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Parametric Model Order Reduction using Bilinear Systems

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Overview

Introduction to Parametric Model Order Reduction

Parametric Dynamical Systems

Dynamical Systems

$$\Sigma(p): \begin{cases} E(p)\dot{x}(t;p) = f(t, x(t;p), u(t), p), \quad x(t_0) = x_0, & \text{(a)} \\ y(t;p) = g(t, x(t;p), u(t), p) & \text{(b)} \end{cases}$$

with

- (generalized) states $x(t;p) \in \mathbb{R}^n$ ($E \in \mathbb{R}^{n \times n}$),
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t;p) \in \mathbb{R}^q$, (b) is called output equation,
- $p \in \Omega \subset \mathbb{R}^d$ is a parameter vector, Ω is bounded.

Applications:

- Repeated simulation for varying material or geometry parameters, boundary conditions,
- control, optimization and design.

Introduction to Parametric Model Order Reduction

Linear Parametric Systems

Linear, time-invariant (parametric) systems

$$\begin{aligned} E(p)\dot{x}(t;p) &= A(p)x(t;p) + B(p)u(t), \quad && A(p), E(p) \in \mathbb{R}^{n\times n}, \\ y(t;p) &= C(p)x(t;p), && B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}. \end{aligned}$$

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Laplace Transformation / Frequency Domain

Application of Laplace transformation $(x(t;p) \mapsto x(s;p),\ \dot{x}(t;p) \mapsto sx(s;p))$ to linear system with $x(0) = 0$:

$$sE(p)x(s;p) = A(p)x(s;p) + B(p)u(s), \quad y(s;p) = C(p)x(s;p),$$

yields I/O-relation in frequency domain:

$$y(s;p) = \Big(\underbrace{C(p)(sE(p) - A(p))^{-1}B(p)}_{=:H(s;p)} \Big) u(s).$$

$H(s;p)$ is the parameter-dependent transfer function of $\Sigma(p)$.

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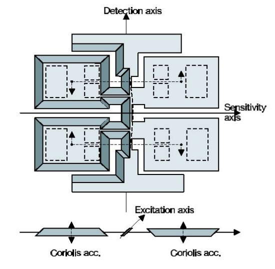
Goal: Fast evaluation of mapping $(u,p) \rightarrow y(s;p)$.

Introduction to Parametric Model Order Reduction

Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

- Application: inertial navigation.



- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order: $N = 17.361 \rightsquigarrow n = 34.722,\ m = 1,\ q = 12.$
- Sensor for position control based on acceleration and rotation.

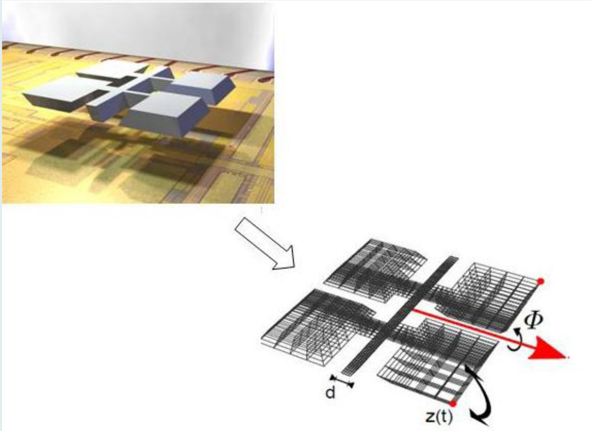
Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Introduction to Parametric Model Order Reduction

Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t).$



Introduction to Parametric Model Order Reduction

Motivating Example: Microsystems/MEMS Design

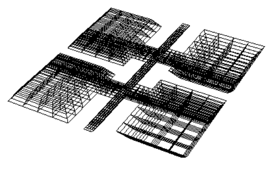
Microgyroscope (butterfly gyro)

