EU Regional School Short Course
Aachen Institute for Advanced Study
in Computational Engineering Science
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Model Reduction Using Rational Approximation Techniques

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Outline

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7. Nonlinear Model Reduction

Introduction

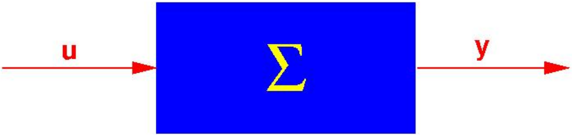
Model Reduction for Dynamical Systems

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.



Model Reduction for Dynamical Systems

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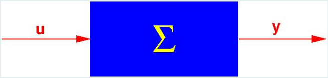

Reduced-Order Model (ROM)

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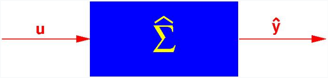

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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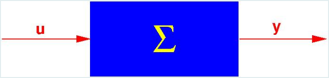

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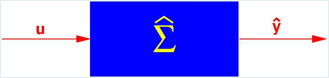

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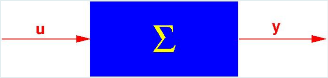

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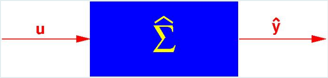

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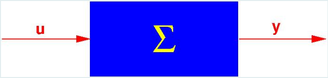

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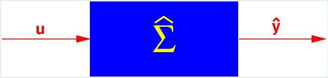

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Model Reduction for Dynamical Systems

Parameter-Dependent Dynamical Systems

Dynamical Systems

$$\Sigma(p) : \begin{cases} E(p)\dot{x}(t;p) = f(t, x(t;p), u(t), p), \quad x(t_0) = x_0, & \text{(a)} \\ y(t;p) = g(t, x(t;p), u(t), p) & \text{(b)} \end{cases}$$

with

- (generalized) states $x(t;p) \in \mathbb{R}^n$ ($E \in \mathbb{R}^{n\times n}$),
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t;p) \in \mathbb{R}^q$, (b) is called output equation,
- $p \in \Omega \subset \mathbb{R}^d$ is a parameter vector, Ω is bounded.

Applications:

- Repeated simulation for varying material or geometry parameters, boundary conditions,
- Control, optimization and design.

Requirement: keep parameters as symbolic quantities in ROM.

Model Reduction for Dynamical Systems

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Model Reduction for Dynamical Systems

Linear Systems

Linear, Time-Invariant (LTI) Systems

$$\begin{array}{rclcll} E\dot{x} & = & f(t,x,u) & = & Ax + Bu, \quad E, A \in \mathbb{R}^{n\times n}, & B \in \mathbb{R}^{n\times m}, \\ y & = & g(t,x,u) & = & Cx + Du, \quad C \in \mathbb{R}^{q\times n}, & D \in \mathbb{R}^{q\times m}. \end{array}$$

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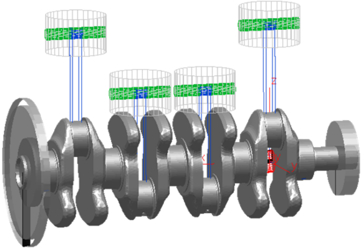
Linear, Time-Invariant Parametric Systems

$$\begin{array}{rcl} E(p)\dot{x}(t;p) & = & A(p)x(t;p) + B(p)u(t), \\ y(t;p) & = & C(p)x(t;p) + D(p)u(t), \end{array}$$

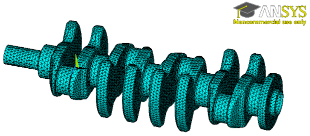
where $A(p), E(p) \in \mathbb{R}^{n \times n}, B(p) \in \mathbb{R}^{n \times m}, C(p) \in \mathbb{R}^{q \times n}, D(p) \in \mathbb{R}^{q \times m}$.

Application Areas

Structural Mechanics / Finite Element Modeling — since ~1960ies



$\rightsquigarrow$



- Resolving complex 3D geometries $\Rightarrow$ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.

Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) $\rightsquigarrow$ Craig-Bampton method.

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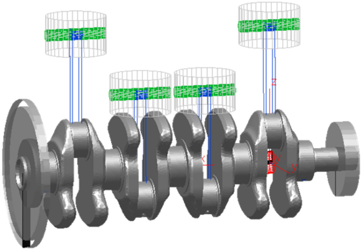

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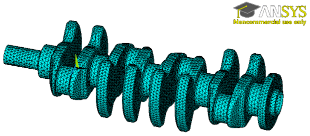

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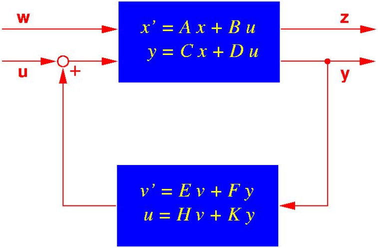
(Optimal) Control **since ∼1980ies**

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N.

$\Longrightarrow$ reduce order of plant (n) and/or controller (N).

Standard MOR techniques in systems and control: balanced truncation and related methods.

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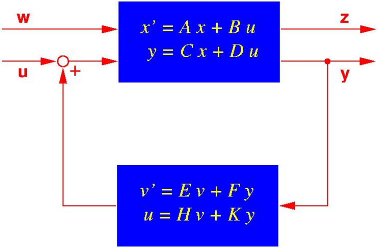

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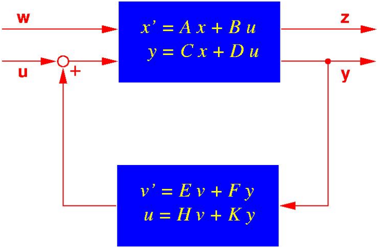

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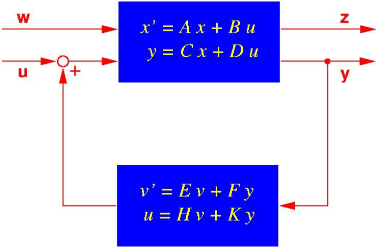

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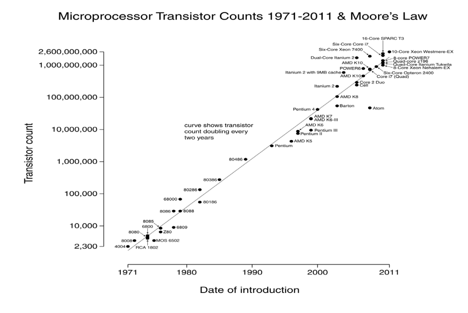
Application Areas

Micro Electronics/Circuit Simulation

since $\sim$1990ies

Progressive miniaturization

- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- **Moore's Law (1965/75)** states that the number of on-chip transistors doubles each 24 months.



Source: http://en.wikipedia.org/wiki/File:Transistor_Count_and_Moore'sLaw_-_2011.svg

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- Increase in packing density and multilayer technology requires modeling of interconncet to ensure that thermic/electro-magnetic effects do not disturb signal transmission.

Intel 4004 (1971)	Intel Core 2 Extreme (quad-core) (2007)
1 layer, 10μ technology	9 layers, $45nm$ technology
2,300 transistors	$> 8,200,000$ transistors
64 kHz clock speed	> 3 GHz clock speed.

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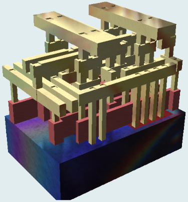

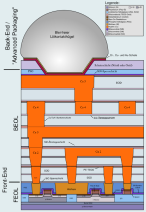

Source: http://en.wikipedia.org/wiki/Image:Silicon_chip_3d.png.

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- Here: mostly MOR for linear systems, they occur in micro electronics through modified nodal analysis (MNA) for RLC networks. e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

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Standard MOR techniques in circuit simulation:
Krylov subspace / Padé approximation / rational interpolation methods.

Application Areas

Many other disciplines in computational sciences and engineering like

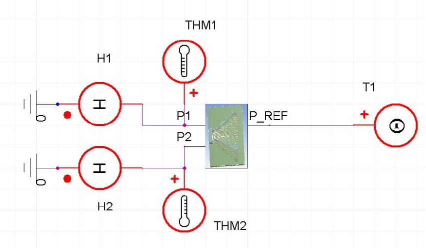
- computational fluid dynamics (CFD),
- computational electromagnetics,
- chemical process engineering,
- design of MEMS/NEMS (micro/nano-electrical-mechanical systems),
- computational acoustics,
- ...

Motivating Examples

Electro-Thermic Simulation of Integrated Circuit (IC)

[Source: Evgenii Rudnyi, CADFEM GmbH]

- SIMPLORER® test circuit with 2 transistors.



- Conservative thermic sub-system in SIMPLORER: voltage ⇝ temperature, current ⇝ heat flow.
- Original model: $n = 270,593$, $m = q = 2 \Rightarrow$ Computing time (on Intel Xeon dualcore 3GHz, 1 Thread):
  - Main computational cost for set-up data $\approx 22min$.
  - Computation of reduced models from set-up data: 44–49sec. ($r = 20$–70).
  - Bode plot (MATLAB on Intel Core i7, 2,67GHz, 12GB): 7.5h for original system, $<$ 1min for reduced system.
  - Speed-up factor: 18 including / $\geq$ 450 excluding reduced model generation!

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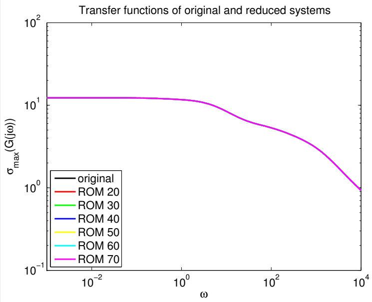
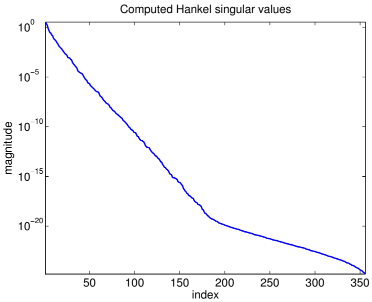
Bode Plot (Amplitude)



Hankel Singular Values



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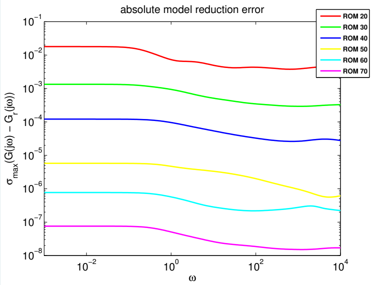
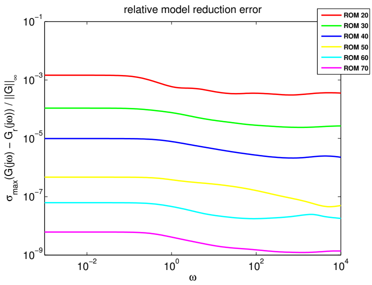
Absolute Error



Relative Error



Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

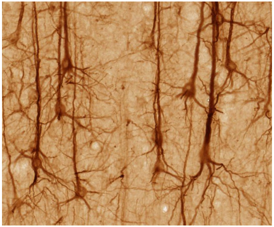
- Simple model for neuron (de-)activation [CHATURANTABUT/SORENSEN 2009]

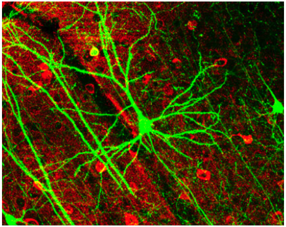
$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1] \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

where $\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ g = 0.05,\ i_0(t) = 50{,}000t^3 \exp(-15t)$.





