CPTS Sitzung
Berlin, 24 October 2014

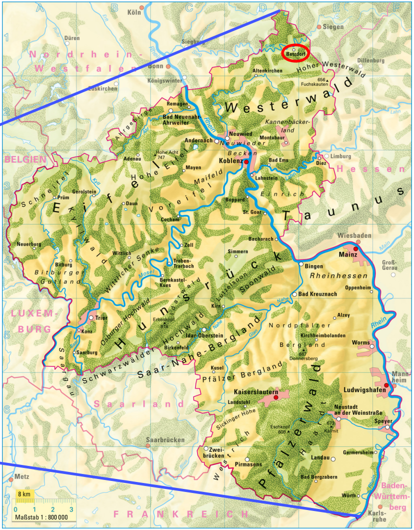
Antrittsrede

Peter Benner

Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

Growing up . . . in Betzdorf/Sieg

1967 . . . until 1986



Growing up ... in Betzdorf/Sieg

1967 ... until 1986

This is where I spent childhood and youth ...

Growing up . . . in Betzdorf/Sieg

1967 . . . **until 1986**

. . . and this is where I actually spent most part of it.

Freiherr-vom-Stein-Gymnasium Betzdorf-Kirchen

1977: entered in 5th grade 1986: graduation (Abitur)

A famous graduate . . .

Freiherr-vom-Stein-Gymnasium Betzdorf-Kirchen

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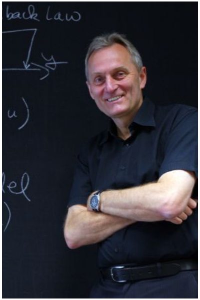
A famous graduate . . .

Jochem Marotzke (Abitur 1977)

Studying Mathematics at RWTH Aachen 1987–1993



Volker Mehrmann

RHEINISCH-
WESTFÄLISCHE
TECHNISCHE
HOCHSCHULE
AACHEN
RWTH

Plesken, Görlich, Neubüser, Oberschelp, Butzer, Jongen, Mehrmann, Dahmen

First research results

Diploma Thesis

. . . and the first paper

The algebraic Riccati equation

$$0 = W + A^T X + XA - XVX =: \mathcal{R}(X),$$

with data $A \in \mathbb{R}^{n\times n}$, $V, W \in \mathbb{R}^{n\times n}$ symmetric positive semidefinite, and unknown $X = X^T \in \mathbb{R}^{n\times n}$.

- Major equation in systems and control theory: feedback, filtering, . . .

[1] P. Benner. Ein orthogonal symplektischer Multishift Algorithmus zur Lösung der algebraischen Riccatigleichung. Diplomarbeit, RWTH Aachen, Institut für Geometrie und Praktische Mathematik, March 1993.

[2] G.S. Ammar, P. Benner, and V. Mehrmann. A multishift algorithm for the numerical solution of algebraic Riccati equations. Electr. Trans. Num. Anal., 1:33–48, 1993.

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- Numerical algorithms need to guarantee the computation of $X_{\max}$ with small *backward error* with at most $\mathcal{O}(n^3)$ flops.
- In [1,2], we describe one of the first algorithms achieving this — at least for small n, say $n < 50$.

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Starting a Math Ph.D. in Kansas

Starting a Math Ph.D. in Kansas

. . . under the guidance of Ralph Byers

Started working on

- variants of Newton's method for AREs,
- large-scale Hamiltonian eigenproblems,
- the sign function method.

Ralph Byers
(1955-2007)

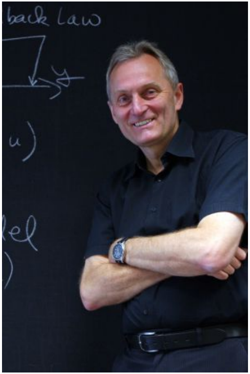
[3] P. Benner and R. Byers. An exact line search method for solving generalized continuous-time algebraic Riccati equations. IEEE Trans. Automat. Control, 43(1):101–107, 1998.

[4] P. Benner and H. Faßbender. An implicitly restarted symplectic Lanczos method for the Hamiltonian eigenvalue problem. Linear Algebra Appl., 263:75–111, 1997.

[5] P. Benner and E.S. Quintana-Ortí. Solving stable generalized Lyapunov equations with the matrix sign function. Numerical Algorithms, 20(1):75–100, 1999.