Parametric FE model:

$$M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t),$$

wobei

$$\begin{aligned} M(d) &= M_1 + dM_2, \\ D(\theta, d, \alpha, \beta) &= \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d), \\ T(d) &= T_1 + \frac{1}{d}T_2 + dT_3, \end{aligned}$$



with

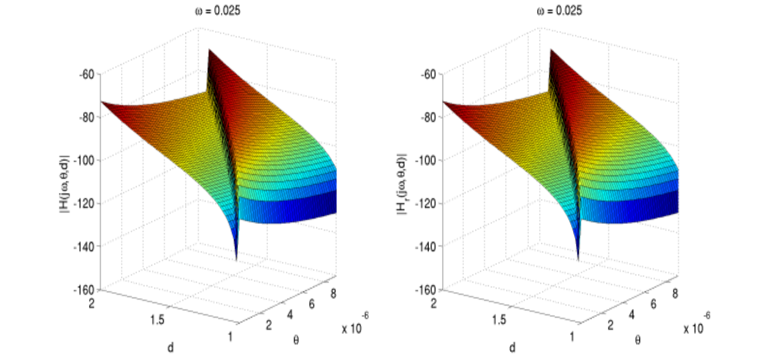
- width of bearing: d,
- angular velocity: θ,
- Rayleigh damping parameters: α, β.

Introduction to Parametric Model Order Reduction

Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Original. . . and reduced-order model.



The Model Order Reduction (MOR) Problem

Problem

Approximate the dynamical system

$$\begin{aligned} E(p)\dot{x} &= A(p)x + B(p)u, \qquad & E(p), A(p) \in \mathbb{R}^{n\times n}, \\ y &= C(p)x, & B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \hat{E}(p)\dot{\hat{x}} &= \hat{A}(p)\hat{x} + \hat{B}(p)u, \qquad & \hat{E}(p), \hat{A}(p) \in \mathbb{R}^{r\times r}, \\ \hat{y} &= \hat{C}(p)\hat{x}, & \hat{B}(p) \in \mathbb{R}^{r\times m}, \hat{C}(p) \in \mathbb{R}^{q\times r}, \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Hu - \hat{H}u\| \leq \|H - \hat{H}\| \cdot \|u\| < \text{tolerance} \cdot \|u\| \quad \forall\, p \in \Omega.$$

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$\Longrightarrow$ Approximation problem: $\displaystyle\min_{\text{order}\,(\hat{H})\leq r} \|H - \hat{H}\|.$

Parametric Systems as Bilinear Systems

Linear Parametric Systems — An Alternative Interpretation

Consider bilinear control systems:

$$\Sigma : \quad \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t) u_i(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

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Key Observation [B./Breiten 2011]

Consider parameters as additional inputs, a linear parametric system

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m_p} a_i(p) A_i x(t) + B_0 u_0(t), \quad y(t) = Cx(t)$$

with $B_0 \in \mathbb{R}^{n \times m_0}$ can be interpreted as bilinear system:

$$u(t) := \begin{bmatrix} a_1(p) & \dots & a_{m_p}(p) & u_0(t) \end{bmatrix}^T,$$
$$B := \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & B_0 \end{bmatrix} \in \mathbb{R}^{n \times m}, \quad m = m_p + m_0.$$

Parametric Systems as Bilinear Systems

Linear Parametric Systems — An Alternative Interpretation

Linear parametric systems can be interpreted as bilinear systems.

Parametric Systems as Bilinear Systems

Linear Parametric Systems — An Alternative Interpretation

Linear parametric systems can be interpreted as bilinear systems.

Consequence

Model order reduction techniques for bilinear systems can be applied to linear parametric systems!

Here:

- Balanced truncation,
- $\mathcal{H}_2$ optimal model reduction.

Balanced Truncation for Linear Systems

Idea (for simplicity, $E = I_n$)

- $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization (needs P, Q!) of the system via state-space transformation

  $$\begin{aligned} \mathcal{T} : (A, B, C) \quad &\mapsto \quad (TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.

Balanced Truncation for Linear Systems

Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_{\hat{n}}$.

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- Adaptive choice of r via computable error bound:

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Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!
- No $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

Balanced Truncation for Bilinear Systems

Bilinear Control Systems — Theory and Background

Bilinear control systems:

$$\Sigma : \quad \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t) u_i(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

where $A, A_i \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{q \times n}$.

Properties:

- Approximation of (weakly) nonlinear systems by Carleman linearization yields bilinear systems.
- Appear naturally in boundary control problems, control via coefficients of PDEs, Fokker-Planck equations, . . .
- Due to the close relation to linear systems, a lot of successful concepts can be extended, e.g. transfer functions, Gramians, Lyapunov equations, . . .
- Linear stochastic control systems possess an equivalent structure and can be treated alike [B./Damm 2011].