Source: http://en.wikipedia.org/wiki/Neuron

Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

- Simple model for neuron (de-)activation [Chaturantabut/Sorensen 2009]

$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

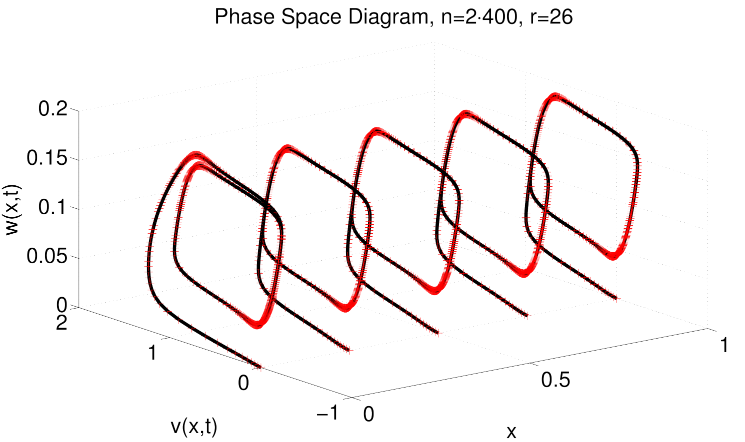
$$\begin{aligned} &v(x,0) = 0, && w(x,0) = 0, && x \in [0,1] \\ &v_x(0,t) = -i_0(t), && v_x(1,t) = 0, && t \geq 0, \end{aligned}$$

where $\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ g = 0.05,\ i_0(t) = 50,000t^3 \exp(-15t)$.

- Parameter g handled as an additional input.
- Original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$.

Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

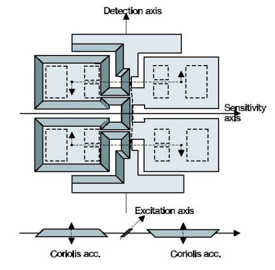


Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

- Application: inertial navigation.



- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order: $N = 17.361 \rightsquigarrow n = 34.722,\ m = 1,\ q = 12.$
- Sensor for position control based on acceleration and rotation.

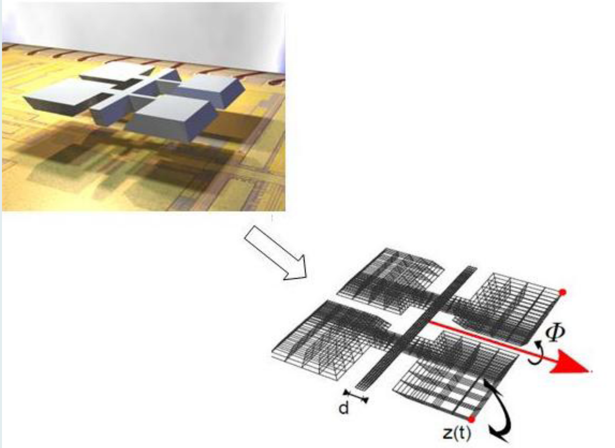
Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t).$



Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model:

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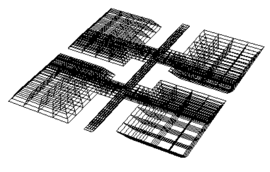
wobei



$$\begin{aligned} M(d) &= M_1 + dM_2, \\ D(\theta, d, \alpha, \beta) &= \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d), \\ T(d) &= T_1 + \frac{1}{d}T_2 + dT_3, \end{aligned}$$

with

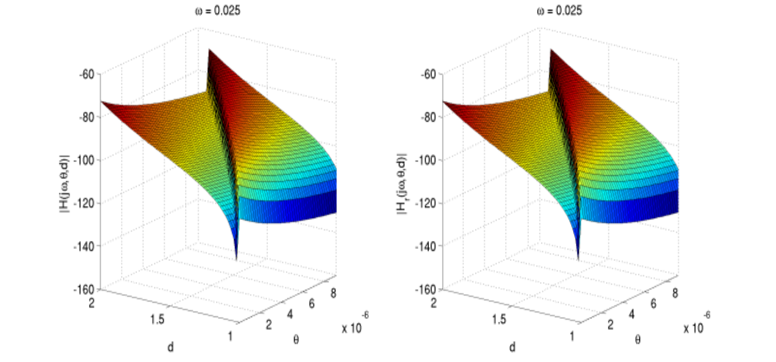
- width of bearing: d,
- angular velocity: θ,
- Rayleigh damping parameters: α, β.

Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Original. . . and reduced-order model.



Some Background
The Laplace transform

Definition

The Laplace transform of a time domain function $f \in L_{1,\mathrm{loc}}$ with $\mathrm{dom}\,(f) = \mathbb{R}_0^+$ is

$$\mathcal{L} : f \mapsto F, \quad F(s) := \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s \in \mathbb{C}.$$

F is a function in the (Laplace or) frequency domain.

Note: for frequency domain evaluations ("frequency response analysis"), one takes $\mathrm{re}\, s = 0$ and $\mathrm{im}\, s \geq 0$. Then $\omega := \mathrm{im}\, s$ takes the role of a frequency (in [rad/s], i.e., $\omega = 2\pi v$ with v measured in [Hz]).

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Lemma

$$\mathcal{L}\{\dot{f}(t)\}(s) = sF(s).$$

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Lemma

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Note: for ease of notation, in the following we will use lower-case letters for both, a function and its Laplace transform!

Some Background
The Model Reduction Problem as Approximation Problem in Frequency Domain

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s))$ to linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(0) = 0$ yields:

$$sEx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),$$

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$$y(s) = \Big(\underbrace{C(sE - A)^{-1}B + D}_{=: G(s)}\Big)u(s).$$

$G(s)$ is the transfer function of Σ.

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$G(s)$ is the transfer function of Σ.

Goal: Fast evaluation of mapping $u \rightarrow y$.

Some Background

The Model Reduction Problem as Approximation Problem in Frequency Domain

Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{array}{rcl} E\dot{x} & = & Ax + Bu, \qquad E, A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y & = & Cx + Du, \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}, \end{array}$$

by reduced-order system

$$\begin{array}{rcl} \hat{E}\dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u, \quad \hat{E}, \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{q\times r},\ \hat{D} \in \mathbb{R}^{q\times m} \end{array}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

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The Model Reduction Problem as Approximation Problem in Frequency Domain

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of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\displaystyle\min_{\text{order}\,(\hat{G}) \leq r} \|G - \hat{G}\|.$

Some Background
Properties of linear systems

Definition

A linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \mid \mathrm{re}(z) < 0\}$.

Lemma

Sufficient for asymptotic stability is that A is asymptotically stable (or Hurwitz), i.e., the spectrum of $A - \lambda E$, denoted by $\Lambda(A, E)$, satisfies $\Lambda(A, E) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

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Some Background

Realizations of Linear Systems (with $E = I_n$ for simplicity)

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad \text{with transfer function} \quad G(s) = C(sI - A)^{-1}B + D,$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

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Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Some Background
Balanced Realizations

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1\text{)}.$$

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When does a balanced realization exist?

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When does a balanced realization exist?
Assume A to be Hurwitz, i.e. $\Lambda(A) \subset \mathbb{C}^-$. Then:

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U\Sigma V^T$ is the SVD of SR^T.

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$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

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Theorem

The infinite controllability/observability Gramians P/Q satisfy the Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

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Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

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Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\mathrm{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

Qualitative and Quantitative Study of the Approximation Error

System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the L_2-norm

$$\|u\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega.$$

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Hardy space $\mathcal{H}_\infty$

Function space of matrix-/scalar-valued functions that are analytic and bounded in $\mathbb{C}^+$.
The $\mathcal{H}_\infty$-norm is

$$\|F\|_\infty := \sup_{\operatorname{re} s > 0} \sigma_{\max}\left(F(s)\right) = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = q = 1$);
- $\mathcal{H}_\infty^{q\times m}$ in the MIMO case (multi-input, multi-output, $m > 1, q > 1$).

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Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty,\infty) \cong \mathcal{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

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Consequently, 2-norms in time and frequency domains coincide!

$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_2 = \|Gu - \hat{G}u\|_2 \leq \|G - \hat{G}\|_\infty \|u\|_2.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_\infty < tol$!

Note: error bound holds in time- and frequency domain due to Paley-Wiener!

Qualitative and Quantitative Study of the Approximation Error

System Norms

Consider stable transfer function

$$G(s) = C\,(sI - A)^{-1}\,B, \quad \text{i.e. } D = 0.$$

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\begin{aligned} \|F\|_2 \quad &:= \quad \frac{1}{2\pi}\left(\sup_{\operatorname{re}\sigma>0}\int_{-\infty}^{\infty}\|F(\sigma+\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}} \\ &= \quad \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\|F(\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}}. \end{aligned}$$

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Qualitative and Quantitative Study of the Approximation Error
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$\mathcal{H}_2$ approximation error for impulse response $(u(t) = u_0\delta(t))$

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_2 = \|Gu_0\delta - \hat{G}u_0\delta\|_2 \leq \|G - \hat{G}\|_2\|u_0\|.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_2 < tol$!

Qualitative and Quantitative Study of the Approximation Error

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$$\|F\|_2 \quad = \quad \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2 \, d\omega\right)^{\frac{1}{2}}.$$

Theorem (Practical Computation of the $\mathcal{H}_2$-norm)

$$\|F\|_2^2 = \operatorname{tr}\left(B^T Q B\right) = \operatorname{tr}\left(C P C^T\right),$$

where P, Q are the controllability and observability Gramians of the corresponding LTI system.

Qualitative and Quantitative Study of the Approximation Error

Approximation Problems

Output errors in time-domain

$$
\begin{aligned}
\|y-\hat{y}\|_2 &\leq \|G-\hat{G}\|_\infty \|u\|_2 \qquad &\Longrightarrow \|G-\hat{G}\|_\infty < \mathrm{tol}\\
\|y-\hat{y}\|_\infty &\leq \|G-\hat{G}\|_2 \|u\|_2 \qquad &\Longrightarrow \|G-\hat{G}\|_2 < \mathrm{tol}
\end{aligned}
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Qualitative and Quantitative Study of the Approximation Error

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\end{aligned}
$$

$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

Introduction
Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
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$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("*system does not generate energy*").

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- Automatic generation of compact models.
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Outline

Model Reduction by Projection

Projection Basics

Definition 3.1 (Projector)

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$.
If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector (aka: Petrov-Galerkin projection).

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- If $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is an orthogonal projector onto $\mathcal{V}$.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\, w_1, \ldots, w_r \,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

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Model Reduction by Projection

Projection Methods

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. many more...

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

Model Reduction by Projection

Projection Methods

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Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, where

$$\mathrm{range}\,(V) = \mathcal{V}, \quad \mathrm{range}\,(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Model Reduction by Projection

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Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T \left(\dot{\tilde{x}} - A\tilde{x} - Bu \right) = W^T \left(VW^T \dot{x} - AVW^T x - Bu \right)$$

Model Reduction by Projection

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Model Reduction by Projection

Projection and Rational Interpolation

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) \quad = \quad \left(C(sI_n - A)^{-1} B + D \right) - \left(\hat{C}(sI_r - \hat{A})^{-1} \hat{B} + \hat{D} \right)$$

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If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $P(s_*)$ is a projector onto $\mathcal{V}$ $\Longrightarrow$

if $(s_* I_n - A)^{-1}B \in \mathcal{V}$, *then* $(I_n - P(s_*))(s_* I_n - A)^{-1}B = 0$,

Hence

$$G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*), \text{ i.e., } \hat{G} \text{ interpolates } G \text{ in } s_*!$$

Model Reduction by Projection

Projection and Rational Interpolation

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If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $Q(s)^H$ is a projector onto $\mathcal{W} \Longrightarrow$

if $(s_* I_n - A)^{-*} C^T \in \mathcal{W}$, *then* $C(s_* I_n - A)^{-1}(I_n - Q(s_*)) = 0.$

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Model Reduction by Projection

Projection and Rational Interpolation

Theorem [Grimme '97, Villemagne/Skelton '87]

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

and $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, if either

- $(s_* I_n - A)^{-1} B \in \mathrm{range}(V)$, or
- $(s_* I_n - A)^{-*} C^T \in \mathrm{range}(W)$,

then the interpolation condition

$$G(s_*) = \hat{G}(s_*).$$

in s_* holds.

Note: extension to Hermite interpolation conditions later!

