hat{\epsilon}_\varpi$, where

$$\hat{\epsilon}_\varpi = \min_{\lambda_u \in \Lambda(A) \cap \mathbb{C}_0^+} \left\{ \frac{(\varpi - \Im(\lambda_u))^2}{\Re(\lambda_u)} + \Re(\lambda_u) \right\}.$$

c) (A, B) controllable $\Longrightarrow$ (A_ϖ, B_ϖ) controllable.

d) (A, C) observable $\Longrightarrow$ (A_ϖ, C_ϖ) observable.

The Frequency-dependent KYP Lemma

Properties of the frequency-dependent systems

Theorem 1

a) G stable $\Longrightarrow$ G_ϖ is stable for all $\epsilon > 0$.

b) If G is unstable, then G_ϖ is stable for $0 < \epsilon < \hat{\epsilon}_\varpi$, where

$$\hat{\epsilon}_\varpi = \min_{\lambda_u \in \Lambda(A) \cap \mathbb{C}_0^+} \left\{ \frac{(\varpi - \Im(\lambda_u))^2}{\Re(\lambda_u)} + \Re(\lambda_u) \right\}.$$

c) (A, B) controllable $\Longrightarrow$ (A_ϖ, B_ϖ) controllable.

d) (A, C) observable $\Longrightarrow$ (A_ϖ, C_ϖ) observable.

e) (A, B, C, D) is a minimal realization of G $\Longrightarrow$ $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ is a minimal realization of G_ϖ.

The Frequency-dependent KYP Lemma

Properties of the frequency-dependent systems

Theorem 1

a) G stable $\Longrightarrow$ G_ϖ is stable for all $\epsilon > 0$.

b) If G is unstable, then G_ϖ is stable for $0 < \epsilon < \hat{\epsilon}_\varpi$, where

$$\hat{\epsilon}_\varpi = \min_{\lambda_u \in \Lambda(A) \cap \mathbb{C}_0^+} \left\{ \frac{(\varpi - \Im(\lambda_u))^2}{\Re(\lambda_u)} + \Re(\lambda_u) \right\}.$$

c) (A, B) controllable $\Longrightarrow$ (A_ϖ, B_ϖ) controllable.

d) (A, C) observable $\Longrightarrow$ (A_ϖ, C_ϖ) observable.

e) (A, B, C, D) is a minimal realization of G $\Longrightarrow$ $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ is a minimal realization of G_ϖ.

f) $G_\varpi(\jmath\varpi) = G(\jmath\varpi)$.

The Frequency-dependent KYP Lemma

Properties of the frequency-dependent systems

Theorem 1

a) G stable $\Longrightarrow$ G_ϖ is stable for all $\epsilon > 0$.

b) If G is unstable, then G_ϖ is stable for $0 < \epsilon < \hat{\epsilon}_\varpi$, where

$$\hat{\epsilon}_\varpi = \min_{\lambda_u \in \Lambda(A) \cap \mathbb{C}_0^+} \left\{ \frac{(\varpi - \Im(\lambda_u))^2}{\Re(\lambda_u)} + \Re(\lambda_u) \right\}.$$

c) (A, B) controllable $\Longrightarrow$ (A_ϖ, B_ϖ) controllable.

d) (A, C) observable $\Longrightarrow$ (A_ϖ, C_ϖ) observable.

e) (A, B, C, D) is a minimal realization of $G \Longrightarrow$ $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ is a minimal realization of G_ϖ.

f) $G_\varpi(\jmath\varpi) = G(\jmath\varpi)$.

g) $\|G\|_{\mathcal{H}_\infty} \leq \gamma \Longrightarrow \|G_\varpi\|_{\mathcal{H}_\infty} \leq \gamma$.