Balanced Truncation for Bilinear Systems

The concept of balanced truncation can be generalized to the case of bilinear systems, where we need the solutions of the generalized Lyapunov equations:

$$AP + PA^T + \sum_{i=1}^{m} A_i P A_i^T + BB^T = 0,$$
$$A^T Q + QA^T + \sum_{i=1}^{m} A_i^T Q A_i + C^T C = 0.$$

- These equations also appear for stochastic control systems, see [B./Damm 2011].
- "Twice-the-trail-of-the-HSVs" error bound does not hold [B./Damm 2014], stability preservation not yet proved.

Balanced Truncation for Bilinear Systems

Some basic facts and assumptions

$$AX + XA^T + \sum_{i=1}^{m} A_i X A_i^T + BB^T = 0. \tag{1}$$

- Need a positive semi-definite symmetric solution X.

Balanced Truncation for Bilinear Systems

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$$AX + XA^T + \sum_{i=1}^{m} A_i X A_i^T + BB^T = 0. \quad (1)$$

- Need a positive semi-definite symmetric solution X.
- In standard Lyapunov case, existence and uniqueness guaranteed if A stable ($\Lambda(A) \subset \mathbb{C}^-$); this is not sufficient here: (1) is equivalent to

$$\left(I_n \otimes A + A \otimes I_n + \sum_{i=1}^{m} A_i \otimes A_i \right) \mathrm{vec}(X) = -\mathrm{vec}(BB^T).$$

  One sufficient condition for stable A is smallness of A_i (related to stability radius of $\mathcal{A}$)

  $\rightsquigarrow$ bounded-input bounded-output (BIBO) stability of bilinear systems.

  This will be assumed from here on, hence: existence and uniqueness of positive semi-definite solution $X = X^T$.

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- Want: solution methods for large scale problems, i.e., only matrix-matrix multiplication with A, A_j, solves with (shifted) A allowed!

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- Want: solution methods for large scale problems, i.e., only matrix-matrix multiplication with A, A_j, solves with (shifted) A allowed!
- Requires to compute data-sparse approximation to generally dense X; here: $X \approx ZZ^T$ with $Z \in \mathbb{R}^{n \times n_Z}$, $n_Z \ll n$!

Balanced Truncation for Bilinear Systems

Existence of low-rank approximations

Q: Can we expect low-rank approximations $ZZ^T \approx X$ to the solution of

$$AX + XA^T + \sum_{j=1}^{m} A_j X A_j^T + BB^T = 0\ ?$$

Balanced Truncation for Bilinear Systems

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Theorem [B./Breiten 2013]

Assume existence and uniqueness assumption with stable A and $A_j = U_j V_j^T$, with $U_j, V_j \in \mathbb{R}^{n \times r_j}$. Set $r = \sum_{j=1}^{m} r_j$.
Then the solution X of

$$AX + XA^T + \sum_{j=1}^{m} A_j X A_j^T + BB^T = 0$$

can be approximated by X_k of rank $(2k+1)(m+r)$, with an error satisfying

$$\|X - X_k\|_2 \lesssim \exp(-\sqrt{k}).$$

Balanced Truncation for Bilinear Systems

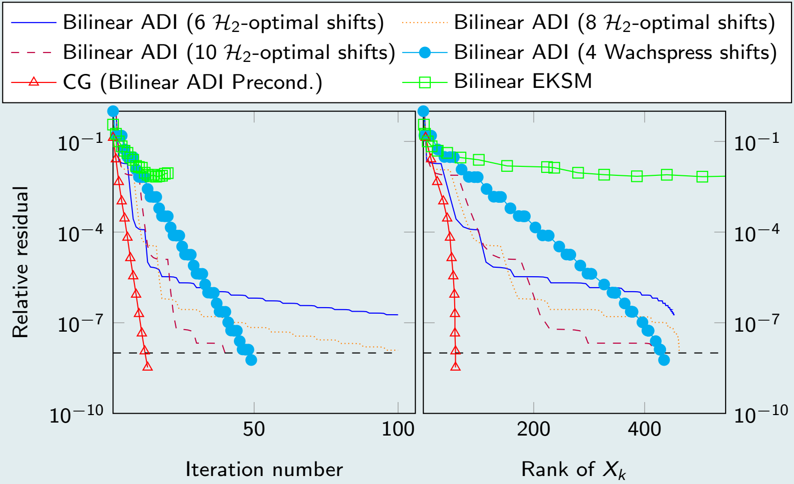
Numerical Methods

- Generalized Alternating Directions Iteration (ADI) method.
  1. Computing square solution matrix ($\sim n^2$ unknowns) [Damm 2008].
  2. Computing low-rank factors of solutions ($\sim n$ unknowns) [B./Breiten 2013].
- Generalized Extended (or rational) Krylov Subspace Method (EKSM) [B./Breiten 2013].
- Tensorized versions of standard Krylov subspace methods, e.g., PCG, PBiCGStab [Kressner/Tobler 2011, B./Breiten 2013].

