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LQG-Balanced Truncation Low-Order Controller for Stabilization of Laminar Flows

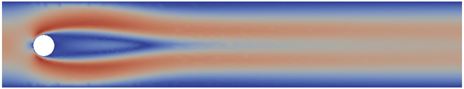
Peter Benner   Jan Heiland

Max Planck Institute for Dynamics of Complex Technical Systems
— Computational Methods in Systems and Control —

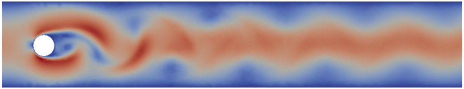
Outline

1. Introduction
   - Problem Statement
   - LQGBT Based Controller
2. Controller Design
   - LQG Balanced Truncation
   - State Equations for Flow Control
   - Controller Definition
   - Numerical Realization
3. Numerical Example
   - Setup
   - Results

Problem Statement



- Cylinder wake at moderate *Reynolds* numbers.
- The steady state is a solution, but unstable.
- Goal: stabilizing feedback controller that works in experiments.
- Thus, the simulation needs to cope with:
  - ➙ limited measurements,
  - ➙ short evaluation times,
  - ➙ external perturbations,
  - ➙ actuation at the boundary.



Model Based and Reduced Controller

We propose a controller, that results from a simultaneous application of

- a linearization about the steady state
  - ➙ to directly attack the deviations;
- a *Kalman filter*
  - ➙ to estimate the state using a few measurements;
- an *LQG regulator*
  - ➙ to stabilize the linearized system;
- and *Balanced Truncation*
  - ➙ to reduce the linearized and stabilized system.

Expectations and Limitations

The proposed controller is based on a linearized model

➙ we expect a good performance for small deviations

and is designed to work for

- ✓ limited state information,
- ✓ fast and unstable dynamics,
- ✓ high dimensionality,

but still awaits its extension to

- ✗ boundary control.

Related Work

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LQG Balanced Truncation

LQG Control

Consider the minimal, but possibly unstable, linear time-invariant system

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Positive semidefinite solution X_c of the control Riccati equation

$$A^T X_c + X_c A - X_c B B^T X_c + C C^T = 0$$

defines a stabilizing feedback (LQ regulator), i.e.

$$\dot{x} = (A - B B^T X_c)x.$$

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Consider the minimal, but possibly unstable, linear time-invariant system

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Positive semidefinite solution X_o of the filter Riccati equation

$$AX_o + X_oA^T - X_oC^TCX_o + BB^T = 0$$

defines a state estimator (Kalman filter) via

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + X_oC^T(y(t) - C\hat{x}(t)).$$

LQG Balanced Truncation

Basic principle:

- A minimal LTI system, realized by (A, B, C), is called LQG balanced, if the solutions X_c, X_o of the control and filter Riccati equations satisfy:

$$X_c = X_o = \mathrm{diag}(\mu_1, \ldots, \mu_n), \quad \text{where} \quad \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0.$$

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- Compute LQG balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C) \quad &\mapsto \quad (TAT^{-1}, TB, CT^{-1}) \\ &= \quad \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \right) \end{aligned}$$

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- Reduction and stabilization simultaneously in 1 step!

Controller Design

We consider spatially discretized *Navier-Stokes* equations with control u and observation $y = Cv$

$$\begin{aligned} M\dot{v} &= -N(v)v - \frac{1}{Re}Lv + J^T p - Bu + f, \\ 0 &= Jv - g, \\ v(0) &= \alpha, \\ y &= Cv, \end{aligned}$$

where α is the steady-state solution.

The task is to define the feedback law $\mathcal{G}^{-1}$ so that $u = \mathcal{G}^{-1}y$ keeps v in the steady state regime.

Defining the Controller

1. Consider the linearization about α

$$\begin{aligned} M\dot{v} &= A_\alpha v + J^T p - Bu + f, \quad v(0) = \alpha, \\ 0 &= Jv, \\ y &= Cv. \end{aligned}$$

2. Compute X_c and X_o which solve the associated *control* and *filter Riccati equations* to define the state estimate $\hat{x}$ and the regulator u as

$$\begin{aligned} M\dot{\hat{x}} &= \hat{A}_\alpha \hat{x} + X_o M C^T (y - C\alpha), \\ u &= -B^T M X_c \hat{x}, \end{aligned}$$

   with $\hat{x}(0) = 0$ and $\hat{A}_\alpha$ denoting the observer dynamics.