Outline

Modal Truncation

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \mathrm{span}(t_1, \ldots, t_r)$, t_k = eigenvectors corresp. to "dominant" modes / eigenvalues of A. Then with

$$V = T(:, 1:r) = [\, t_1, \ldots, t_r \,], \quad \tilde{W}^H = T^{-1}(1:r, :), \quad W = \tilde{W}(V^H \tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^H A V = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_r\}, \quad \hat{B} := W^H B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\, \hat{C}, \, C_2 \,], \quad \hat{D} = D.$$

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Properties:

Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.

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Properties:

Error bound:

$$\|G - \hat{G}\|_\infty \leq \|C_2\|\|B_2\|\frac{1}{\min_{\lambda\in\Lambda(A_2)}|\mathrm{Re}(\lambda)|}.$$

Proof:

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B + D = CTT^{-1}(sI-A)^{-1}TT^{-1}B + D \\ &= CT(sI - T^{-1}AT)^{-1}T^{-1}B + D \\ &= [\,\hat{C},\; C_2\,]\begin{bmatrix} (sI_r-\hat{A})^{-1} & \\ & (sI_{n-r}-A_2)^{-1} \end{bmatrix}\begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix} + D \\ &= \hat{G}(s) + C_2(sI_{n-r}-A_2)^{-1}B_2, \end{aligned}$$

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Proof:

$$G(s) \quad = \quad \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,$$

observing that $\|G - \hat{G}\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2)$, and

$$C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2 = C_2 \mathrm{diag}\left(\frac{1}{\jmath\omega - \lambda_{r+1}}, \ldots, \frac{1}{\jmath\omega - \lambda_n}\right) B_2.$$

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Difficulties:

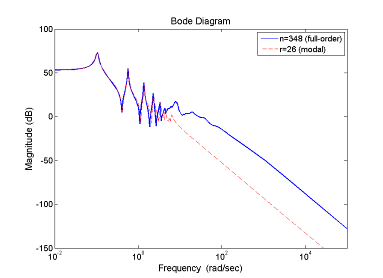
- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
  ([Litz '79] use Jordan canoncial form; otherwise merely heuristic criteria, e.g., [Varga '95]. Recent improvement: dominant pole algorithm.)
- Error bound not computable for really large-scale problems.

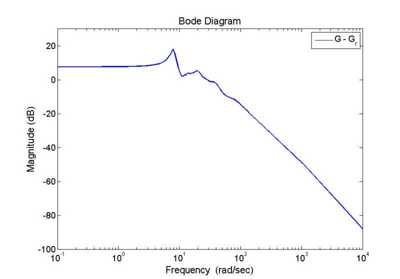
Modal Truncation

Example

BEAM, SISO system from SLICOT Benchmark Collection for Model Reduction, $n = 348$, $m = q = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_\infty \leq 1.21 \cdot 10^3$

Bode plots of transfer functions and error function





Modal Truncation

Extensions

Base enrichment

Static modes are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \quad \Longrightarrow \quad x(t) \equiv -A^{-1}b_j.$$

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[\, b_1, \ldots, b_m \,] = A^{-1}B$.

Interpolation-projection framework $\Longrightarrow G(0) = \hat{G}(0)!$

If two sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \Longrightarrow G'(0) = \hat{G}'(0)!$ (If $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$).

Modal Truncation

Extensions

Guyan reduction (static condensation)

Partition states in masters $x_1 \in \mathbb{R}^r$ and slaves $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)
Assume stationarity, i.e., $\dot{x} = 0$ and solve for x_2 in

$$
\begin{aligned}
0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\
\Rightarrow \quad x_2 &= -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u.
\end{aligned}
$$

Inserting this into the first part of the dynamic system

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u, \quad y = C_1 x_1 + C_2 x_2$$

then yields the reduced-order model

$$
\begin{aligned}
\dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2) u \\
y &= (C_1 - C_2 A_{22}^{-1} A_{21}) x_1 - C_2 A_{22}^{-1} B_2 u.
\end{aligned}
$$

Modal Truncation

Dominant Poles

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

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Note: this follows using the spectral decomposition $A = XDX^{-1}$, with $X = [\,x_1, \ldots, x_n]$ the right and $X^{-1} =: Y = [y_1, \ldots, y_n]^H$ the left eigenvector matrices:

$$\begin{aligned}
G(s) &= C(sI - XDX^{-1})^{-1}B = CX(sI - \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\})^{-1}YB \\
&= [\,Cx_1, \ldots, Cx_n\,] \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} y_1^H B \\ \vdots \\ y_n^H B \end{bmatrix} \\
&= \sum_{k=1}^{n} \frac{(Cx_k)(y_k^H B)}{s - \lambda_k}.
\end{aligned}$$

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Note: $R_k = (Cx_k)(y_k^H B)$ are the residues of G in the sense of the residue theorem of complex analysis:

$$\mathrm{res}\,(G, \lambda_\ell) = \lim_{s \to \lambda_\ell} (s - \lambda_\ell) G(s) = \sum_{k=1}^{n} \underbrace{\lim_{s \to \lambda_\ell} \frac{s - \lambda_\ell}{s - \lambda_k}}_{= \begin{cases} 0 \text{ for } k \neq \ell \\ 1 \text{ for } k = \ell \end{cases}} R_k = R_\ell.$$

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Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

As projection basis use spaces spanned by right/left eigenvectors corresponding to dominant poles, i.e.. $(\lambda_j,\ x_j,\ y_j)$ with largest

$$\|R_k\| / |\operatorname{re}(\lambda_k)|.$$

Modal Truncation

Dominant Poles

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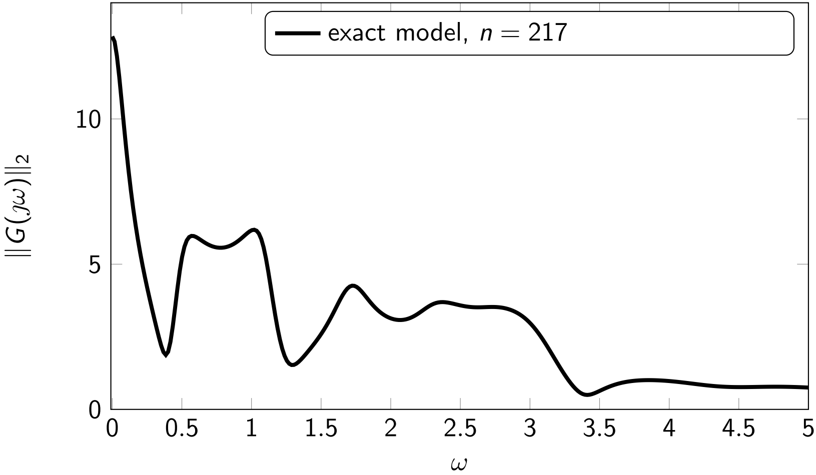
Remark

The dominant modes have most important influence on the input-output behavior of the system and are responsible for the "peaks"' in the frequency response.

Dominant Poles

Random SISO Example ($B,\ C^T \in \mathbb{R}^n$)



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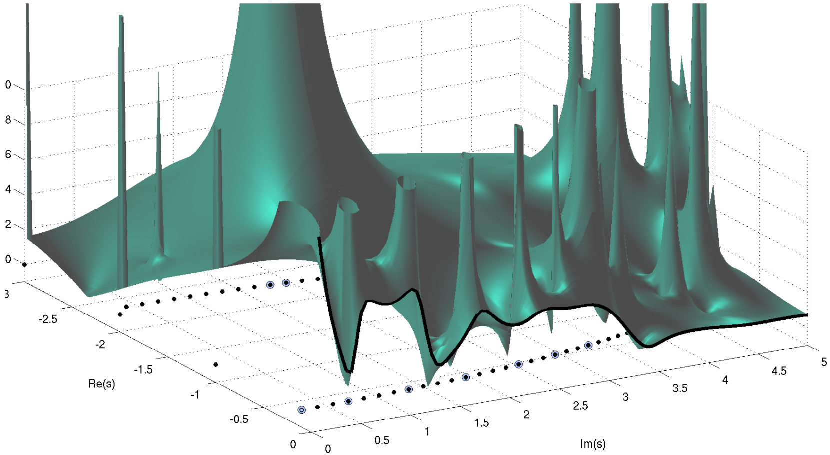

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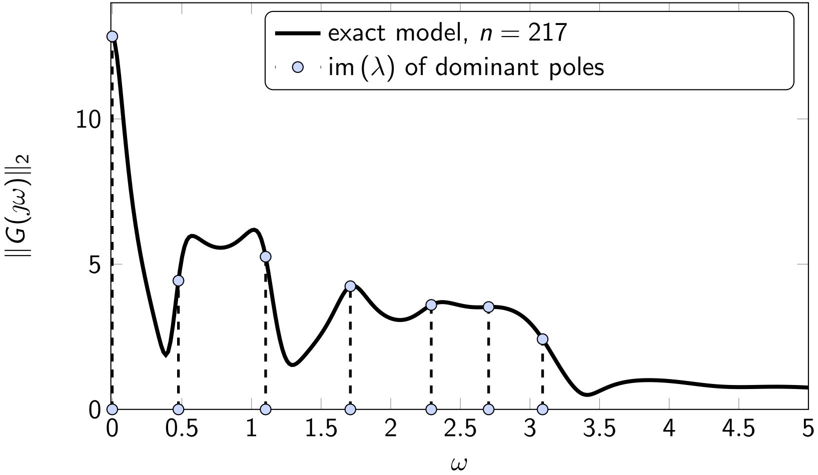

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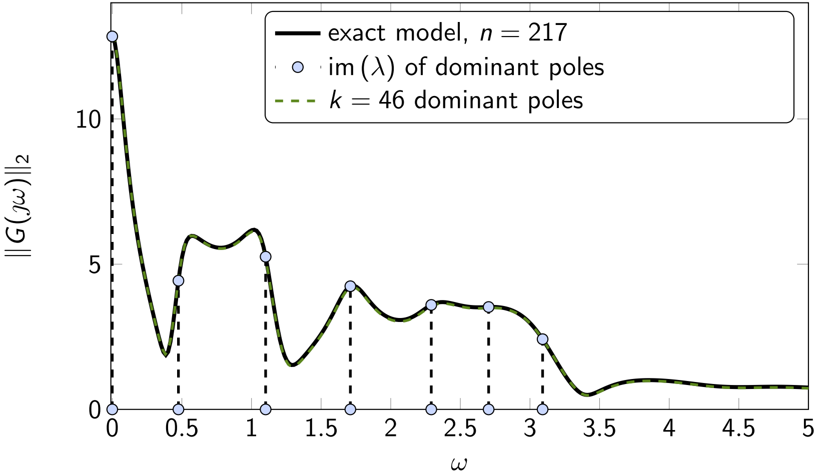

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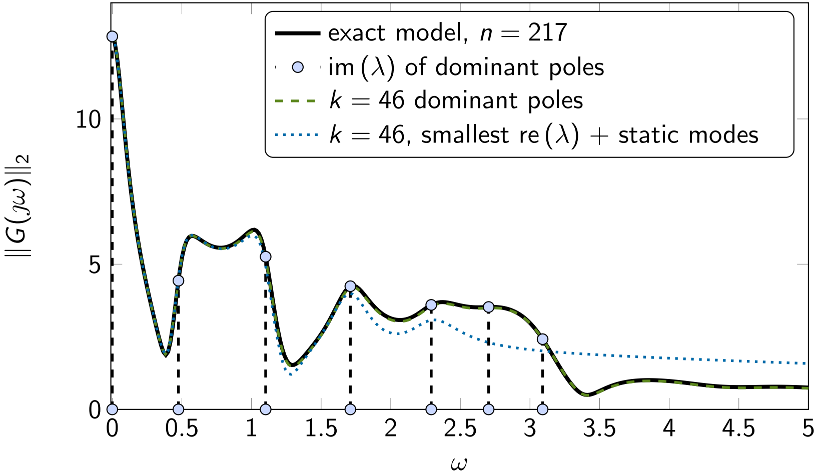
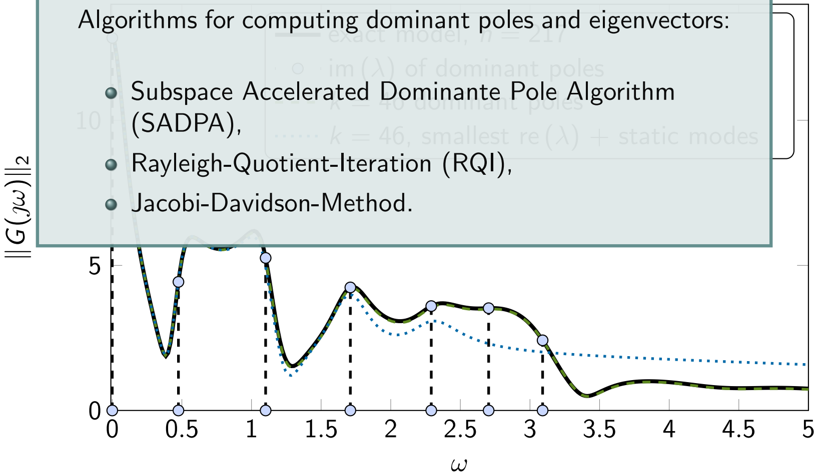


Dominant Poles

Random SISO Example ($B,\ C^T \in \mathbb{R}^n$)

Algorithms for computing dominant poles and eigenvectors:

- Subspace Accelerated Dominante Pole Algorithm (SADPA),
- Rayleigh-Quotient-Iteration (RQI),
- Jacobi-Davidson-Method.