Balanced Truncation for Bilinear Systems

Numerical Examples: Heat Equation with Boundary Control

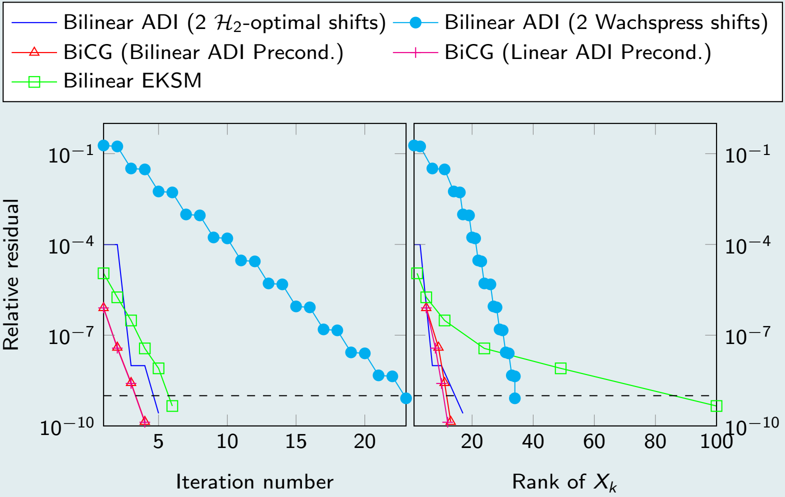
Comparison of low rank solution methods for $n = 562,500$.



Balanced Truncation for Bilinear Systems

Numerical Examples: Fokker-Planck Equation

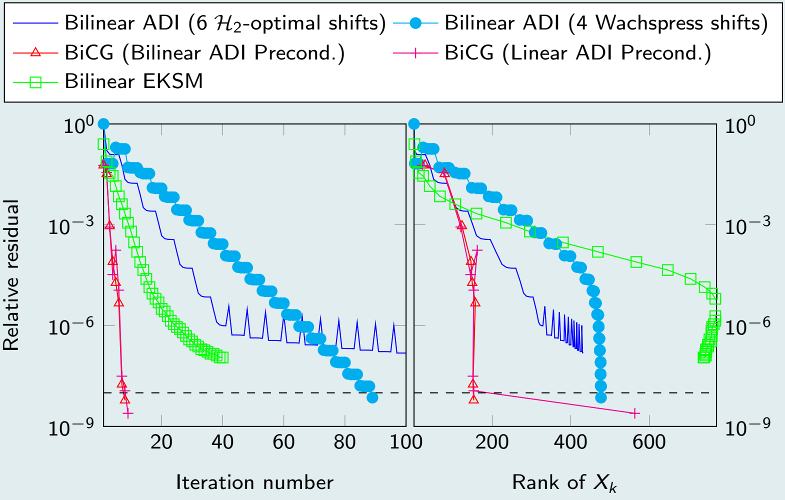
Comparison of low rank solution methods for $n = 10,000$.



Balanced Truncation for Bilinear Systems

Numerical Examples: RC Circuit Simulation

Comparison of low rank solution methods for $n = 250,000$.



Balanced Truncation for Bilinear Systems

Numerical Examples: Comparison

Comparison of CPU times

	Heat equation	RC circuit	Fokker-Planck
Bilin. ADI 2 $\mathcal{H}_2$ shifts	-	-	1.733 (1.578)
Bilin. ADI 6 $\mathcal{H}_2$ shifts	144,065 (2,274)	20,900 (3091)	-
Bilin. ADI 8 $\mathcal{H}_2$ shifts	135,711 (3,177)	-	-
Bilin. ADI 10 $\mathcal{H}_2$ shifts	33,051 (4,652)	-	-
Bilin. ADI 2 Wachspress shifts	-	-	6.617 (4.562)
Bilin. ADI 4 Wachspress shifts	41,883 (2,500)	18,046 (308)	-
CG (Bilin. ADI precond.)	15,640	-	-
BiCG (Bilin. ADI precond.)	-	16,131	11.581
BiCG (Linear ADI precond.)	-	12,652	9.680
EKSM	7,093	19,778	8.555

Numbers in brackets: computation of shift parameters.

