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Parametric Model Order Reduction of Dynamical Systems: Survey and Recent Advances

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H₂-Model Reduction for Bilinear Systems



Overview

Introduction to Parametric Model Order Reduction

- Dynamical systems
- Parametric Systems as Bilinear Systems
- 2 Balanced Truncation for Bilinear Systems
 - Balanced Truncation for Linear Systems
 - Bilinear Systems
 - Existence of low-rank approximations
 - Numerical Methods
 - Application to Parametric MOR
- 3 \mathcal{H}_2 -Model Reduction for Bilinear Systems
 - \bullet $\mathcal{H}_2\text{-}\mathsf{Model}$ Reduction for Linear Systems
 - Industrial Case Study: Thermal Analysis of Electrical Motor



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Introduction to Parametric Model Order Reduction Parametric Dynamical Systems

Dynamical Systems

$$\Sigma(p): \begin{cases} E(p)\dot{x}(t;p) &= f(t,x(t;p),u(t),p), \quad x(t_0) = x_0, \quad (a) \\ y(t;p) &= g(t,x(t;p),u(t),p) \quad (b) \end{cases}$$

with

- (generalized) states $x(t; p) \in \mathbb{R}^n$ $(E \in \mathbb{R}^{n \times n})$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t; p) \in \mathbb{R}^q$, (b) is called output equation,
- $p \in \Omega \subset \mathbb{R}^d$ is a parameter vector, Ω is bounded.

Applications:

- Repeated simulation for varying material or geometry parameters, boundary conditions,
- control, optimization and design.

Introduction to Parametric Model Order Reduction

Linear Parametric Systems

Linear, time-invariant (parametric) systems

$$\begin{array}{rcl} E(p)\dot{x}(t;p) &=& A(p)x(t;p)+B(p)u(t), & A(p), E(p) \in \mathbb{R}^{n \times n}, \\ y(t;p) &=& C(p)x(t;p), & B(p) \in \mathbb{R}^{n \times m}, C(p) \in \mathbb{R}^{q \times n}. \end{array}$$

Introduction to Parametric Model Order Reduction



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Laplace Transformation / Frequency Domain

Application of Laplace transformation $(x(t; p) \mapsto x(s; p), \dot{x}(t; p) \mapsto sx(s; p))$ to linear system with x(0) = 0:

 $sE(p)x(s;p) = A(p)x(s;p) + B(p)u(s), \quad y(s;p) = C(p)x(s;p),$

yields I/O-relation in frequency domain:

$$y(s;p) = \left(\underbrace{C(p)(sE(p) - A(p))^{-1}B(p)}_{=:H(s;p)}\right)u(s).$$

H(s; p) is the parameter-dependent transfer function of $\Sigma(p)$.

Introduction to Parametric Model Order Reduction

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H(s; p) is the parameter-dependent transfer function of $\Sigma(p)$. Goal: Fast evaluation of mapping $(u, p) \rightarrow y(s; p)$.

H₂-Model Reduction for Bilinear Systems

Introduction to Parametric Model Order Reduction Motivating Example: Microsystems/MEMS Design

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Microgyroscope (butterfly gyro)



- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order: $N = 17.361 \rightsquigarrow n = 34.722, m = 1, q = 12.$
- Sensor for position control based on acceleration and rotation

• Application: inertial navigation.



Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Balanced Truncation for Bilinear Systems

H₂-Model Reduction for Bilinear Systems

Conclusions and Outlool

Introduction to Parametric Model Order Reduction Motivating Example: Microsystems/MEMS Design



Microgyroscope (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t)$.



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Conclusions and Outlool

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Introduction to Parametric Model Order Reduction Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model:

$$M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t),$$

wobei

$$\begin{array}{lll} \mathcal{M}(d) &=& \mathcal{M}_1 + d\mathcal{M}_2, \\ \mathcal{D}(\theta, d, \alpha, \beta) &=& \theta(\mathcal{D}_1 + d\mathcal{D}_2) + \alpha \mathcal{M}(d) + \beta \mathcal{T}(d), \\ \mathcal{T}(d) &=& \mathcal{T}_1 + \frac{1}{d} \mathcal{T}_2 + d\mathcal{T}_3, \end{array}$$

with

- width of bearing: *d*,
- angular velocity: θ,
- Rayleigh damping parameters: α, β .





H₂-Model Reduction for Bilinear Systems

Conclusions and Outlook

Introduction to Parametric Model Order Reduction Motivating Example: Microsystems/MEMS Design





Balanced Truncation for Bilinear Systems

H₂-Model Reduction for Bilinear Systems

The Model Order Reduction (MOR) Problem

Problem

Approximate the dynamical system

$$\begin{array}{rcl} E(p)\dot{x} &=& A(p)x + B(p)u, \\ y &=& C(p)x, \end{array}$$