Outline

Padé Approximation

Idea:

- Consider (even for possibly singular E if $\lambda E - A$ regular):

$$E\dot{x} = Ax + Bu, \quad y = Cx$$

with transfer function $G(s) = C(sE - A)^{-1}B$.

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$$G(s) \quad = \quad C\big((s_0E - A) + (s - s_0)E\big)^{-1}B$$

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$$\begin{aligned} G(s) &= C\big((s_0E - A) + (s - s_0)E\big)^{-1}B \\ &= C\Big(I + (s - s_0)\underbrace{(s_0E - A)^{-1}E}_{:=\tilde{A}}\Big)^{-1}\underbrace{(s_0E - A)^{-1}B}_{:=\tilde{B}} \end{aligned}$$

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Neumann Lemma. $\|F\| < 1 \;\Rightarrow\; I - F$ invertible, $(I - F)^{-1} = \sum_{k=0}^{\infty} F^k$.

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&= C\left(\sum_{k=0}^{\infty}(-1)^k (s - s_0)^k \tilde{A}^k\right)\tilde{B} \\
&= \sum_{k=0}^{\infty} \underbrace{(-1)^k C\tilde{A}^k\tilde{B}}_{=:\, m_k}(s - s_0)^k
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with $m_k = (-1)^k C\tilde{A}^k\tilde{B}$.

  - For $s_0 = 0$: $m_k := -C(A^{-1}E)^k A^{-1}B \rightsquigarrow$ moments.
    $(m_k = -CA^{-(k+1)}B$ for $E = I_n)$
  - For $s_0 = \infty$ and $E = I_n$: $m_0 = 0$, $m_k := CA^{k-1}B$ for $k \geq 1 \rightsquigarrow$ Markov parameters.

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- As reduced-order model use rth Padé approximant $\hat{G}$ to G:

$$G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^{2r}),$$

i.e., $m_k = \widehat{m}_k$ for $k = 0, \ldots, 2r - 1$

$\rightsquigarrow$ moment matching if $s_0 < \infty$,

$\rightsquigarrow$ partial realization if $s_0 = \infty$.

Padé Approximation

The Padé-Lanczos Connection [Gallivan/Grimme/Van Dooren 1994, Freund/Feldmann 1994]

Theorem [Grimme '97, Villemagne/Skelton '87]

Let $s_* \notin \Lambda(A, E)$ and

$$\begin{aligned} \tilde{A} &:= (s_* E - A)^{-1} E, \qquad \tilde{B} := (s_* E - A)^{-1} B, \\ \tilde{A}^* &:= (s_* E - A)^{-T} E^T, \qquad \tilde{C} := (s_* E - A)^{-T} C^T. \end{aligned}$$

If the reduced-order model is obtained by oblique projection onto $\mathcal{V} \subset \mathbb{R}^n$ along $\mathcal{W} \subset \mathbb{R}^n$, and

$$\begin{aligned} \operatorname{span}\left\{\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{K-1}\tilde{B}\right\} &\subset \mathcal{V}, \\ \operatorname{span}\left\{\tilde{C}, \tilde{A}^*\tilde{C}, \ldots, (\tilde{A}^*)^{K-1}\tilde{C}\right\} &\subset \mathcal{W}, \end{aligned}$$

then $G(s_*) = \hat{G}(s_*)$, $\frac{d^k}{ds^k} G(s_*) = \frac{d^k}{ds^k} \hat{G}(s_*)$ for $k = 1, \ldots, \ell - 1$, where

$$\ell \geq 2K$$

Padé Approximation

The Padé-Lanczos Connection [Gallivan/Grimme/Van Dooren 1994, Freund/Feldmann 1994]

Padé-via-Lanczos Method (PVL)

- Padé approximation/moment matching yield:

$$m_k = \frac{1}{k!} G^{(k)}(s_0) = \frac{1}{k!} \hat{G}^{(k)}(s_0) = \hat{m}_k, \quad k = 0, \ldots, 2K-1,$$

  i.e., Hermite interpolation in s_0.

- Recall interpolation via projection result $\Rightarrow$ moments need not be computed explicitly; moment matching is equivalent to projecting state-space onto

$$\mathcal{V} = \mathrm{span}(\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{K-1}\tilde{B}) =: \mathcal{K}_K(\tilde{A}, \tilde{B})$$

  (where $\tilde{A} = (s_0 E - A)^{-1} E$, $\tilde{B} = (s_0 E - A)^{-1} B$) along

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- Computation via unsymmetric Lanczos method.

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Remark: Arnoldi (PRIMA) yields only $G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^r)$.

Padé Approximation

The Padé-Lanczos Connection [Gallivan/Grimme/Van Dooren 1994, Freund/Feldmann 1994]

Padé-via-Lanczos Method (PVL)

Difficulties:

- Computable error estimates/bounds for $\|y - \hat{y}\|_2$ often very pessimistic or expensive to evaluate.
- Mostly heuristic criteria for choice of expansion points.
  Optimal choice for second-order systems with proportional/Rayleigh damping (BEATTIE/GUGERCIN '05).
- Good approximation quality only locally.
- Preservation of physical properties only in special cases (e.g. PRIMA/Arnoldi: $V^T AV$ is stable if A is negative definite or dissipative $\rightsquigarrow$ exercises); usually requires post processing which (partially) destroys moment matching properties.

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Interpolatory Model Reduction

A Change of Perspective: Rational Interpolation

Theorem (simplified) [Grimme '97, Villemagne/Skelton '87]

If

$$\begin{aligned} \operatorname{span}\left\{(s_1 I_n - A)^{-1}B, \ldots, (s_k I_n - A)^{-1}B\right\} &\subset \operatorname{Ran}(V), \\ \operatorname{span}\left\{(s_1 I_n - A)^{-T}C^T, \ldots, (s_k I_n - A)^{-T}C^T\right\} &\subset \operatorname{Ran}(W), \end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds}G(s_j) = \frac{d}{ds}\hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

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Remark:

computation of V, W from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme '97],
- Iterative Rational Krylov-Algo. [Antoulas/Beattie/Gugercin '07].

$\mathcal{H}_2$-Optimal Model Reduction

Best $\mathcal{H}_2$-norm approximation problem

$$\text{Find} \quad \arg\min_{\hat{G}\in\mathcal{H}_2 \text{ of order } \leq r} \|G - \hat{G}\|_2.$$

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$\rightsquigarrow$ First-order necessary $\mathcal{H}_2$-optimality conditions:

For SISO systems

$$\begin{aligned} G(-\mu_i) &= \hat{G}(-\mu_i), \\ G'(-\mu_i) &= \hat{G}'(-\mu_i), \end{aligned}$$

where μ_i are the poles of the reduced transfer function $\hat{G}$.

$\mathcal{H}_2$-Optimal Model Reduction

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$\rightsquigarrow$ First-order necessary $\mathcal{H}_2$-optimality conditions:

For MIMO systems

$$\begin{aligned}
G(-\mu_i)\tilde{B}_i &= \hat{G}(-\mu_i)\tilde{B}_i, && \text{for } i = 1, \dots, r,\\
\tilde{C}_i^T G(-\mu_i) &= \tilde{C}_i^T \hat{G}(-\mu_i), && \text{for } i = 1, \dots, r,\\
\tilde{C}_i^T G'(-\mu_i)\tilde{B}_i &= \tilde{C}_i^T \hat{G}'(-\mu_i)\tilde{B}_i, && \text{for } i = 1, \dots, r,
\end{aligned}$$

where $T^{-1}\hat{A}T = \operatorname{diag}\{\mu_1, \dots, \mu_r\}$ = spectral decomposition and

$$\tilde{B} = \hat{B}^T T^{-T}, \quad \tilde{C} = \hat{C}T.$$

$\rightsquigarrow$ tangential interpolation conditions.

Interpolatory Model Reduction

Interpolation of the Transfer Function by Projection

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{G}(s) = CV\left(sI - W^T AV\right)^{-1} W^T B,$$

where V and W are given as the rational Krylov subspaces

$$V = \left[(-\mu_1 I - A)^{-1}B, \ldots, (-\mu_r I - A)^{-1}B\right],$$
$$W = \left[(-\mu_1 I - A^T)^{-1}C^T, \ldots, (-\mu_r I - A^T)^{-1}C^T\right].$$

Then

$$G(-\mu_i) = \hat{G}(-\mu_i) \quad \text{and} \quad G'(-\mu_i) = \hat{G}'(-\mu_i),$$

for $i = 1, \ldots, r$ as desired.

$\rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. '06], [Bunse-Gerstner et al. '07], [Van Dooren et al. '08]

Interpolatory Model Reduction

Interpolation of the Transfer Function by Projection

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{G}(s) = CV\left(sI - W^T AV\right)^{-1} W^T B,$$

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$\mathcal{H}_2$-Optimal Model Reduction

The Basic IRKA Algorithm

Algorithm 1 IRKA (MIMO version/MIRIAm)

Input: A stable, B, C, $\hat{A}$ stable, $\hat{B}$, $\hat{C}$, $\delta > 0$.
Output: A^{opt}, B^{opt}, C^{opt}

while $\left(\max_{j=1,\ldots,r}\left\{\frac{|\mu_j - \mu_j^{\text{old}}|}{|\mu_j|}\right\} > \delta\right)$ **do**

  $\operatorname{diag}\{\mu_1,\ldots,\mu_r\} := T^{-1}\hat{A}T =$ spectral decomposition,
  $\tilde{B} = \hat{B}^H T^{-T}$, $\tilde{C} = \hat{C}T$.

  $V = \left[(-\mu_1 I - A)^{-1}B\tilde{B}_1,\ldots,(-\mu_r I - A)^{-1}B\tilde{B}_r\right]$

  $W = \left[(-\mu_1 I - A^T)^{-1}C^T\tilde{C}_1,\ldots,(-\mu_r I - A^T)^{-1}C^T\tilde{C}_r\right]$

  $V = \operatorname{orth}(V)$, $W = \operatorname{orth}(W)$, $W = W(V^H W)^{-1}$

  $\hat{A} = W^H A V$, $\hat{B} = W^H B$, $\hat{C} = CV$

end while

$A^{opt} = \hat{A}$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$

Outline

Motivation

Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory (in single precision): $4 \cdot n_x \cdot n_y$ bytes.