Continuing (and finishing) the Ph.D. in Chemnitz 1997

. . . with Volker Mehrmann (and Ralph Byers)



Volker Mehrmann

[6] P. Benner. *Contributions to the Numerical Solution of Algebraic Riccati Equations and Related Eigenvalue Problems.*
Dissertation, Fakultät für Mathematik, TU Chemnitz–Zwickau, 09107 Chemnitz (Germany), February 1997.

Van Loan's curse

The Hamiltonian Eigenproblem

Definition

Let $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, then $H \in \mathbb{R}^{2n \times 2n}$ is called Hamiltonian if $(HJ)^T = HJ$.

Note: $J^{-1} = J^T = -J$.

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Explicit block form of Hamiltonian matrices

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}, \quad \text{where } A, B, C \in \mathbb{R}^{n \times n} \quad \text{and } B = B^T,\ C = C^T.$$

For mathematicians: Hamiltonian matrices form the Lie algebra associated to the symplectic matrix group and the skew bilinear form $\langle\, .,.\, \rangle_J$.

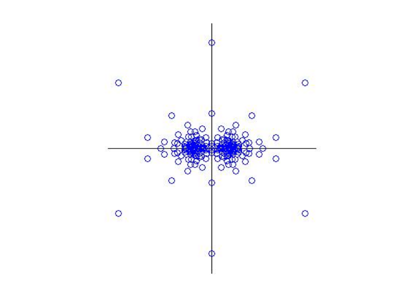
The Hamiltonian Eigenproblem

Spectral Properties

Hamiltonian Eigensymmetry

Hamiltonian matrices exhibit the Hamiltonian eigensymmetry:
if λ is a finite eigenvalue of H, then $\bar{\lambda}, -\lambda, -\bar{\lambda}$ are eigenvalues of H, too.

Typical Hamiltonian spectrum



The Hamiltonian Eigenproblem

Goal

Structure-preserving algorithm, i.e., if $\tilde{\lambda}$ is a computed eigenvalue of H, then $\bar{\tilde{\lambda}}, -\tilde{\lambda}, -\bar{\tilde{\lambda}}$ should also be computed eigenvalues.

Goal cannot be achieved by general methods for matrices or matrix pencils like the QR/QZ, Lanczos, Arnoldi algorithms!

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For an algorithm based on similarity transformations, the goal is achieved if the Hamiltonian structure is preserved.

Definition

$V \in \mathbb{R}^{2n \times 2n}$ is symplectic if $\quad V^T J V = J, \quad$ i.e., $V^{-1} = J^T V^T J$.

$V_k \in \mathbb{R}^{2n \times 2k}$ is symplectic or a J-isometry if $\quad V_k^T J_n V_k = J_k$.

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Lemma

If H is Hamiltonian and V is symplectic, then $V^{-1} H V$ is Hamiltonian, too.

Van Loan's curse

Applications

Hamiltonian eigenproblems arise in many different applications, e.g.:

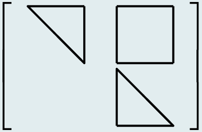
- Systems and control:
  - Solution methods for algebraic and differential Riccati equations.
  - Design of LQR/LQG/H_2/H_∞ controllers and filters for continuous-time linear control systems.
  - Stability radii and system norm computations; optimization of system norms.
  - Passivity-preserving model reduction based on balancing.
  - Reduced-order control for infinite-dimensional systems based on inertial manifolds.
- Computational physics:
  exponential integrators for Hamiltonian dynamics.
- Quantum chemistry:
  computing excitation energies in many-particle systems using random phase approximation (RPA); Bethe-Salpeter equation.
- Quadratic eigenvalue problems:
  in particular, gyroscopic systems.

Van Loan's curse

Theorem (Paige/Van Loan 1981)

If H has no purely imaginary eigenvalues, then the Hamiltonian Schur form exists: there exists a symplectic and *orthogonal similarity transformation*

$$V^T H V =$$



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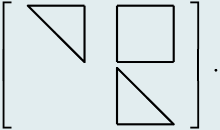

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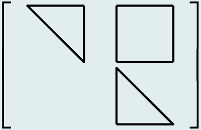

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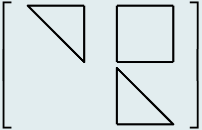

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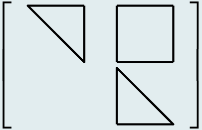

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Problem: *no constructive algorithm given in* [7]!