Application to Parametric MOR

Fast Simulation of Cyclic Voltammogramms [Feng/Koziol/Rudnyi/Korvink '06]

$$E\dot{x}(t) = (A + p_1(t)A_1 + p_2(t)A_2)x(t) + B,$$
$$y(t) = Cx(t), \quad x(0) = x_0 \neq 0,$$

- Rewrite as system with zero initial condition,
- FE model: $n = 16,912, \; m = 3, \; q = 1,$
- $p_j \in [0, 10^9]$ time-varying voltage functions,
- transfer function $H(i\omega, p_1, p_2)$,
- reduced system dimension $r = 67$,
- $\max\limits_{\substack{\omega \in \{\omega_{min},\dots,\omega_{max}\} \\ p_j \in \{p_{min},\dots,p_{max}\}}} \frac{||H - \hat{H}||_2}{||H||_2} < 6 \cdot 10^{-4},$
- evaluation times: FOM 4.5h, ROM 38s
  $\rightsquigarrow$ speed-up factor ≈ 426.

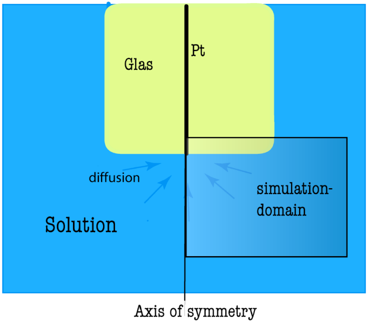
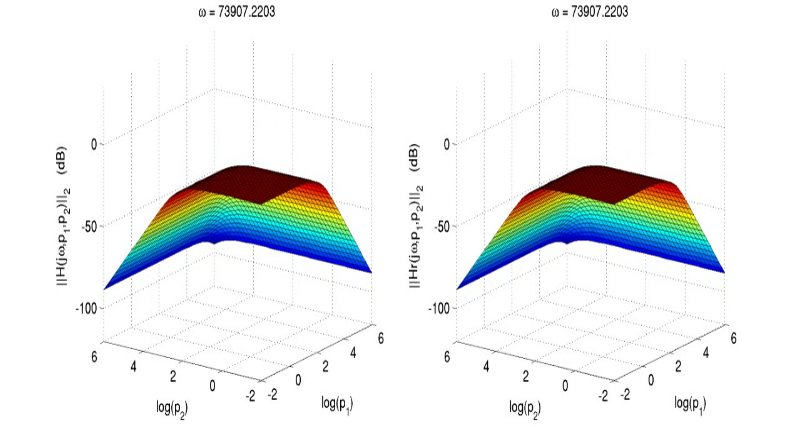


Figure : [Feng et al. '06]

Application to Parametric MOR

Fast Simulation of Cyclic Voltammogramms [Feng/Koziol/Rudnyi/Korvink '06]

Original. . . and reduced-order model.



Application to Parametric MOR

2D Model of an Anemometer [Baur et al. '10]

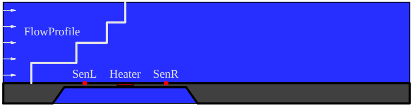


Figure : [Baur et al. '10]

Consider an anemometer, a flow sensing device located on a membrane used in context of minimizing heat dissipation.

$$E\dot{x}(t) = (A + pA_1)x(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = 0,$$

- FE model: $n = 29,008, \ m = 1, \ q = 3$,
- $p_1 \in [0, 1]$ fluid velocity,
- transfer function $H(i\omega, p_1)$, reduced system dimension $r = 146$,
- $\max\limits_{\substack{\omega \in \{\omega_{min}, \dots, \omega_{max}\} \\ p_1 \in \{p_{min}, \dots, p_{max}\}}} \frac{\|H(\omega,p) - \hat{H}(\omega,p)\|_2}{\|H(\omega,p)\|_2} < 3 \cdot 10^{-5}$,
- evaluation times: FOM 51min, ROM 21s.