Theorem (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U\Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\|X - \widehat{X}\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r, \sigma_1 v_1, \ldots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

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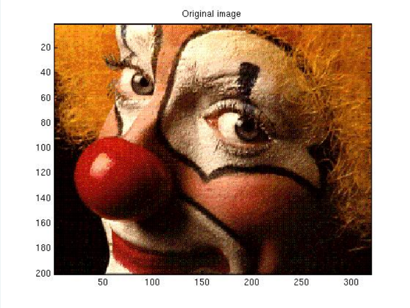
Example: Image Compression by Truncated SVD

Example: Clown



320×200 pixel

$\rightsquigarrow \quad \approx 256$ kB

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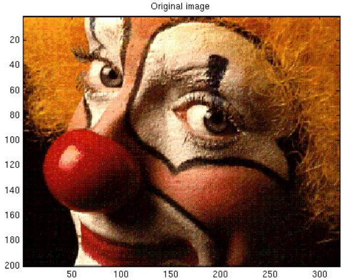

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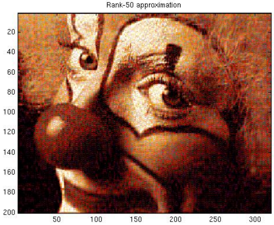
- rank $r = 50$, ≈ 104 kB



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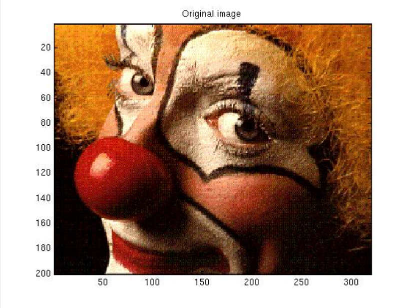

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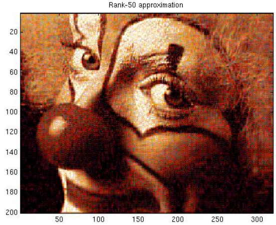
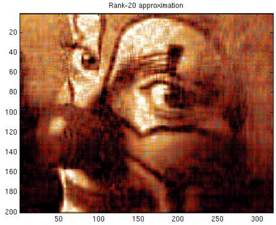


- rank $r = 20$, ≈ 42 kB



Dimension Reduction via SVD

Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:

James H. Wilkinson, Wallace Givens,

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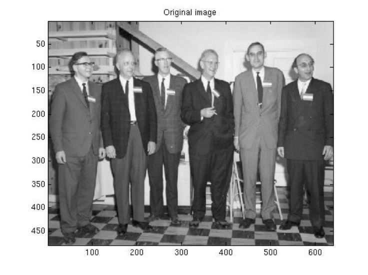

640×480 pixel, ≈ 1229 kB

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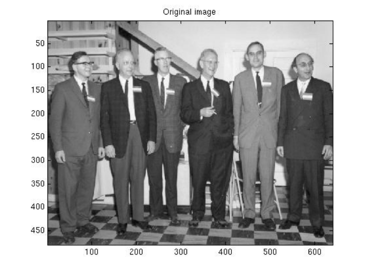
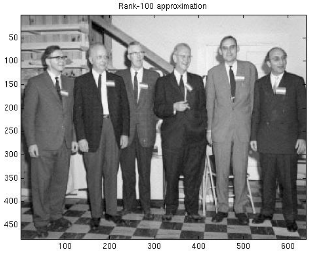


640×480 pixel, ≈ 1229 kB

- rank $r = 100$, ≈ 448 kB



- rank $r = 50$, ≈ 224 kB

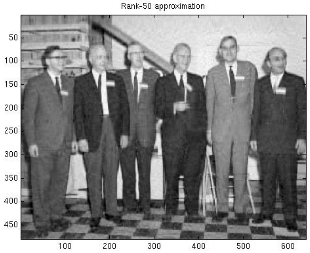
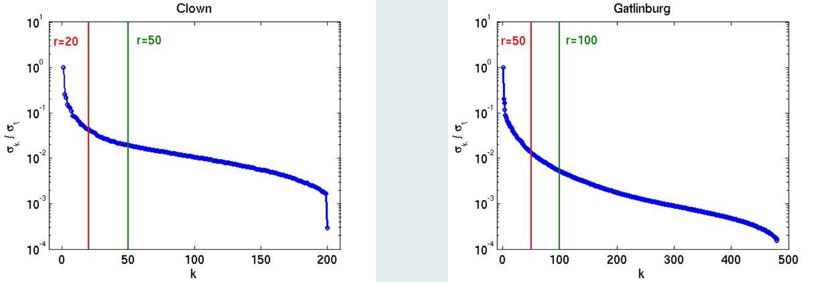


Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices



A different viewpoint

Linear Mapping

A matrix $A \in \mathbb{R}^{\ell \times k}$ represents a linear mapping

$$\mathcal{A} : \mathbb{R}^k \to \mathbb{R}^\ell \; : \; x \to y := Ax.$$

The truncated SVD ignores small Hankel singular values and thus the related left and right singular vectors.

Consequence:

- Vectors (almost) in the kernel of A do not contribute to $\mathrm{range}\,(A)$ and can hardly or not at all be reconstructed from the input-output relation ("A^{-1}") $\rightsquigarrow$ "unobservable" states.
- Vectors (almost) in $\mathrm{range}\,(A)^\perp$ cannot be "reached" from any $x \in \mathbb{R}^k$ $\rightsquigarrow$ "unreachable/uncontrollable" states.
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Balanced Truncation

The basic method

Basic principle:

- Recall: a stable system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

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- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

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Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S, \; Q = R^T R.$
2. Compute SVD $SR^T = [\, U_1, \, U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$
3. ROM is $(W^T AV, W^T B, CV, D)$, where

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$$\begin{aligned} V^T W &= (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}}) = \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}} \\ &= \Sigma_1^{-\frac{1}{2}} [\, I_r, \, 0 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \Sigma_1^{-\frac{1}{2}} \end{aligned}$$

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$\Longrightarrow$ VW^T is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

Balanced Truncation

The basic method

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

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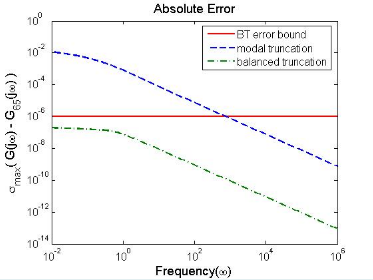
Balanced Truncation

Numerical examples for BT: Optimal Cooling of Steel Profiles

$n = 1,357$, Absolute Error



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- MT w/o static condensation, same order as BT model.

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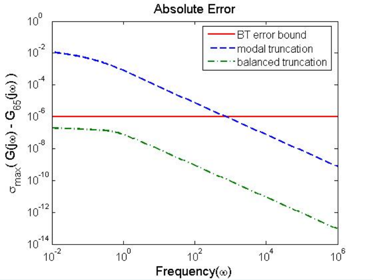

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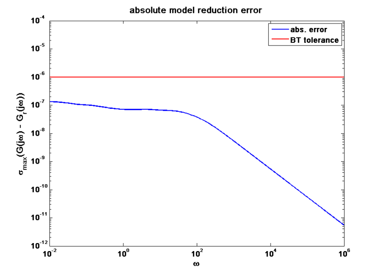
- MT w/o static condensation, same order as BT model.

$n = 79,841$, Absolute Error



- BT model computed using M.E.S.S. in MATLAB,
- Computation time: $\approx$1 min.

Balanced Truncation

Numerical examples for BT: Microgyroscope (Butterfly Gyro)

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722$, $m = 1$, $q = 12$.
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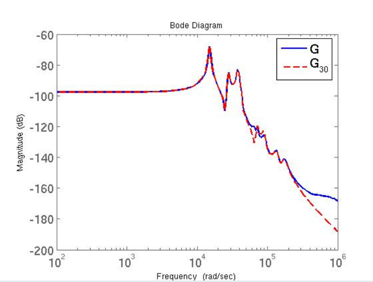

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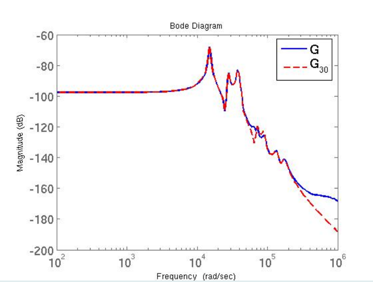
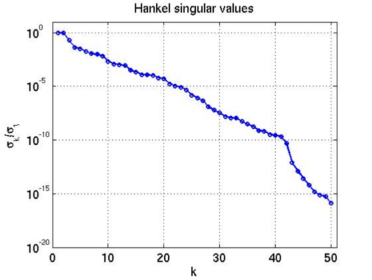


Hankel Singular Values



Balanced Truncation
Software

Lyapack [Penzl 2000]

MATLAB toolbox for solving

- Lyapunov equations and algebraic Riccati equations,
- model reduction and LQR problems.

Main work horse: Low-rank ADI and Newton-ADI iterations.

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M.E.S.S. – Matrix Equations Sparse Solvers [B./Köhler/Saak '08–]

- Extended and revised version of LYAPACK.
- Includes solvers for large-scale differential Riccati equations (based on Rosenbrock and BDF methods).
- Many algorithmic improvements:
  - new ADI parameter selection,
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  - treatment of generalized systems without factorization of the mass matrix,
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Outline

Parametric Model Order Redution

Affine Representation

Parametric Systems

$$\Sigma(p) : \left\{ \begin{array}{rcl} E(p)\dot{x}(t;p) & = & A(p)x(t;p) + B(p)u(t)), \\ y(t;p) & = & C(p)x(t;p). \end{array} \right.$$

Assume

$$\begin{array}{rcl} E(p) & = & E_0 + e_1(p)E_1 + \ldots + e_{q_E}(p)E_{q_E}, \\ A(p) & = & A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \\ B(p) & = & B_0 + b_1(p)B_1 + \ldots + b_{q_B}(p)B_{q_B}, \\ C(p) & = & C_0 + c_1(p)C_1 + \ldots + c_{q_C}(p)C_{q_C}. \end{array}$$

Parametric Model Order Redution

Structure-Preservation

Petrov-Galerkin-type projection

For given projection matrices $V, W \in \mathbb{R}^{n\times r}$ with $W^T V = I_r$
($\rightsquigarrow (VW^T)^2 = VW^T$ is projector), compute

$$
\begin{aligned}
\hat{E}(p) &= W^T E_0 V + e_1(p) W^T E_1 V + \ldots + e_{q_E}(p) W^T E_{q_E} V, \\
&= \hat{E}_0 + e_1(p)\hat{E}_1 + \ldots + e_{q_E}(p)\hat{E}_{q_E}, \\
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PMOR based on Multi-Moment Matching

Idea: choose appropriate frequency parameter $\hat{s}$ and parameter vector $\hat{p}$, expand into multivariate power series about $(\hat{s}, \hat{p})$ and compute reduced-order model, so that

$$G(s,p) = \hat{G}(s,p) + \mathcal{O}\left(|s-\hat{s}|^K + \|p-\hat{p}\|^L + |s-\hat{s}|^k\|p-\hat{p}\|^\ell\right),$$

i.e., first $K, L, k+\ell$ (mostly: $K = L = k+\ell$) coefficients (multi-moments) of Taylor/Laurent series coincide.

Algorithms:

- [Daniel et al. '04]: explicit computation of moments, numerically unstable.
- [Farle et al. '06/'07]: Krylov subspace approach, only polynomial parameter-dependance, numerical properties not clear, but appears to be robust.
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PMOR based on Multi-Moment Matching

Numerical Examples

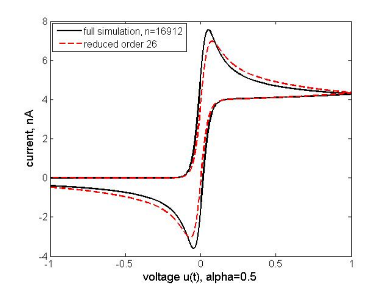
Electro-chemical SEM:

compute cyclic voltammogram based on FEM model

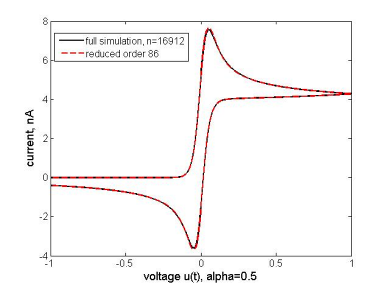
$$E\dot{x}(t) = (A_0 + p_1 A_1 + p_2 A_2)x(t) + Bu(t), \quad y(t) = c^T x(t),$$

where $n = 16,912$, $m = 3$, A_1, A_2 diagonal.