[7] C.C. Paige and C.F. Van Loan. A Schur decomposition for Hamiltonian matrices. Linear Algebra Appl., 41:11–32, 1981.

Van Loan's curse

Van Loan's curse

Compute all eigenvalues of a Hamiltonian matrix using a numerical algorithm that is

- backward stable,
- preserves the Hamiltonian eigensymmetry
- requires at most $\mathcal{O}(n^3)$ operations.

Despite progress in [8,9], the problem remained open until 1997/98.

Charlie Van Loan
(Cornell U.)

[8] R. Byers. A Hamiltonian QR-algorithm. SIAM J. Sci. Statist. Comput., 7:212–229, 1986.

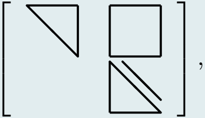
[9] C.F. Van Loan. A symplectic method for approximating all the eigenvalues of a Hamiltonian matrix. Linear Algebra Appl., 61:233–251, 1984.

Van Loan's curse . . . solved 1997/98

Breaking the curse

- Use relation to skew-Hamiltonian $N := H^2$;
- do not compute N explicitly (as in [Van Loan 1984]), but use a clever implicit strategy to compute symplectic URV decomposition:

$$UHV = \begin{bmatrix} H_1 & \widetilde{H} \\ & H_2^T \end{bmatrix} =$$



then apply product QR algorithm to $H_2 H_1$ (no explicit product needed!) to get eigenvalues.

- Further tricks to get eigenvectors/invariant subspaces.
- backward stable, preserves Hamiltonian eigensymmetry, $\mathcal{O}(n^3)$ (and 40% faster than unstructured solver!).

Now part of, e.g., MATLAB.

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[10] P. Benner, V. Mehrmann, and H. Xu. A numerically stable, structure preserving method for computing the eigenvalues of real Hamiltonian or symplectic pencils. Numer. Math., 78(3):329–358, 1998.

[11] P. Benner, V. Mehrmann, and H. Xu. A new method for computing the stable invariant subspace of a real Hamiltonian matrix. J. Comput. Appl. Math., 86:17–43, 1997.

Wanderings

Habilitation in Bremen 2001

Building up the Zentrum für Technomathematik.

Wanderings

Lecturer at TU Berlin 2002–2003

First phase of DFG Research Center MATHEON.

(Visiting associate professor for one term at TU Hamburg-Harburg.)

Wanderings

Full professor at TU Chemnitz 2003–2010

Head of research group
"Mathematics in Industry and Technology"

Research on

- model reduction,
- feedback control,
- HPC algorithms

for large-scale instationary dynamical systems.

Wanderings

. . . come to an end

Appointment as director at MPI Magdeburg 2010

Head of search group (department)
Computational Methods in Systems and Control Theory

Managing Director 2013/14

MPG

Managing Director in Magdeburg — nonscientific challenges

Political quarrels in Sachsen-Anhalt

Gemeinsame Presseerklärung der außeruniversitären Wissenschaftseinrichtungen der Landeshauptstadt Magdeburg

Wer an Bildung und Forschung spart, spart an der Zukunft des Landes

Die Einschätzung des Finanzministers des Landes Sachsen-Anhalt, die Hochschulen seien mittelmäßig, ist nicht nur fatal für die Innen- und Außensicht des Landes, sondern auch eine Fehleinschätzung. Die aus dieser Fehleinschätzung abgeleiteten Sparpläne gefährden die Forschung nicht nur an den Hochschulen, sondern insgesamt im Land – sie bringen Sachsen-Anhalt um die Chance, von der bisherigen Forschungsförderung langfristig profitieren zu können. Wer an Bildung und Forschung spart, spart an der Zukunft des Landes.

Süddeutsche Zeitung vom 6. Mai 2014

Abbau Ost

Rektoren in den neuen Ländern warnen vor Niedergang der Hochschulen

Armin Willingmann findet das Verhalten der Landesregierung inzwischen nur noch „skurril". Sie betreibe ein „borniertes Einsparen – das belastet die Wissenschaft hier schwer", sagt der Vorsitzende der Landesrektorenkonferenz in Sachsen-Anhalt und Chef der Hochschule Harz. Willingmann ist eine gewichtige Stimme, aber nur eine von vielen, die sich derzeit für die Wissenschaft in Ostdeutschland erheben. Professoren und Studenten sind sich einig: Hochschulen und Forschungseinrichtungen trocknen finanziell aus.