3. Balance and truncate X_o and X_c to define a reduced observer.

Computational Challenges

The major effort lies in the computation of X_o and X_c, because of

1. high-dimensionality: X_c, $X_o \in \mathbb{R}^{n_v,n_v}$
   $\rightarrow$ n_v is the dimension of the state $v(t)$;
2. nonlinearity of the Riccati equation
   $\rightarrow$ a good initial guess for a Newton iteration is needed;
3. differential algebraic structure of the state equations
   $\rightarrow$ X_c, X_o need to obey divergence constraints.

Low-Rank Approximations

How to obtain approximations to X_c, X_o:

1. Consider, e.g., the constrained filter Riccati equation,

$$A_\alpha^T XM + M^T XA_\alpha - M^T X\mathcal{P}B(\mathcal{P}B)^T XM + CC^T = 0,$$

   with the discrete *Leray* projector

$$\mathcal{P} := I - M^{-1}J(J^T M^{-1}J)^{-1}J$$

2. and apply a *low-rank Newton-ADI iteration* to it.

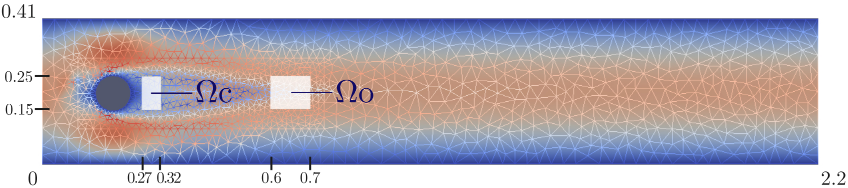
Reduced Closed Loop System

After the truncation, we arrive at

$$
\begin{aligned}
M\dot{v} &= -N(v)v - \frac{1}{Re}Lv + J^T p - BB_k^T X_{ck}\hat{x}_k + f,\\
0 &= Jv - g,\\
v(0) &= \alpha,\\
y &= Cv,\\
\dot{\hat{x}}_k &= (A_{\alpha k} - X_{ok}C_k^T C_k - B_k B_k^T X_{ck})\hat{x}_k + X_{ok}C_k^T(y - y_\alpha),\\
\hat{x}_k(0) &= 0,
\end{aligned}
$$

where, in particular, $A_{\alpha k}$, B_k, C_k, X_{ck}, X_{ok} define the reduced system for $\hat{x}_k(t) \in \mathbb{R}^{n_k}$ with $n_k \ll n_v$ (dimension of $v(t)$).

Simulation Setup



- 2D cylinder wake
- Navier-Stokes Equations
- $Re = 200$
- *Taylor-Hood* finite elements
- $20,000$ velocity nodes

- Distributed control and observation with 6 degrees of freedom each;
- LQGBT-reduced order observer and controller of state dimension $n_k = 27$;
- Target: stabilization of the steady-state solution.

Simulation Results

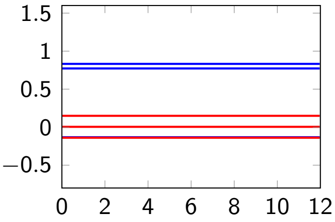
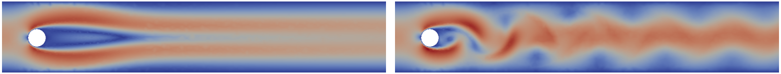


Figure : Measured signal y versus time $t \in [0, 12]$ of the closed loop system with a reduced controller of dimension $n_k = 27$ (left), compared to the response of the uncontrolled system (right). Blue corresponds to the x-component of the velocity and red to y-component. Below, a snapshot of the magnitude of the velocity solutions at $t = 12$.



Summary and Conclusion

- The general LQGBT approach has been applied to controller design for Navier-Stokes equations.
- The resulting controller is of very small dimension and works for limited state information.
- The numerical approximation of the controller requires advanced methods for solving large-scale Riccati equations.
- Successful application in distributed control of the cylinder wake.
- Extension to boundary control is ongoing.

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Thank you for your attention!

More Literature

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