$K = L = k + \ell = 4 \;\Rightarrow\; r = 26$



$K = L = k + \ell = 9 \;\Rightarrow\; r = 86$



PMOR based on Rational Interpolation

Theory: Interpolation of the Transfer Function

Theorem [Baur/Beattie/B./Gugercin '07/'11]

Let
$$\begin{aligned}\hat{G}(s,p) &:= \hat{C}(p)(s\hat{E}(p)-\hat{A}(p))^{-1}\hat{B}(p)\\ &= C(p)V(sW^TE(p)V-W^TA(p)V)^{-1}W^TB(p)\end{aligned}$$

and suppose $\hat{p} = [\hat{p}_1, \ldots, \hat{p}_d]^T$ and $\hat{s} \in \mathbb{C}$ are chosen such that both $\hat{s}\,E(\hat{p}) - A(\hat{p})$ and $\hat{s}\,\hat{E}(\hat{p}) - \hat{A}(\hat{p})$ are invertible.

If
$$(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\,B(\hat{p}) \in \mathrm{Ran}(V)$$

or
$$\left(C(\hat{p})\,(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\right)^T \in \mathrm{Ran}(W),$$

then $G(\hat{s},\hat{p}) = \hat{G}(\hat{s},\hat{p})$.

Note: result extends to MIMO case using tangential interpolation:
Let $0 \neq b \in \mathbb{R}^m$, $0 \neq c \in \mathbb{R}^q$ be arbitrary.

a) If $(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\,B(\hat{p})b \in \mathrm{Ran}(V)$, then $G(\hat{s},\hat{p})b = \hat{G}(\hat{s},\hat{p})b$;

b) If $\left(c^T C(\hat{p})\,(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\right)^T \in \mathrm{Ran}(W)$, then $c^T G(\hat{s},\hat{p}) = c^T\hat{G}(\hat{s},\hat{p})$.

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PMOR based on Rational Interpolation

Theory: Interpolation of the Parameter Gradient

Theorem [Baur/Beattie/B./Gugercin '07/'11]

Suppose that $E(p)$, $A(p)$, $B(p)$, $C(p)$ are C^1 in a neighborhood of $\hat{p} = [\hat{p}_1, \ldots, \hat{p}_d]^T$ and that both $\hat{s}\, E(\hat{p}) - A(\hat{p})$ and $\hat{s}\, \hat{E}(\hat{p}) - \hat{A}(\hat{p})$ are invertible. If

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then

$$\nabla_p G(\hat{s}, \hat{p}) = \nabla_p G_r(\hat{s}, \hat{p}), \qquad \frac{\partial}{\partial s} G(\hat{s}, \hat{p}) = \frac{\partial}{\partial s} \hat{G}(\hat{s}, \hat{p}).$$

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1. Assertion of theorem satisfies necessary conditions for surrogate models in trust region methods [Alexandrov/Dennis/Lewis/Torczon '98].
2. Approximation of gradient allows use of reduced-order model for sensitivity analysis.

PMOR based on Rational Interpolation
Algorithm

Generic implementation of interpolatory PMOR

Define $\mathcal{A}(s,p) := sE(p) - A(p)$.

1. Select "frequencies" $s_1, \ldots, s_k \in \mathbb{C}$ and parameter vectors $p^{(1)}, \ldots, p^{(\ell)} \in \mathbb{R}^d$.
2. Compute (orthonormal) basis of
$$\mathcal{V} = \operatorname{span}\left\{\mathcal{A}(s_1, p^{(1)})^{-1} B(p^{(1)}), \ldots, \mathcal{A}(s_k, p^{(\ell)})^{-1} B(p^{(\ell)})\right\}.$$
3. Compute (orthonormal) basis of
$$\mathcal{W} = \operatorname{span}\left\{\mathcal{A}(s_1, p^{(1)})^{-H} C(p^{(1)})^T, \ldots, \mathcal{A}(s_k, p^{(\ell)})^{-T} C(p^{(\ell)})^T\right\}.$$
4. Set $V := [v_1, \ldots, v_{k\ell}]$, $\tilde{W} := [w_1, \ldots, w_{k\ell}]$, and $W := \tilde{W}(\tilde{W}^T V)^{-1}$. (Note: $r = k\ell$).
5. Compute $\begin{cases} \hat{A}(p) := W^T A(p) V, & \hat{B}(p) := W^T B(p) V, \\ \hat{C}(p) := W^T C(p) V, & \hat{E}(p) := W^T E(p) V. \end{cases}$

PMOR based on Rational Interpolation

Remarks

- If directional derivatives w.r.t. p are included in $\mathrm{Ran}(V)$, $\mathrm{Ran}(W)$, then also the Hessian of $G(\hat{s}, \hat{p})$ is interpolated by the Hessian of $\hat{G}(\hat{s}, \hat{p})$.
- Choice of optimal interpolation frequencies s_k and parameter vectors $p^{(k)}$ in general is an open problem.
- For prescribed parameter vectors $p^{(k)}$, we can use corresponding $\mathcal{H}_2$-optimal frequencies $s_{k,\ell}$, $\ell = 1, \ldots, r_k$ computed by IRKA, i.e., reduced-order systems $\hat{G}_*^{(k)}$ so that

$$\|G(., p^{(k)}) - \hat{G}_*^{(k)}(.)\|_{\mathcal{H}_2} = \min_{\substack{\mathrm{order}(\hat{G})=r_k \\ \hat{G} \ \mathrm{stable}}} \|G(., p^{(k)}) - \hat{G}^{(k)}(.)\|_{\mathcal{H}_2},$$

  where

$$\|G\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \|G(\jmath\omega)\|_{\mathrm{F}}^2 d\omega\right)^{1/2}.$$

- Optimal choice of interpolation frequencies s_k and parameter vectors $p^{(k)}$ possible for special parameterized SISO systems.

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PMOR based on Rational Interpolation

Numerical Example: 2D Convection-Diffusion Equation

- FD discretization ($n = 400$, $m = q = 1$) yields

$$\dot{x}(t) = (p_0 A_0 + p_1 A_1 + p_2 A_2)\, x(t) + B\, u(t),$$

where $p_0 =$ diffusion coefficient; p_i, $i = 1, 2$, convection in x_i direction, $p \in [0, 1]^3$.

- Parameter vectors for interpolation:

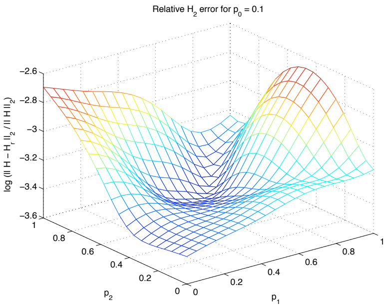
$$\begin{array}{lll} p^{(1)} = (0.8, 0.5, 0.5), & p^{(2)} = (0.8, 0, 0.5), & p^{(3)} = (0.8, 1, 0.5), \\ p^{(4)} = (0.1, 0.5, 0.5), & p^{(5)} = (0.1, 0, 1), & p^{(6)} = (0.1, 1, 1). \end{array}$$

- Compare implementations:
  - generic rational PMOR ($\equiv$ fix interpolation frequencies),
  - IRKA-based rational PMOR ($\equiv$ optimize interpolation frequencies).
- Reduced-order model: $r_1 = r_2 = r_3 = 3$, $r_4 = r_5 = r_6 = 4 \Rightarrow r = 21.$

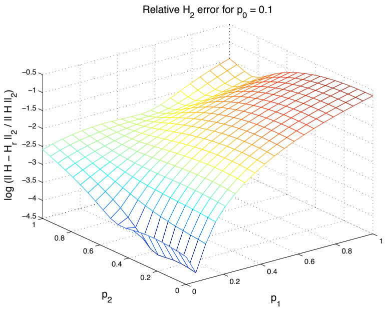
PMOR based on Rational Interpolation

Numerical Example: 2D Convection-Diffusion Equation

Relative $\mathcal{H}_2$ Error for $p_0 = 0.1$



IRKA, 5 steps

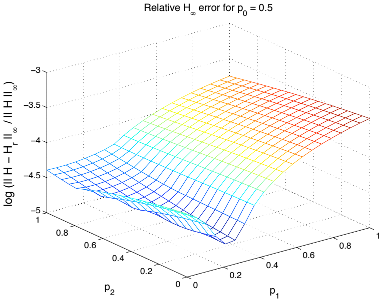


generic

PMOR based on Rational Interpolation

Numerical Example: 2D Convection-Diffusion Equation

Relative $\mathcal{H}_\infty$ Error for $p_0 = 0.1$



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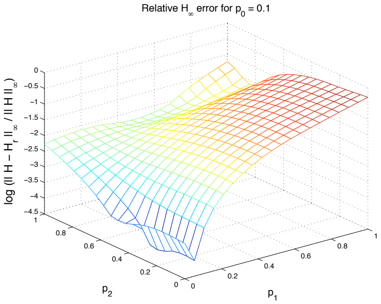


generic

PMOR based on Rational Interpolation

Numerical Example: Thermal Conduction in a Semiconductor Chip

- Important requirement for a compact model of thermal conduction is boundary condition independence.
- The thermal problem is modeled by the heat equation, where heat exchange through device interfaces is modeled by convection boundary conditions containing film coefficients $\{p_i\}_{i=1}^{3}$, to describe the heat exchange at the ith interface.
- Spatial semi-discretization leads to

$$E\dot{x}(t) = (A_0 + \sum_{i=1}^{3} p_i A_i)x(t) + bu(t), \quad y(t) = c^T x(t),$$

where $n = 4,257$, A_i, $i = 1, 2, 3$, are diagonal.

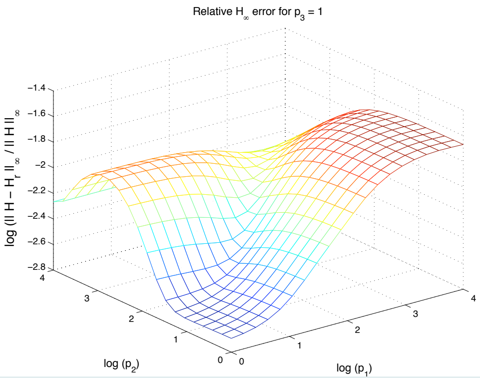
Source: C.J.M Lasance, *Two benchmarks to facilitate the study of compact thermal modeling phenomena*, IEEE. Trans. Components and Packaging Technologies, Vol. 24, No. 4, pp. 559–565, 2001.

PMOR based on Rational Interpolation

Numerical Example: Thermal Conduction in a Semiconductor Chip

Choose 2 interpolation points for parameters ("important" configurations), 8/7 interpolation frequencies are picked H_2 optimal by IRKA. $\Longrightarrow$ $k = 2, \ell = 8, 7$, hence $r = 15$.

$p_3 = 1,\ p_1, p_2 \in [1, 10^4]$.



Outline

Nonlinear Model Reduction

A Brief Introduction

Given a large-scale control-affine nonlinear control system of the form

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t)) + bu(t), \\ y(t) = c^T x(t), \quad x(0) = x_0, \end{cases}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ nonlinear and $b, c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$.

MOR

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \hat{f}(\hat{x}(t)) + \hat{b}u(t), \\ \hat{y}(t) = \hat{c}^T \hat{x}(t), \quad \hat{x}(0) = \hat{x}_0, \end{cases}$$

with $\hat{f} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}}$ and $\hat{b}, \hat{c} \in \mathbb{R}^{\hat{n}}$, $x \in \mathbb{R}^{\hat{n}}$, $u \in \mathbb{R}$ and $\hat{y} \approx y \in \mathbb{R}$, $\hat{n} \ll n$.