Sechs große Institute, darunter das Fraunhofer-Institut, greifen offen Landesfinanzminister Jens Bullerjahn (SPD) an, die Vorgaben der Regierung werden „den Standort langfristig schädigen", schreiben sie in einer gemeinsamen Erklärung. Vergangene Woche protestierten 7000 Studenten in Halle. Aber auch in weiteren neuen Ländern erhebt sich Protest: „Das kann so nicht weitergehen", schimpft der Rektor der Universität Rostock, Wolfgang Schareck, über die Ausstattung durch die Regierung in Mecklenburg-Vorpommern. Und auch in Brandenburg fordern Städte und Hochschulen den Landtag auf, endlich mehr Geld bereitzustellen. Steht ein Niedergang der ostdeutschen Hochschulen bevor? Oder ist das alles Lobbyistengetöse?

Bildung hat Vorrang, so ist es stets in Politikerreden zu hören, auch in Ostdeutschland. Bessere Bildung gilt gemeinhin als Schlüssel – für Wirtschaftswachstum, soziale Gleichheit, Integration von Migranten. Doch gleichzeitig müssen gerade die ostdeutschen Länder mit immer weniger Geld auskommen. Die finanzielle Förderung aus dem Westen schmilzt zusammen, Menschen und damit Steuerzahler wandern zunehmend ab, zudem greift die

kommt ein harsches Sparprogramm zu: In Sachsen-Anhalt sollen sie von 2015 an jedes Jahr fünf Millionen Euro einsparen, zehn Jahre lang. 2025 sollen sie mit jährlich 50 Millionen weniger auskommen.

Doch wo soll das Geld herkommen? 80 bis 90 Prozent ihres Etats geben die Hochschulen laut Willingmann für ihre Mitarbeiter aus; genau dort wird man nun sparen müssen. „Am stärksten gefährdet sind alle mit befristeten Verträgen; aber auch Stellen von Professoren, die in den Ruhestand gehen", sagt Sachsen-Anhalts Rektorenoberhaupt. Welcher Lehrstuhl wegfällt, hängt davon ab, wie alt der jeweilige

Professor ist – ein Umbau nach dem Zufallsprinzip. Dabei ist eine durchdachte Reform durchaus greifbar: Im Juli legt der Wissenschaftsrat seine Empfehlungen für die Reform der Hochschulen in Sachsen-Anhalt vor. Doch so ein Umbau würde Zeit brauchen. Bei einem Treffen der Hochschulchefs mit dem neuen Wissenschaftsminister Hartmut Möllring (CDU), dessen Vorgängerin im Streit über das Sparen entlassen wurde, einigte man sich, den Bericht abzuwarten. Doch dies ist ein Scheinkompromiss: Die Sparauflagen bleiben bestehen. „Wir haben dann zwar die Empfehlungen für den Umbau, aber nicht die Zeit, ihn umzusetzen", sagt Willingmann.

Wer das zuständige Wissenschaftsministerium in Magdeburg darauf anspricht, wird rasch auf das Finanzministerium verwiesen. Dort werden die Vorgaben gemacht: Das Land hat bis 2020 wohl zwei Milliarden Euro einzusparen, da müsse sich die Wissenschaft schon beteiligen, sagt ein Sprecher: „Das Problem löst sich nicht durch Proteste." Die Wissenschaftler „sperren sich ja nicht gegen Einsparungen", entgegnet Willingmann. Aber für den womöglich dramatischsten Umbau der Hochschulen seit der Wende brauche es eben ein paar Jahre mehr.