The Frequency-dependent KYP Lemma

Properties of the frequency-dependent systems

Theorem 1

a) G stable $\Longrightarrow$ G_{ϖ} is stable for all $\epsilon > 0$.

b) If G is unstable, then G_{ϖ} is stable for $0 < \epsilon < \hat{\epsilon}_{\varpi}$, where

$$\hat{\epsilon}_{\varpi} = \min_{\lambda_u \in \Lambda(A) \cap \mathbb{C}_0^+} \left\{ \frac{(\varpi - \Im(\lambda_u))^2}{\Re(\lambda_u)} + \Re(\lambda_u) \right\}.$$

c) (A, B) controllable $\Longrightarrow$ (A_{ϖ}, B_{ϖ}) controllable.

d) (A, C) observable $\Longrightarrow$ (A_{ϖ}, C_{ϖ}) observable.

e) (A, B, C, D) is a minimal realization of G $\Longrightarrow$ $(A_{\varpi}, B_{\varpi}, C_{\varpi}, D_{\varpi})$ is a minimal realization of G_{ϖ}.

f) $G_{\varpi}(\jmath\varpi) = G(\jmath\varpi)$.

g) $\|G\|_{\mathcal{H}_\infty} \leq \gamma \Longrightarrow \|G_{\varpi}\|_{\mathcal{H}_\infty} \leq \gamma$.

h) $\|G_{\varpi}\|_{\mathcal{H}_\infty} \leq \gamma_{\varpi} \Longrightarrow \sigma_{\max}(G(\jmath\varpi)) \leq \gamma_{\varpi}$.

The Frequency-dependent KYP Lemma

Properties of the frequency-dependent systems

Theorem 2

Suppose the LTI system (A, B, C, D) is Hurwitz and minimal, and denote its controllability, observability, and balanced Gramians as P, Q, Σ, then for any ϖ-dependent extended system $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ with Gramians $P_\varpi, Q_\varpi, \Sigma_\varpi$:

a) $P \succ P_\varpi, \quad Q \succ Q_\varpi, \quad \Sigma \succ \Sigma_\varpi.$

b) $\lim_{\varepsilon\to 0} P_\varpi = 0, \quad \lim_{\varepsilon\to 0} Q_\varpi = 0, \quad \lim_{\varepsilon\to 0} \Sigma_\varpi = 0.$

c) $\lim_{\varepsilon\to \infty} P_\varpi = P, \quad \lim_{\varepsilon\to \infty} Q_\varpi = Q, \quad \lim_{\varepsilon\to \infty} \Sigma_\varpi = \Sigma.$

Frequency-dependent Balanced Truncation (FDBT)

Apply the generic balancing procedure to $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$, i.e., solve

$$A_\varpi P_\varpi + P_\varpi A_\varpi^H + B_\varpi B_\varpi^H = 0, \quad A_\varpi^H Q_\varpi + Q_\varpi A_\varpi + C_\varpi^H C_\varpi = 0,$$

and compute the balancing transformation T_ϖ so that

$$T_\varpi P_\varpi T_\varpi^H = T_\varpi^{-H} Q_\varpi T_\varpi^{-1} = \Sigma_\varpi = \mathrm{diag}\,(\sigma_{\varpi,1}, \ldots, \sigma_{\varpi,n}), \quad \text{with } \sigma_{\varpi,k} \geq \sigma_{\varpi,k+1}.$$

Frequency-dependent Balanced Truncation (FDBT)

Apply the generic balancing procedure to $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$, i.e., solve

$$A_\varpi P_\varpi + P_\varpi A_\varpi^H + B_\varpi B_\varpi^H = 0, \quad A_\varpi^H Q_\varpi + Q_\varpi A_\varpi + C_\varpi^H C_\varpi = 0,$$

and compute the balancing transformation T_ϖ so that

$$T_\varpi P_\varpi T_\varpi^H = T_\varpi^{-H} Q_\varpi T_\varpi^{-1} = \Sigma_\varpi = \mathrm{diag}\,(\sigma_{\varpi,1}, \ldots, \sigma_{\varpi,n}), \quad \text{with } \sigma_{\varpi,k} \geq \sigma_{\varpi,k+1}.$$

Balance the system:

$$\begin{aligned}(T_\varpi A_\varpi T_\varpi^{-1}, T_\varpi B_\varpi, C_\varpi T_\varpi^{-1}, D_\varpi) \\ = \left(\begin{bmatrix} A_{\varpi,11} & A_{\varpi,12} \\ A_{\varpi,21} & A_{\varpi,22} \end{bmatrix}, \begin{bmatrix} B_{\varpi,1} \\ B_{\varpi,2} \end{bmatrix}, \begin{bmatrix} C_{\varpi,1} & C_{\varpi,2} \end{bmatrix}, D_\varpi \right).\end{aligned}$$

Frequency-dependent Balanced Truncation (FDBT)