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Nonlinear Model Reduction

Common Reduction Techniques

Proper Orthogonal Decomposition (POD)

- Take computed or experimental 'snapshots' of full model: $[x(t_1), x(t_2), \ldots, x(t_N)] =: X$,
- perform SVD of snapshot matrix: $X = VSW^T \approx V_{\hat{n}} S_{\hat{n}} W_{\hat{n}}^T$.
- Reduction by POD-Galerkin projection: $\dot{\hat{x}} = V_{\hat{n}}^T f(V_{\hat{n}} \hat{x}) + V_{\hat{n}}^T Bu$.
- Requires evaluation of f
  $\rightsquigarrow$ discrete empirical interpolation [Sorensen/Chaturantabut '09].
- Input dependency due to 'snapshots'!

Trajectory Piecewise Linear (TPWL)

- Linearize f along trajectory,
- reduce resulting linear systems,
- construct reduced model by weighted sum of linear systems.
- Requires simulation of original model and several linear reduction steps, many heuristics.

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Nonlinear Model Reduction by Generalized Moment-Matching

Quadratic-Bilinear Differential Algebraic Equations (QBDAEs)

Consider the class of quadratic-bilinear differential algebraic equations

$$\Sigma : \quad \begin{cases} E\dot{x}(t) = A_1 x(t) + A_2 x(t) \otimes x(t) + N x(t) u(t) + B u(t), \\ \quad y(t) = C x(t), \quad x(0) = x_0, \end{cases}$$

where $E, A_1, N \in \mathbb{R}^{n \times n}$, $A_2 \in \mathbb{R}^{n \times n^2}$ (Hessian tensor), $B, C^T \in \mathbb{R}^n$ are quite helpful.

- A large class of smooth nonlinear control-affine systems can be transformed into the above type of control system.
- The transformation is exact, but a slight increase of the state dimension has to be accepted.
- Input-output behavior can be characterized by generalized transfer functions $\rightsquigarrow$ enables us to use Krylov-/rational interpolation-based reduction techniques.

Nonlinear Model Reduction by Generalized Moment-Matching

Transformation to QBDAE form via McCormick relaxation

Theorem [Gu '09]

Assume that the state equation of a nonlinear system Σ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking derivatives and adding algebraic equations, respectively, Σ can be transformed into a system of QBDAEs.

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Example

- $\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \quad \dot{x}_2 = -x_2 + u.$
- $z_1 := \exp(-x_2),$
- $\dot{x}_1 = z_1 \cdot z_2,$

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  $\dot{z}_2 = \frac{2 \cdot x_1 \cdot z_1 \cdot z_2}{2 \cdot z_2} = x_1 \cdot z_1.$

Nonlinear Model Reduction by Generalized Moment-Matching
Variational Analysis and Linear Subsystems

Analysis of nonlinear systems by variational equation approach:

- consider input of the form $\alpha u(t)$,
- nonlinear system is assumed to be a series of homogeneous nonlinear subsystems, i.e. response should be of the form

$$x(t) = \alpha x_1(t) + \alpha^2 x_2(t) + \alpha^3 x_3(t) + \ldots.$$

- Comparison of terms $\alpha^i, i = 1, 2, \ldots$ leads to series of systems

$$\begin{aligned}
E\dot{x}_1 &= A_1 x_1 + Bu,\\
E\dot{x}_2 &= A_1 x_2 + A_2 x_1 \otimes x_1 + N x_1 u,\\
E\dot{x}_3 &= A_1 x_3 + A_2 \left(x_1 \otimes x_2 + x_2 \otimes x_1\right) + N x_2 u\\
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\end{aligned}$$

- although i-th subsystem is coupled nonlinearly to preceding systems, linear systems are obtained if terms x_j, $j < i$, are interpreted as pseudo-inputs.

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Nonlinear Model Reduction by Generalized Moment-Matching

Generalized Transfer Functions

In a similar way, a series of generalized symmetric transfer functions can be obtained via formal multivariate Laplace transforms:

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$$H_1(s_1) = C \underbrace{(s_1 E - A_1)^{-1} B}_{G_1(s_1)},$$

Nonlinear Model Reduction by Generalized Moment-Matching

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$$\begin{aligned} H_2(s_1, s_2) = \frac{1}{2!} C \left((s_1 + s_2)E - A_1\right)^{-1} \big[& N\left(G_1(s_1) + G_1(s_2)\right) \\ & + A_2\left(G_1(s_1) \otimes G_1(s_2) + G_1(s_2) \otimes G_1(s_1)\right)\big], \end{aligned}$$

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&\qquad \left. + A_2 \left(G_1(s_1) \otimes G_1(s_2) + G_1(s_2) \otimes G_1(s_1)\right)\right], \\
H_3(s_1, s_2, s_3) &= \frac{1}{3!} C \left((s_1 + s_2 + s_3)E - A_1\right)^{-1} \\
&\qquad \Big[N(G_2(s_1, s_2) + G_2(s_2, s_3) + G_2(s_1, s_3)) \\
&\qquad + A_2 \big(G_1(s_1) \otimes G_2(s_2, s_3) + G_1(s_2) \otimes G_2(s_1, s_3) \\
&\qquad\quad + G_1(s_3) \otimes G_2(s_1, s_3) + G_2(s_2, s_3) \otimes G_1(s_1) \\
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\end{aligned}
$$

Nonlinear Model Reduction by Generalized Moment-Matching
Characterization via Multimoments

For simplicity, focus on the first two transfer functions. For $H_1(s_1)$, choosing σ and making use of the Neumann lemma leads to

$$H_1(s_1) = \sum_{i=0}^{\infty} C \underbrace{\left((A_1 - \sigma E)^{-1} E\right)^i (A_1 - \sigma E)^{-1} B \,(s_1 - \sigma)^i}_{m^i_{s_1,\sigma}}.$$

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Similarly, specifying an expansion point (τ, ξ) yields

$$H_2(s_1, s_2) = \frac{1}{2} \sum_{i=0}^{\infty} C \left((A_1 - (\tau + \xi)E)^{-1} E\right)^i (A_1 - (\tau + \xi)E)^{-1} \, (s_1 + s_2 - \tau - \xi)^i \cdot$$

$$\left[A_2 \left(\sum_{j=0}^{\infty} m^j_{s_1,\tau} \otimes \sum_{k=0}^{\infty} m^k_{s_2,\xi} + \sum_{k=0}^{\infty} m^k_{s_2,\xi} \otimes \sum_{j=0}^{\infty} m^j_{s_1,\tau}\right) + N \left(\sum_{p=0}^{\infty} m^p_{s_1,\tau} + \sum_{p=0}^{\infty} m^q_{s_2,\xi}\right)\right]$$

Nonlinear Model Reduction by Generalized Moment-Matching
Constructing the Projection Matrix

Goal: $\frac{\partial}{\partial s_1^{q-1}} H_1(\sigma) = \frac{\partial}{\partial s_1^{q-1}} \hat{H}_1(\sigma)$, $\quad \frac{\partial}{\partial s_1^l s_2^m} H_2(\sigma,\sigma) = \frac{\partial}{\partial s_1^l s_2^m} \hat{H}_2(\sigma,\sigma)$, $l + m \leq q - 1$.

Construct the following sequence of nested Krylov subspaces

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for $i = 1 : q$

$$V_2^i = \mathcal{K}_{q-i+1} \left((A_1 - 2\sigma E)^{-1} E, (A_1 - 2\sigma E)^{-1} N V_1(:, i) \right),$$

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V_1 = &\mathcal{K}_q\left((A_1-\sigma E)^{-1}E, (A_1-\sigma E)^{-1}b\right)\\
&\textbf{for } i = 1:q\\
&\quad V_2^i = \mathcal{K}_{q-i+1}\left((A_1-2\sigma E)^{-1}E, (A_1-2\sigma E)^{-1}NV_1(:,i)\right),\\
&\quad\quad \textbf{for } j = 1:\min(q-i+1,i)\\
&\quad\quad\quad V_3^{i,j} = \mathcal{K}_{q-i-j+2}\left((A_1-2\sigma E)^{-1}E, (A_1-2\sigma E)^{-1}A_2 V_1(:,i)\otimes V_1(:,j)\right),
\end{aligned}
$$

$V_1(:,i)$ denoting the i-*th* column of V_1.

Nonlinear Model Reduction by Generalized Moment-Matching

Constructing the Projection Matrix

Goal: $\frac{\partial}{\partial s_1^{q-1}} H_1(\sigma) = \frac{\partial}{\partial s_1^{q-1}} \hat{H}_1(\sigma)$, $\frac{\partial}{\partial s_1^l s_2^m} H_2(\sigma,\sigma) = \frac{\partial}{\partial s_1^l s_2^m} \hat{H}_2(\sigma,\sigma)$, $l+m \le q-1$.
Construct the following sequence of nested Krylov subspaces

$$
\begin{aligned}
V_1 =& \mathcal{K}_q\left((A_1-\sigma E)^{-1}E, (A_1-\sigma E)^{-1}b\right) \\
&\textbf{for } i = 1:q \\
&\quad V_2^i = \mathcal{K}_{q-i+1}\left((A_1-2\sigma E)^{-1}E, (A_1-2\sigma E)^{-1}NV_1(:,i)\right), \\
&\quad\quad \textbf{for } j = 1:\min(q-i+1, i) \\
&\quad\quad\quad V_3^{i,j} = \mathcal{K}_{q-i-j+2}\left((A_1-2\sigma E)^{-1}E, (A_1-2\sigma E)^{-1}A_2 V_1(:,i)\otimes V_1(:,j)\right),
\end{aligned}
$$

$V_1(:,i)$ denoting the i-*th* column of V_1. Set $\mathcal{V} = \text{orth}\,[V_1, V_2^i, V_3^{i,j}]$ and construct $\hat{\Sigma}$ by the Galerkin-Projection $\mathcal{P} = \mathcal{V}\mathcal{V}^T$:

$$
\hat{A}_1 = \mathcal{V}^T A_1 \mathcal{V} \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{A}_2 = \mathcal{V}^T A_2 (\mathcal{V}\otimes\mathcal{V}) \in \mathbb{R}^{\hat{n}\times\hat{n}^2},
$$

$$
\hat{N} = \mathcal{V}^T N \mathcal{V} \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{b} = \mathcal{V}^T b \in \mathbb{R}^{\hat{n}}, \quad \hat{c}^T = c^T \mathcal{V} \in \mathbb{R}^{\hat{n}}.
$$

Nonlinear Model Reduction by Generalized Moment-Matching
Two-Sided Projection Methods

- Similarly to the linear case, one can exploit duality concepts, in order to construct two-sided (Petrov-Galerkin) projection methods.
- Construction the dual Krylov subspaces efficiently requires a bit of tensor calculus.

Nonlinear Model Reduction by Generalized Moment-Matching

Two-Sided Projection Methods

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Nonlinear Model Reduction by Generalized Moment-Matching

Two-Sided Projection Methods

Theorem [B./Breiten 2012]

- $\Sigma = (E, A_1, A_2, N, b, c)$ original QBDAE system.
- Reduced system by Petrov-Galerkin projection $\mathcal{P} = \mathcal{V}\mathcal{W}^T$ with

$$V_1 = \mathcal{K}_{q_1}(E, A_1, b, \sigma), \quad W_1 = \mathcal{K}_{q_1}\left(E^T, A_1^T, c, 2\sigma\right)$$

$$\begin{aligned}
&\textbf{for } i = 1 : q_2 \\
&\quad V_2 = \mathcal{K}_{q_2-i+1}(E, A_1, NV_1(:,i), 2\sigma) \\
&\quad W_2 = \mathcal{K}_{q_2-i+1}\left(E^T, A_1^T, N^T W_1(:,i), \sigma\right) \\
&\quad\quad \textbf{for } \; j = 1 : \min(q_2 - i + 1, i) \\
&\quad\quad\quad V_3 = \mathcal{K}_{q_2-i-j+2}(E, A_1, A_2 V_1(:,i) \otimes V_1(:,j), 2\sigma) \\
&\quad\quad\quad W_3 = \mathcal{K}_{q_2-i-j+2}\left(E^T, A_1^T, \mathcal{A}^{(2)} V_1(:,i) \otimes W_1(:,j), \sigma\right).
\end{aligned}$$

Then, it holds:

$$\frac{\partial^i H_1}{\partial s_1^i}(\sigma) = \frac{\partial^i \hat{H}_1}{\partial s_1^i}(\sigma), \quad \frac{\partial^i H_1}{\partial s_1^i}(2\sigma) = \frac{\partial^i \hat{H}_1}{\partial s_1^i}(2\sigma), \quad i = 0, \ldots, q_1 - 1,$$

$$\frac{\partial^{i+j}}{\partial s_1^i s_2^j} H_2(\sigma, \sigma) = \frac{\partial^{i+j}}{\partial s_1^i s_2^j} \hat{H}_2(\sigma, \sigma), \quad i + j \leq 2q_2 - 1.$$

Numerical Examples
Two-Dimensional Burgers Equation

- 2D-Burgers equation on $\underbrace{(0,1)\times(0,1)}_{:=\Omega}\times[0,T]$

$$u_t = -(u\cdot\nabla)\,u + \nu\Delta u$$

with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.