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

$\mathcal{H}_2$-Model Reduction for Linear Systems

First consider stable (i.e. $\Lambda(A) \subset \mathbb{C}^-$) linear systems,

$$\dot{x}(t) = Ax(t) + Bu(t), \; y(t) = Cx(t) \quad \simeq \quad Y(s) = C(sI - A)^{-1}BU(s)$$

System norms

Two common system norms for measuring approximation quality:

- $\mathcal{H}_2$-norm, $\|\Sigma\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\left((H^*(-i\omega)H(i\omega))\right) d\omega\right)^{\frac{1}{2}}$,
- $\mathcal{H}_\infty$-norm, $\|\Sigma\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(H(i\omega))$,

where

$$H(s) = C(sI - A)^{-1}B.$$

Note: $\mathcal{H}_\infty$*-norm approximation* $\rightsquigarrow$ *balanced truncation.*

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Error system and $\mathcal{H}_2$-Optimality **[Meier/Luenberger 1967]**

In order to find an $\mathcal{H}_2$-optimal reduced system, consider the error system $H(s) - \hat{H}(s)$ which can be realized by

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

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Assuming a coordinate system in which $\hat{A}$ is diagonal and taking derivatives of

$$\|H(\,.\,) - \hat{H}(\,.\,)\|^2_{\mathcal{H}_2}$$

with respect to free parameters in $\Lambda(\hat{A}), \hat{B}, \hat{C} \rightsquigarrow$ **first-order necessary $\mathcal{H}_2$-optimality conditions (SISO)**

$$H(-\hat{\lambda}_i) = \hat{H}(-\hat{\lambda}_i),$$
$$H'(-\hat{\lambda}_i) = \hat{H}'(-\hat{\lambda}_i),$$

where $\hat{\lambda}_i$ are the poles of the reduced system $\hat{\Sigma}$.

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First-order necessary $\mathcal{H}_2$-optimality conditions (MIMO):

$$H(-\hat{\lambda}_i)\tilde{B}_i = \hat{H}(-\hat{\lambda}_i)\tilde{B}_i, \qquad \text{for } i = 1, \ldots, \hat{n},$$
$$\tilde{C}_i^T H(-\hat{\lambda}_i) = \tilde{C}_i^T \hat{H}(-\hat{\lambda}_i), \qquad \text{for } i = 1, \ldots, \hat{n},$$
$$\tilde{C}_i^T H'(-\hat{\lambda}_i)\tilde{B}_i = \tilde{C}_i^T \hat{H}'(-\hat{\lambda}_i)\tilde{B}_i \qquad \text{for } i = 1, \ldots, \hat{n},$$

where $\hat{A} = R\hat{\Lambda}R^{-T}$ is the spectral decomposition of the reduced system and $\tilde{B} = \hat{B}^T R^{-T}$, $\tilde{C} = \hat{C}R$.

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

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$$\begin{aligned} H(-\hat{\lambda}_i)\tilde{B}_i &= \hat{H}(-\hat{\lambda}_i)\tilde{B}_i, && \text{for } i = 1,\dots,\hat{n}, \\ \tilde{C}_i^T H(-\hat{\lambda}_i) &= \tilde{C}_i^T \hat{H}(-\hat{\lambda}_i), && \text{for } i = 1,\dots,\hat{n}, \\ \tilde{C}_i^T H'(-\hat{\lambda}_i)\tilde{B}_i &= \tilde{C}_i^T \hat{H}'(-\hat{\lambda}_i)\tilde{B}_i && \text{for } i = 1,\dots,\hat{n}, \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \quad & \operatorname{vec}(I_p)^T \left(e_j e_i^T \otimes C\right) \left(-\hat{\Lambda} \otimes I_n - I_{\hat{n}} \otimes A\right)^{-1} \left(\tilde{B}^T \otimes B\right) \operatorname{vec}(I_m) \\ = & \operatorname{vec}(I_p)^T \left(e_j e_i^T \otimes \hat{C}\right) \left(-\hat{\Lambda} \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A}\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \operatorname{vec}(I_m), \\ & \text{for } i = 1,\dots,\hat{n} \text{ and } j = 1,\dots,p. \end{aligned}$$

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Interpolation of the Transfer Function [Grimme 1997]

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{H}(s) = CV\left(sI - W^T AV\right)^{-1} W^T B,$$

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$$V = \left[(\sigma_1 I - A)^{-1}B, \dots, (\sigma_r I - A)^{-1}B\right],$$
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Then

$$H(\sigma_i) = \hat{H}(\sigma_i) \quad \text{and} \quad H'(\sigma_i) = \hat{H}'(\sigma_i),$$

for $i = 1, \ldots, r$.