Die Lage der mecklenburgischen Hochschulen wirkt da noch komfortabel: Sie müssen mit einem jährlichen Zuwachs von nur 1,5 Prozent auskommen. Das aber sei zu wenig, sagt der Rostocker Rektor Schareck. Die Energiepreise, die Löhne, alles steige schneller als die staatliche Unterstützung. Allein die Uni Rostock fordert für kommendes Jahr 8,5 Millionen Euro mehr Zuwendungen. Die Kritik trifft Bildungsminister Mathias Brodkorb (SPD) offenbar unvorbereitet. Man sei noch dabei, die Finanz-

MAGDEBURGWILLsWISSEN

Magdeburg ist Wissenschaftsstandort: Otto-von-Guericke-Universität, Universitätsmedizin, Hochschule Magdeburg-Stendal, Max-Planck-Institut, Fraunhofer-Institut, Leibniz-Institut für Neurobiologie, Helmholtz-Zentrum für Umweltforschung und Deutsches Zentrum für Neurodegenerative Erkrankungen sorgen mit großer Unterstützung der Stadt für neue, innovative Ideen und Technologien in der Industrie, den mittelständischen Unternehmen und im Gesundheitswesen. Magdeburg ist ohne Wissenschaft nicht vorstellbar!

Mit den von der Landesregierung geplanten drastischen Kürzungen im Hochschulbereich kann eine Abwärtsspirale in Gang gesetzt werden, die Sachsen-Anhalt und Magdeburg im Bereich Forschung, Entwicklung und Lehre unattraktiv macht und dauerschaft schwächt.

MPG

Managing Director in Magdeburg — nonscientific challenges

Elbe flood June 2013

Model Order Reduction

A Key Technology for Complex Dynamical Systems

- Complexity of computer simulation (DNS, direct numerical simulation) and, particularly, of computer-aided control, optimization and design of dynamical processes increases rapidly, due to
  - multiphysics applications (e.g., MEMS),
  - parameter uncertainties,
  - network structures (e.g., nanoelectronics, biochemical/ metabolic networks),
  - complicated 3D geometries (e.g., machine tools) or *n*D problems in molecular dynamics.

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  - complicated 3D geometries (e.g., machine tools) or *n*D problems in molecular dynamics.
- Curse of Dimensionality:
  Hardware acceleration does not compensate for increase in model complexity!

Model Order Reduction

A Key Technology for Complex Dynamical Systems

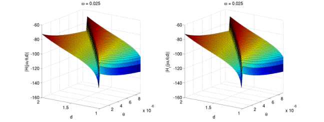
Example: Algorithmic vs. Hardware Acceleration

Engineering design often requires extensive parameter studies in "real-time".

- **Micro gyroscope example:**
  duration of parameter study reduced from

  about 3 days
  to 1 hour
  with 0.1% loss in accuracy

  using advanced mathematical methods.

  $\Longrightarrow$ *Acceleration factor* ≈ 72.

  Another MEMS example (Anemometer): reduction from > 11 days to ~ 90sec. $\rightsquigarrow$ 10.500 times faster!

- CPU clock rate limited (≤ 3 GHz), *hardware acceleration* only achievable using multicore technology.
- Given ideal speed-up: would require $\approx$ 288 (21.000) cores to achieve the same!

Model Order Reduction

A Key Technology for Complex Dynamical Systems

- Complexity of computer simulation (DNS, direct numerical simulation) and, particularly, of computer-aided control, optimization and design of dynamical processes increases rapidly, due to
  - multiphysics applications (e.g., MEMS),
  - parameter uncertainties,
  - network structures (e.g., nanoelectronics, biochemical/ metabolic networks),
  - complicated 3D geometries (e.g., machine tools) or *n*D problems in molecular dynamics.
- Curse of Dimensionality:
  Hardware acceleration does not compensate for increase in model complexity!
- Require goal-oriented reduced-order models (ROMs) and algorithmic improvements!

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$$\Downarrow$$

Key technology: system approximation / model reduction

Model Order Reduction

Parametric Dynamical Systems

Dynamical Systems

$$
\Sigma(p) : \begin{cases} E(p)\dot{x}(t;p) = f(t, x(t;p), u(t); p), \quad x(t_0;p) = x_0(p), & \text{(a)} \\ y(t;p) = g(t, x(t;p), u(t); p) & \text{(b)} \end{cases}
$$

with

- (generalized) states $x(t;p) \in \mathbb{R}^n$ (or a suitable Hilbert space),
- inputs $u(t) \in \mathbb{R}^m$ (or a suitable Hilbert space),
- outputs $y(t;p) \in \mathbb{R}^q$, (b) is called output equation,
- $p \in \Omega \subset \mathbb{R}^d$ is a parameter vector, Ω is bounded.