Numerical Examples

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with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.

- Consider initial and boundary conditions

$$\begin{aligned}
u_x(x,y,0) &= \frac{\sqrt{2}}{2}, & u_y(x,y,0) &= \frac{\sqrt{2}}{2}, && \text{for } (x,y)\in\Omega_1 := (0,0.5],\\
u_x(x,y,0) &= 0, & u_y(x,y,0) &= 0, && \text{for } (x,y)\in\Omega\backslash\Omega_1,\\
u_x &= 0, & u_y &= 0, && \text{for } (x,y)\in\partial\Omega.
\end{aligned}$$

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- Spatial discretization $\rightsquigarrow$ QBDAE system with nonzero I.C. and $N=0$ $\rightsquigarrow$ reformulate as system with zero I.C. and constant input.

Numerical Examples

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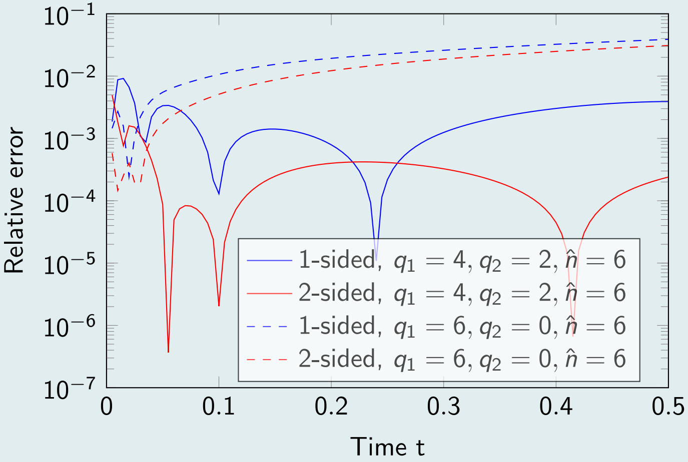
- Spatial discretization $\rightsquigarrow$ QBDAE system with nonzero I.C. and $N=0 \rightsquigarrow$ reformulate as system with zero I.C. and constant input.
- Output C chosen to be average x-velocity.

Numerical Examples

Two-Dimensional Burgers Equation

Comparison of relative time-domain error for $n = 1600$



Numerical Examples
Two-Dimensional Burgers Equation

- 2D-Burgers equation on $\underbrace{(0,1)\times(0,1)}_{:=\Omega}\times[0,T]$

$$u_t = -(u\cdot\nabla)\,u + \nu\Delta u$$

with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.

- Now consider initial and boundary conditions

$$\begin{aligned}
&u_x(x,y,0)=0, && u_y(x,y,0)=0, && \text{for } x,y\in\Omega,\\
&u_x=\cos(\pi t), && u_y=\cos(2\pi t), && \text{for } (x,y)\in\{0,1\}\times(0,1),\\
&u_x=\sin(\pi t), && u_y=\sin(2\pi t), && \text{for } (x,y)\in(0,1)\times\{0,1\}.
\end{aligned}$$

Numerical Examples

Two-Dimensional Burgers Equation

- 2D-Burgers equation on $\underbrace{(0,1)\times(0,1)}_{:=\Omega}\times[0,T]$

$$u_t = -\left(u\cdot\nabla\right)u + \nu\Delta u$$

  with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.

- Now consider initial and boundary conditions

$$\begin{aligned}
&u_x(x,y,0)=0, && u_y(x,y,0)=0, && \text{for } x,y\in\Omega,\\
&u_x=\cos(\pi t), && u_y=\cos(2\pi t), && \text{for } (x,y)\in\{0,1\}\times(0,1),\\
&u_x=\sin(\pi t), && u_y=\sin(2\pi t), && \text{for } (x,y)\in(0,1)\times\{0,1\}.
\end{aligned}$$

- Spatial discretization $\rightsquigarrow$ QBDAE system with zero I.C. and 4 inputs $B\in\mathbb{R}^{n\times 4}$, N_1,N_2,N_3,N_4, ROM with $q_1=5, q_2=2, \sigma=0, \hat{n}=52$.

Numerical Examples
Two-Dimensional Burgers Equation

- 2D-Burgers equation on $\underbrace{(0,1)\times(0,1)}_{:=\Omega}\times[0,T]$

$$u_t = -\left(u\cdot\nabla\right)u + \nu\Delta u$$

  with $u(x,y,t)\in\mathbb{R}^2$ describing the motion of a compressible fluid.
- Now consider initial and boundary conditions

$$\begin{aligned} u_x(x,y,0) &= 0, & u_y(x,y,0) &= 0, & &\text{for } x,y\in\Omega,\\ u_x &= \cos(\pi t), & u_y &= \cos(2\pi t), & &\text{for } (x,y)\in\{0,1\}\times(0,1),\\ u_x &= \sin(\pi t), & u_y &= \sin(2\pi t), & &\text{for } (x,y)\in(0,1)\times\{0,1\}. \end{aligned}$$

- Spatial discretization $\rightsquigarrow$ QBDAE system with zero I.C. and 4 inputs $B\in\mathbb{R}^{n\times 4}$, N_1,N_2,N_3,N_4, ROM with $q_1=5, q_2=2, \sigma=0, \hat{n}=52$.
- State reconstruction by reduced model $x\approx V\hat{x}$, max. rel. err $<3\%$.

Numerical Examples
The Chafee-Infante equation

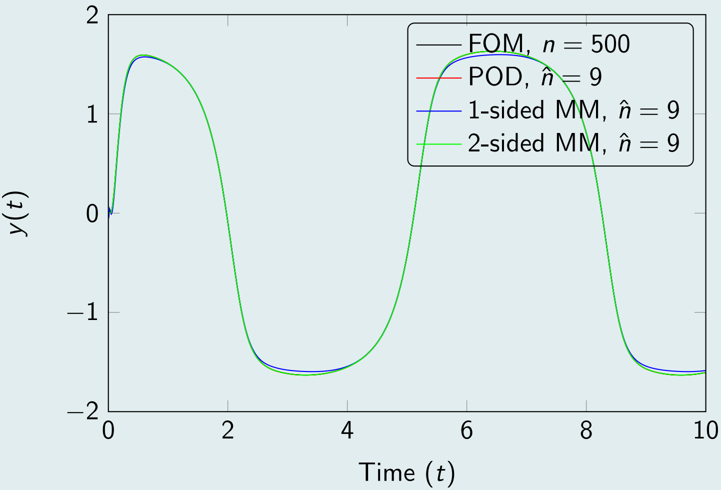
- Consider PDE with a cubic nonlinearity:

$$\begin{aligned} v_t + v^3 &= v_{xx} + v, && \text{in } (0,1)\times(0,T),\\ v(0,\cdot) &= u(t), && \text{in } (0,T),\\ v_x(1,\cdot) &= 0, && \text{in } (0,T),\\ v(x,0) &= v_0(x), && \text{in } (0,1) \end{aligned}$$

- original state dimension $n = 500$, QBDAE dimension $N = 2 \cdot 500$, reduced QBDAE dimension $r = 9$

Numerical Examples
The Chafee-Infante equation

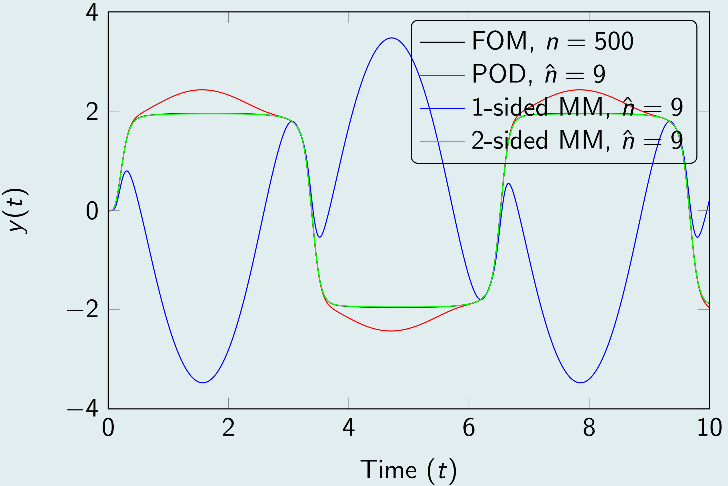
Comparison between moment-matching and POD ($u(t) = 5\cos(t)$)



Numerical Examples

The Chafee-Infante equation

Comparison between moment-matching and POD ($u(t) = 50 \sin(t)$)



Numerical Examples
The FitzHugh-Nagumo System

- FitzHugh-Nagumo system modeling a neuron

[Chaturantabut, Sorensen '09]

$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1], \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

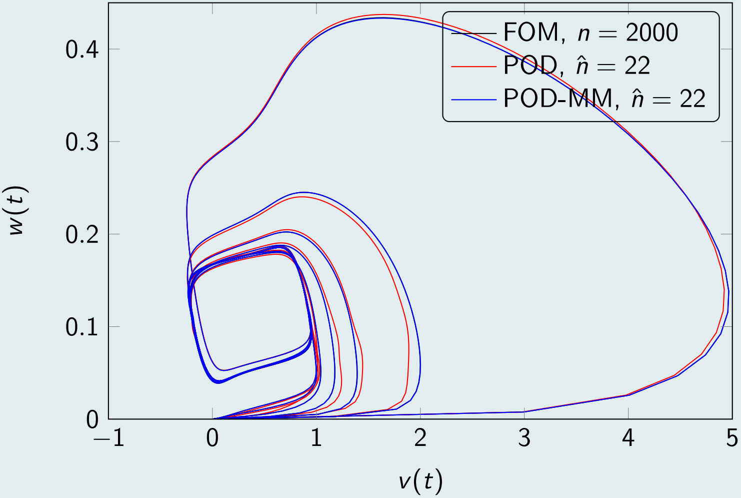
where
$\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ g = 0.05,\ i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)$

- original state dimension $n = 2 \cdot 1000$, QBDAE dimension $N = 3 \cdot 1000$, reduced QBDAE dimension $r = 20$

Numerical Examples

The FitzHugh-Nagumo System

POD via moment-matching (varying input)



Topics Not Covered

Linear Systems:

- Balanced residualization (singular perturbation approximation), yields $G(0) = \hat{G}(0)$.
- Balancing-related methods.
- Special methods for second-order (mechanical) systems.
- Extensions to bilinear and stochastic systems.
- MOR methods for discrete-time systems.
- Extensions to descriptor systems $E\dot{x} = Ax + Bu$, E singular.

Nonlinear Systems:

- Other MOR techniques like POD, RB, Empirical Gramians.
- Simulation-free methods for parametric systems is widely open!

Further Reading

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Model Reduction for Circuit Simulation.
LECTURE NOTES IN ELECTRICAL ENGINEERING, Vol. 74,
Springer-Verlag, Dordrecht, 2011.

Course Material

Material, including a video of all lectures and exercises with solutions, corresponding to an extended (10 hours) version Matrix Equations and Model Reduction of this course. held at Gene Golub SIAM Summer School 2013 "Matrix Functions and Matrix Equations", Fudan University, Shanghai, China is available at

`http://g2s3.cs.ucdavis.edu/course.html`.