Balance the system:

$$
\begin{aligned}
&(T_\varpi A_\varpi T_\varpi^{-1}, T_\varpi B_\varpi, C_\varpi T_\varpi^{-1}, D_\varpi) \\
&\quad = \left(\begin{bmatrix} A_{\varpi,11} & A_{\varpi,12} \\ A_{\varpi,21} & A_{\varpi,22} \end{bmatrix}, \begin{bmatrix} B_{\varpi,1} \\ B_{\varpi,2} \end{bmatrix}, \begin{bmatrix} C_{\varpi,1} & C_{\varpi,2} \end{bmatrix}, D_\varpi \right).
\end{aligned}
$$

Reduced-order model is then obtained by truncation and back transformation: select r such that $\sigma_{\varpi,r} > \sigma_{\varpi,r+1}$ and set

$$
\begin{aligned}
\hat{A} &= \jmath\varpi I_r - \epsilon(\jmath\varpi I_r - A_{\varpi,11})\left((\epsilon - \jmath\varpi)I_r + A_{\varpi,11}\right)^{-1}, \\
\hat{B} &= \frac{1}{\epsilon}((\epsilon + \jmath\varpi)I_r - \hat{A})B_{\varpi,1}, \\
\hat{C} &= \frac{1}{\epsilon}C_{\varpi,1}((\epsilon + \jmath\varpi)I_r - \hat{A}), \\
\hat{D} &= D_\varpi - \frac{1}{\epsilon^2}C_{\varpi,1}((\epsilon + \jmath\varpi)I_r - \hat{A})B_{\varpi,1}.
\end{aligned}
$$

Frequency-dependent Balanced Truncation (FDBT)

Reduced-order model is then obtained by truncation and back transformation: select r such that $\sigma_{\varpi,r} > \sigma_{\varpi,r+1}$ and set

$$\begin{aligned}
\hat{A} &= \jmath\varpi I_r - \epsilon(\jmath\varpi I_r - A_{\varpi,11})\left((\epsilon - \jmath\varpi)I_r + A_{\varpi,11}\right)^{-1}, \\
\hat{B} &= \frac{1}{\epsilon}((\epsilon + \jmath\varpi)I_r - \hat{A})B_{\varpi,1}, \\
\hat{C} &= \frac{1}{\epsilon}C_{\varpi,1}((\epsilon + \jmath\varpi)I_r - \hat{A}), \\
\hat{D} &= D_{\varpi} - \frac{1}{\epsilon^2}C_{\varpi,1}((\epsilon + \jmath\varpi)I_r - \hat{A})B_{\varpi,1}.
\end{aligned}$$

Theorem 3 (Local Error Bound)

The reduced-order transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$ satisfies:

$$\sigma_{\max}\left(G(\jmath\varpi) - \hat{G}(\jmath\varpi)\right) \leq 2\sum_{k=r+1}^{n} \sigma_{\varpi,k}.$$

Proof: use proof for BT error bound based on FD-KYP instead of KYP.

Numerical Examples

RLC ladder network

Simple example of electronic circuit from [Sorensen '05]

- input $\equiv$ voltage u, output $\equiv$ current y,
- scaled inductances, capacities, and resistance: $L_j = 1$, $C_j = 1$ for all j; $R_1 = 0.5$, $R_2 = 0.2$.
- $n = 5$, $m = p = 1$.

Numerical Examples

RLC ladder network

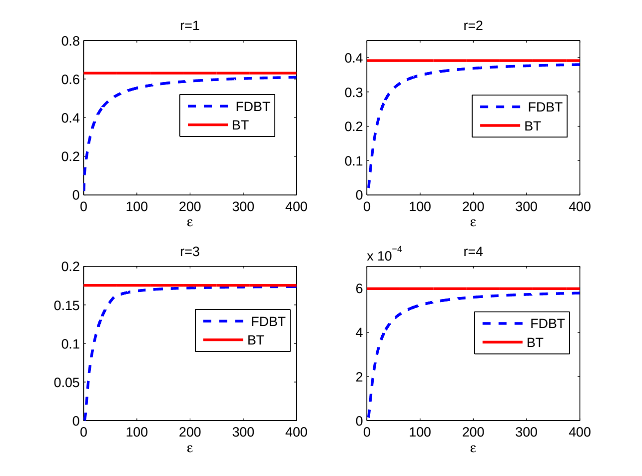
Comparison of FDBT and BT ($\bar{\omega} = 0, \ \varepsilon = 1$)

r	FDBT		BT	
	bound	true error	bound	true error
4	1.2201×10^{-7}	1.2201×10^{-7}	0.0006	0.0006
3	8.7426×10^{-5}	8.7182×10^{-5}	0.1752	0.1740
2	5.5028×10^{-4}	3.7568×10^{-4}	0.3914	0.0421
1	0.0584	0.0582	0.6311	0.1975