Applications ("multi-query context"):

- Repeated simulation for varying material or geometry parameters, boundary conditions,
- control, optimization and design.

Dynamical Systems

Original System

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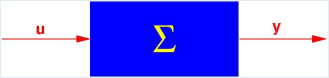

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible inputs u and all $p \in \Omega$.

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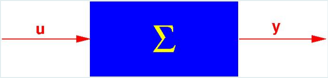

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t;p) = \widehat{f}(t, \hat{x}(t), u(t); p), \\ \hat{y}(t;p) = \widehat{g}(t, \hat{x}(t), u(t); p). \end{cases}$$

- states $\hat{x}(t;p) \in \mathbb{R}^r$, $r \ll n$
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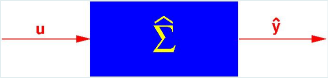

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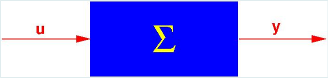

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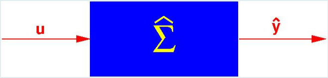

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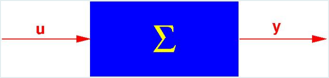

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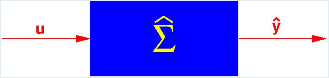

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible inputs u and all $p \in \Omega$.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Model Order Reduction

Linear Parametric Systems

Linear, time-invariant (parametric) systems

$$
\begin{aligned}
E(p)\dot{x}(t;p) &= A(p)x(t;p) + B(p)u(t), & A(p), E(p) \in \mathbb{R}^{n\times n},\\
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Laplace Transformation / Frequency Domain

Application of Laplace transformation $(x(t;p) \mapsto x(s;p), \dot{x}(t;p) \mapsto sx(s;p))$ to linear system with $x(0;p) \equiv 0$:

$$sE(p)x(s;p) = A(p)x(s;p) + B(p)u(s), \quad y(s;p) = C(p)x(s;p),$$

yields I/O-relation in frequency domain:

$$y(s;p) = \Big(\underbrace{C(p)(sE(p) - A(p))^{-1}B(p)}_{=:G(s;p)}\Big)u(s).$$

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Goal: Fast evaluation of mapping $(u,p) \ \rightarrow \ y(s;p)$.

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$G(s;p)$ is the parameter-dependent transfer function of $\Sigma(p)$.

Goal: Approx. $G(s;p)$ by rational matrix $\hat{G}(s;p)$ of much lower degree!

Exemplary Results: Linear Parametric Systems

Rational Interpolation by Projection onto Krylov Subspaces

Theorem [10]

Suppose that $E(p)$, $A(p)$, $B(p)$, $C(p)$ are C^1 in a neighborhood of $\hat{p} = [\hat{p}_1, \ldots, \hat{p}_d]^T \in \Omega$ and that both $\hat{s}\,E(\hat{p}) - A(\hat{p})$ and $\hat{s}\,\hat{E}(\hat{p}) - \hat{A}(\hat{p})$ are invertible. If

$$(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\,B(\hat{p}) \in \mathrm{Ran}(V),$$
$$\left(C(\hat{p})\,(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\right)^T \in \mathrm{Ran}(W),$$

and

$$\hat{A}(s,p) = W^T A(s,p) V,\ \hat{B}(s,p) = W^T B(s,p),\ \hat{C}(s,p) = C(s,p)V, \ldots$$

then

$$G(\hat{s},\hat{p}) = \hat{G}(\hat{s},\hat{p})$$
$$\nabla_p G(\hat{s},\hat{p}) = \nabla_p \hat{G}(\hat{s},\hat{p}), \qquad \frac{\partial}{\partial s} G(\hat{s},\hat{p}) = \frac{\partial}{\partial s} \hat{G}(\hat{s},\hat{p}).$$

[10] U. Baur, C. Beattie, P. Benner, and S. Gugercin. Interpolatory Projection Methods for Parameterized Model Reduction. SIAM J. Sci. Comp., 33:2489–2518, 2011.

Exemplary Results: Nonlinear Systems

Matching Volterra Moments

Quadratic-bilinear systems

$$\Sigma: \quad \begin{cases} E\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

where $E, A, N \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{n \times n^2}$ (Hessian tensor), $B, C^T \in \mathbb{R}^n$.