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for $i = 1, \ldots, r$.
Starting with an initial guess for $\hat{\Lambda}$ and setting $\sigma_i \equiv -\hat{\lambda}_i \rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. 2006/08], [Bunse-Gerstner et al. 2007], [Van Dooren et al. 2008]

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Some background

Consider bilinear system

$$\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t) u_i(t) + Bu(t), \quad y(t) = Cx(t). \end{cases}$$

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Output Characterization (SISO): Volterra series

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} K(t_1, \dots, t_k) u(t - t_1 - \dots - t_k) \cdots u(t - t_k) dt_k \cdots dt_1,$$

with kernels $K(t_1, \dots, t_k) = Ce^{At_k} A_1 \cdots e^{At_2} A_1 e^{At_1} B$.

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Multivariate Laplace-transform:

$$H_k(s_1, \dots, s_k) = C(s_k I - A)^{-1} N_1 \cdots (s_2 I - A)^{-1} N_1 (s_1 I - A)^{-1} B.$$

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Bilinear $\mathcal{H}_2$-norm: [Zhang/Lam 2002]

$$||\Sigma||_{\mathcal{H}_2} := \left(\mathrm{tr} \left(\left(\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} \overline{H_k(i\omega_1,\dots,i\omega_k)} H_k^T(i\omega_1,\dots,i\omega_k) \right) \right) \right)^{\frac{1}{2}}.$$

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Measuring the Approximation Error

Lemma [B./Breiten 2012]

Let Σ denote a bilinear system. Then, the $\mathcal{H}_2$-norm is given as:

$$||\Sigma||_{\mathcal{H}_2}^2 = (\mathrm{vec}(I_p))^T (C \otimes C) \left(-A \otimes I - I \otimes A - \sum_{i=1}^{m} A_i \otimes A_i \right)^{-1} (B \otimes B) \, \mathrm{vec}(I_m).$$

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Error System

In order to find an $\mathcal{H}_2$-optimal reduced system, define the error system $\Sigma^{err} := \Sigma - \hat{\Sigma}$ as follows:

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad A_i^{err} = \begin{bmatrix} A_i & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

$\mathcal{H}_2$-Model Reduction

$\mathcal{H}_2$-Optimality Conditions

Let us assume $\hat{\Sigma}$ is given by its eigenvalue decomposition:

$$\hat{A} = R\Lambda R^{-1}, \quad \tilde{A}_i = R^{-1}\hat{A}_i R, \quad \tilde{B} = R^{-1}\hat{B}, \quad \tilde{C} = \hat{C}R.$$

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Using Λ, $\tilde{A}_i$, $\tilde{B}$, $\tilde{C}$ as optimization parameters, we can derive necessary conditions for $\mathcal{H}_2$-optimality, e.g.:

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$$(\mathrm{vec}(I_q))^T \left(e_j e_\ell^T \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^m \tilde{A}_i \otimes A_i\right)^{-1} \left(\tilde{B} \otimes B\right) \mathrm{vec}(I_m)$$

$$= (\mathrm{vec}(I_q))^T \left(e_j e_\ell^T \otimes \hat{C}\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A} - \sum_{i=1}^m \tilde{A}_i \otimes \hat{A}_i\right)^{-1} \left(\tilde{B} \otimes \hat{B}\right) \mathrm{vec}(I_m).$$

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Connection to interpolation of transfer functions?

$\mathcal{H}_2$-Model Reduction

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For $A_i \equiv 0$, this is equivalent to

$$H(-\lambda_\ell)\tilde{B}_\ell^T = \hat{H}(-\lambda_\ell)\tilde{B}_\ell^T$$

$\rightsquigarrow$ tangential interpolation at mirror images of reduced system poles!

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Note: [Flagg 2011] shows equivalence to interpolating the Volterra series!