Numerical Examples

RLC ladder network

Comparison of FDBT and BT ($\bar{\omega} = 0$, varying ε)

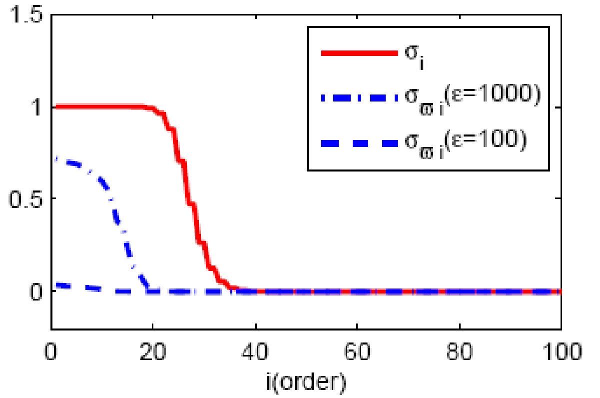


Numerical Examples

Butterworth filter

Bandstop filter `[A,B,C,D]= butter(50,[90 110],stop,s)`

Hankel singular values

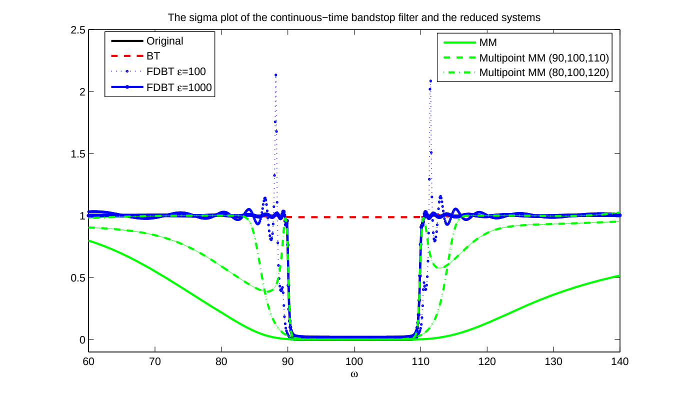


Numerical Examples

Butterworth filter

Bandstop filter [A,B,C,D]= butter(50,[90 110],stop,s)

Transfer functions



Conclusions and Future Work

Summary:

- Relations of KYP lemma to balanced truncation.
- Frequency-dependent KYP lemma suggests new frequency-dependent balanced truncation (FDBT) method.
- FDBT offers alternative to interpolation-based method if good local approximation quality is desired.
- Continuous- and discrete-time FDBT derived.
- FDBT is stability preserving and has local error bound, which is often much better than global BT bound.

Conclusions and Future Work

Summary:

- Relations of KYP lemma to balanced truncation.
- Frequency-dependent KYP lemma suggests new frequency-dependent balanced truncation (FDBT) method.
- FDBT offers alternative to interpolation-based method if good local approximation quality is desired.
- Continuous- and discrete-time FDBT derived.
- FDBT is stability preserving and has local error bound, which is often much better than global BT bound.

Future work:

- Details for non-minimal systems.
- Large-scale implementation and testing.
- Computational feasible method for frequency bands.
- Extension to descriptor systems.

References

V. Balakrishnan and L. Vandenberghe.
Semidefinite programming duality and linear time-invariant systems.
IEEE Transactions on Automatic Control 48(1):30–41, 2003.
doi: 10.1109/TAC.2002.806652

X. Du, P. Benner, G. Yang, and D. Ye.
Balanced Truncation of Linear Time-Invariant Systems at a Single Frequency.
Max Planck Institute Magdeburg Preprints, MPIMD/13–02, January 2013.

T. Iwasaki and S. Hara.
Generalized KYP lemma: unified frequency domain inequalities with design applications.
IEEE Transactions on Automatic Control 50(1):41–59, 2005.

J. Willems.
Dissipative dynamical systems.
Archive for Rationale Mechanics Analysis 45:321–393, 1972.