- A large number of smooth nonlinear systems can be transformed into QB systems using additional variables.
- Solution is represented by Volterra series, or, by a sequence of formal linear systems.
- Coefficients in Volterra series can be interpolated using concept of generalized transfer functions.

Exemplary Results: Nonlinear Systems

Matching Volterra Moments

Theorem [11]

Let $\hat{E} = W^T E V$, $\hat{A} = W^T A V$, $\hat{H} = W^T H (V \otimes V)$, $\hat{N} = W^T N V$, $\hat{b} = W^T b$, $\hat{c} = V^T c$ with full-rank $V, W \in \mathbb{R}^{n \times n_r}$ s.t. $\hat{E}$ is invertible and define $F(s) = sE - A$, $b_i := F(\sigma_i)^{-1} b$, $c_i := F(2\sigma_i)^{-T} c$,

$$\text{span } V \supset \operatorname*{span}_{i=1,\ldots,k} \left\{ b_i,\ F(2\sigma_i)^{-1} \left[H(b_i \otimes b_i) + N b_i \right] \right\},$$

$$\text{span } W \supset \operatorname*{span}_{i=1,\ldots,k} \left\{ c_i,\ F(\sigma_i)^{-T} \left[\mathcal{H}^{(2)}(f_i \otimes c_i) + \frac{1}{2} N^T c_i \right] \right\},$$

with $\sigma_i \notin \{\Lambda(A, E), \Lambda(\hat{A}, \hat{E})\}$. Then for $i = 1, \ldots, k$:

$$G_1(\sigma_i) = \hat{G}_1(\sigma_i), \qquad G_1(2\sigma_i) = \hat{G}_1(2\sigma_i),$$

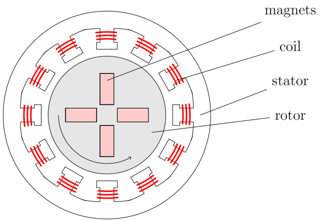
$$G_2(\sigma_i, \sigma_i) = \hat{G}_2(\sigma_i, \sigma_i), \qquad \frac{\partial}{\partial s_j} G_2(\sigma_i, \sigma_i) = \frac{\partial}{\partial s_j} \hat{G}_2(\sigma_i, \sigma_i), \quad j = 1, 2.$$

[11] P. Benner and T. Breiten. Two-Sided Projection Methods for Nonlinear Model Order Reduction. SIAM J. Sci. Comp., provisionally accepted.

Numerical Examples

Industrial Case Study: Thermal Analysis of Electrical Motor

- Thermal simulations to detect whether temperature changes lead to fatigue or deterioration of employed materials.
- Main heat source: thermal losses resulting from current stator coil/rotor.
- Many different current profiles need to be considered to predict whether temperature on certain parts of the motor remans in feasible region.
- Finite element analysis on rather complicated geometries ⇝ large-scale linear models with many (here: 7/13) parameters.



Schematic view of an electrical motor.

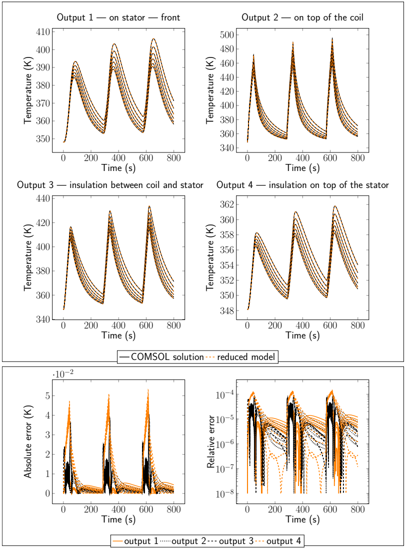
Bosch integrated motor generator used in hybrid variants of Porsche Cayenne, VW Touareg.

Pictures: BOSCH

Numerical Examples

Industrial Case Study: Thermal Analysis of Electrical Motor

- FEM analysis of thermal model $\rightsquigarrow$ linear parametric systems with $n = 41,199$, $m = 4$ inputs, and $d = 13$ parameters,
- measurements taken at $q = 4$ heat sensors;
- time for 1 transient simulation in COMSOL® $\sim$ 90min;
- ROM order $\hat{n} = 300$, time for 1 transient simulation $\sim$ 15sec.
- Legend: Temperature curves for six different values (5, 25, 45, 65, 85, 100[W/m^2K]) of the heat transfer coefficient on the coil.