A First Iterative Approach

Algorithm 1 Bilinear IRKA

Input: A, A_i, B, C, $\hat{A}$, $\hat{A}_i$, $\hat{B}$, $\hat{C}$

Output: A^{opt}, A_i^{opt}, B^{opt}, C^{opt}

while (change in $\Lambda > \epsilon$) **do**

$R\Lambda R^{-1} = \hat{A}$, $\tilde{B} = R^{-1}\hat{B}$, $\tilde{C} = \hat{C}R$, $\tilde{A}_i = R^{-1}\hat{A}_i R$

$$\text{vec}(V) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^{m} \tilde{A}_i \otimes A_i\right)^{-1} \left(\tilde{B} \otimes B\right) \text{vec}(I_m)$$

$$\text{vec}(W) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A^T - \sum_{i=1}^{m} \tilde{A}_i^T \otimes A_i^T\right)^{-1} \left(\tilde{C}^T \otimes C^T\right) \text{vec}(I_q)$$

$V = \text{orth}(V)$, $W = \text{orth}(W)$

$\hat{A} = \left(W^T V\right)^{-1} W^T A V$, $\hat{A}_i = \left(W^T V\right)^{-1} W^T A_i V$,
$\hat{B} = \left(W^T V\right)^{-1} W^T B$, $\hat{C} = CV$

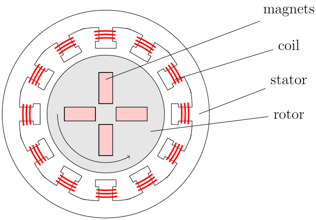
end while

$A^{opt} = \hat{A}$, $A_i^{opt} = \hat{A}_i$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Industrial Case Study: Thermal Analysis of Electrical Motor

- Thermal simulations to detect whether temperature changes lead to fatigue or deterioration of employed materials.
- Main heat source: thermal losses resulting from current stator coil/rotor.
- Many different current profiles need to be considered to predict whether temperature on certain parts of the motor remans in feasible region.
- Finite element analysis on rather complicated geometries $\rightsquigarrow$ large-scale linear models with many (here: 7/13) parameters.



Schematic view of an electrical motor.

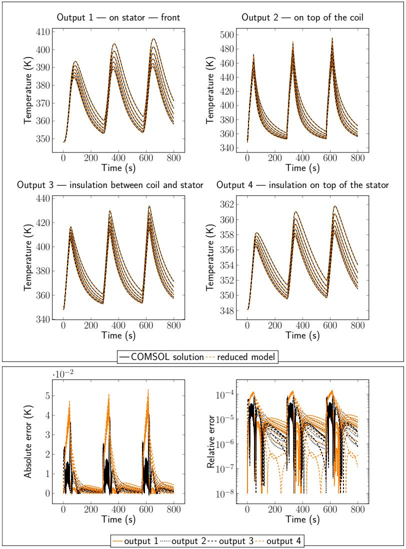
Bosch integrated motor generator used in hybrid variants of Porsche Cayenne, VW Touareg.

Pictures: BOSCH

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Industrial Case Study: Thermal Analysis of Electrical Motor

- FEM analysis of thermal model $\rightsquigarrow$ linear parametric systems with $n = 41,199$, $m = 4$ inputs, and $d = 13$ parameters,
- measurements taken at $q = 4$ heat sensors;
- time for 1 transient simulation in COMSOL© $\sim$ 90min;
- ROM order $\hat{n} = 300$, time for 1 transient simulation $\sim$ 15sec.
- Legend: Temperature curves for six different values (5, 25, 45, 65, 85, 100$[W/m^2K]$) of the heat transfer coefficient on the coil.



Conclusions and Outlook

- We have established a connection between special linear parametric and bilinear systems that automatically yields structure-preserving model reduction techniques for linear parametric systems.

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  - Under certain assumptions, we can expect the existence of low-rank approximations to the solution of generalized Lyapunov equations.
  - Solutions strategies via extending the ADI iteration to bilinear systems and EKSM as well as using preconditioned iterative solvers like CG or BiCGstab up to dimensions $n \sim 500,000$ in MATLAB®.
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  - Optimal choice of shift parameters for ADI is a nontrivial task.
  - Existence of low-rank solutions in case of A_i being full rank?
- $\mathcal{H}_2$ **optimal model reduction:**
  - Yields competitive approach, proven in industrial context.
  - Still high offline cost (= time for generating reduced-order model).
  - May need to switch to one-sided projection ($W = V$) to preserve stability.

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