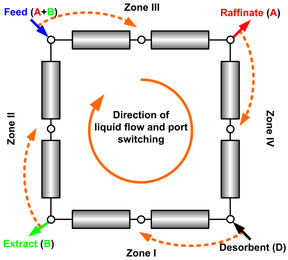


[12] P. Benner and A. Bruns. Parametric Model Order Reduction of Thermal Models Using the Bilinear Interpolatory Rational Krylov Algorithm. Math. Comp. Model. Dyn. Syst., 2014 (online).

Numerical Examples

Model Reduction and ROM-based Optimization of SMB Chromatography

Goal: optimization of *simulated moving bed (SMB) chromatography* (multi-column separation process).



Challenges:

- periodic shifting of inlets and outlets to realize continuous counter-current movement between liquid and adsorbent phases,
- mixed continuous and discrete behavior,
- columns modeled by coupled system of nonlinear PDEs (dispersion, Langmuir isotherm),
- inherent cyclic steady state (CSS) nature.

Achievements:

- Applied trust-region POD and Krylov subspace-based model reduction techniques.
- Reduced computing time by factors 5–50.

[13] S. Li, L. Feng, P. Benner, and A. Seidel-Morgenstern. Using Surrogate Models for Efficient Optimization of Simulated Moving Bed Chromatography. Comp. & Chem. Engrg., 67:121–132, 2014.

[14] S. Li, Y. Yue, L. Feng, A. Seidel-Morgenstern, and P. Benner. Model Reduction for Linear Simulated Moving Bed Chromatography Systems Using Krylov-Subspace Methods. AIChE Journal, 2014 (online).

Numerical Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

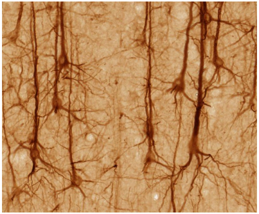
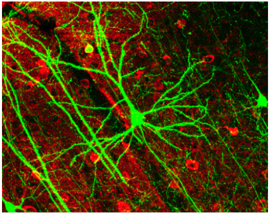
- Simple model for neuron (de-)activation [Chaturantabut/Sorensen 2009]

$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1], \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

where $\epsilon = 0.015, h = 0.5, \gamma = 2, g = 0.05, i_0(t) = 50000t^3 \exp(-15t)$.

Source: http://en.wikipedia.org/wiki/Neuron

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- Parameter g handled as an additional input.
- Original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$.

Numerical Examples

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Thanks to . . .

. . . **the** *Computational Methods in Systems and Control Theory* **team**

15 PostDocs, 17 Ph.D. students, 13 nationalities, 16 kids

Current and Future Research

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- Combine mathematical algorithm development with efficient use of new hardware accelerators.

The LQR Problem

R.E. Kalman. Contributions to the theory of optimal control.
BOLETIN SOCIEDAD MATEMATICA MEXICANA, **5:102–119, 1960**

The linear-quadratic regulator (LQR) problem

$$\min_{u \in L_2[0,\infty]} \int_0^\infty x(t)^T W x(t) + u(t)^T R u(t)\, dt \quad (= \mathcal{V}(x_0)) \tag{1}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $0 \le W = W^T \in \mathbb{R}^{n\times n}$, $0 \le R = R^T \in \mathbb{R}^{m\times m}$.

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BOLETIN SOCIEDAD MATEMATICA MEXICANA, **5:102–119, 1960**

The linear-quadratic regulator (LQR) problem

$$\min_{u \in L_2[0,\infty]} \int_0^\infty x(t)^T W x(t) + u(t)^T R u(t)\, dt \quad (= \mathcal{V}(x_0)) \tag{1}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $0 \leq W = W^T \in \mathbb{R}^{n\times n}$, $0 \leq R = R^T \in \mathbb{R}^{m\times m}$.

Solution: optimal control/Riccati feedback [KALMAN 1960]

$$u_*(t) = -R^{-1}B^T X_* x(t), \tag{3}$$

where $X_* = X_*^T \in \mathbb{R}^{n\times n}$ is the unique positive semidefinite solution of the algebraic Riccati equation (ARE)

$$0 = Q + A^T X + XA - XBR^{-1}B^T X =: \mathcal{R}(X). \tag{4}$$

return