 $E(p), A(p) \in \mathbb{R}^{n \times n}, \ B(p) \in \mathbb{R}^{n \times m}, C(p) \in \mathbb{R}^{q \times n},$

by reduced-order system

$$\begin{array}{rcl} \hat{E}(p)\dot{\hat{x}} &=& \hat{A}(p)\hat{x}+\hat{B}(p)u, \quad \hat{E}(p), \hat{A}(p)\in \mathbb{R}^{r\times r}, \\ \hat{y} &=& \hat{C}(p)\hat{x}, \qquad \qquad \hat{B}(p)\in \mathbb{R}^{r\times m}, \hat{C}(p)\in \mathbb{R}^{q\times r}, \end{array}$$

of order $r \ll n$, such that

$$\|y-\hat{y}\| = \|Hu-\hat{H}u\| \leq \|H-\hat{H}\|\cdot\|u\| < ext{tolerance}\cdot\|u\| \quad orall \ p\in \Omega$$



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of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Hu - \hat{H}u\| \le \|H - \hat{H}\| \cdot \|u\| < ext{tolerance} \cdot \|u\| \quad \forall \ p \in \Omega.$$

 \implies Approximation problem: $\min_{\text{order}(\hat{H}) \leq r} \|H - \hat{H}\|.$

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Parametric Systems as Bilinear Systems



Linear Parametric Systems — An Alternative Interpretation

Consider bilinear control systems:

$$\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t) u_i(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

where $A, A_i \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{q \times n}$.

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Balanced Truncation for Bilinear Systems

 \mathcal{H}_2 -Model Reduction for Bilinear Systems



Parametric Systems as Bilinear Systems

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where $A, A_i \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \ C \in \mathbb{R}^{q \times n}$.

Key Observation

[B./BREITEN 2011]

Consider parameters as additional inputs, a linear parametric system $\dot{x}(t) = Ax(t) + \sum_{i=1}^{m_p} a_i(p)A_ix(t) + B_0u_0(t), \quad y(t) = Cx(t)$

with $B_0 \in \mathbb{R}^{n \times m_0}$ can be interpreted as bilinear system:

$$\begin{aligned} u(t) &:= \begin{bmatrix} a_1(p) & \dots & a_{m_p}(p) & u_0(t) \end{bmatrix}^T, \\ B &:= \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & B_0 \end{bmatrix} \in \mathbb{R}^{n \times m}, \quad m = m_p + m_0. \end{aligned}$$

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Parametric Systems as Bilinear Systems

Linear Parametric Systems — An Alternative Interpretation

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Linear parametric systems can be interpreted as bilinear systems.

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Parametric Systems as Bilinear Systems

Linear Parametric Systems — An Alternative Interpretation

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Linear parametric systems can be interpreted as bilinear systems.

Consequence

Model order reduction techniques for bilinear systems can be applied to linear parametric systems!

Here:

- Balanced truncation,
- \mathcal{H}_2 optimal model reduction.

Balanced Truncation for Bilinear Systems

H₂-Model Reduction for Bilinear Systems

Conclusions and Outlool

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Balanced Truncation for Linear Systems

Idea (for simplicity,
$$E = I_n$$
)

•
$$\Sigma$$
:

$$\begin{cases}
\dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t), \\
\text{is balanced, if system Gramians, i.e., solutions } P, Q \text{ of the Lyapunov} \\
\text{equations}
\end{cases}$$

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

satisfy:
$$P = Q = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$$
 with $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_n > 0$.

Balanced Truncation for Bilinear Systems

H₂-Model Reduction for Bilinear Systems

Conclusions and Outlool

Balanced Truncation for Linear Systems

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$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$$
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• $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ .

Balanced Truncation for Bilinear Systems

H₂-Model Reduction for Bilinear Systems

Conclusions and Outlool

Balanced Truncation for Linear Systems



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- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ .
- Compute balanced realization (needs *P*, *Q*!) of the system via state-space transformation

$$\begin{aligned} \mathcal{T}: (A, B, C) &\mapsto (TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

Balanced Truncation for Bilinear Systems

H₂-Model Reduction for Bilinear Systems

Balanced Truncation for Linear Systems



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• Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1).$

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Conclusions and Outlool

Balanced Truncation for Linear Systems



Properties

• Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_{\hat{n}}$.

Balanced Truncation for Bilinear Systems

H₂-Model Reduction for Bilinear Systems

Balanced Truncation for Linear Systems



Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_{\hat{n}}$.
- Adaptive choice of *r* via computable error bound:

$$\|y - \hat{y}\|_2 \le \left(2\sum_{k=\hat{n}+1}^n \sigma_k\right) \|u\|_2.$$

Balanced Truncation for Bilinear Systems

H₂-Model Reduction for Bilinear Systems

Conclusions and Outlool

Balanced Truncation for Linear Systems



Properties

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- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \le \left(2\sum_{k=\hat{n}+1}^n \sigma_k\right) \|u\|_2.$$

Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S!$
- No $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

H₂-Model Reduction for Bilinear Systems

Conclusions and Outlool

Balanced Truncation for Bilinear Systems



Bilinear Control Systems — Theory and Background

Bilinear control systems:

$$\Sigma: \quad \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t) u_i(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

where $A, A_i \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \ C \in \mathbb{R}^{q \times n}$.

Properties:

- Approximation of (weakly) nonlinear systems by Carleman linearization yields bilinear systems.
- Appear naturally in boundary control problems, control via coefficients of PDEs, Fokker-Planck equations, ...
- Due to the close relation to linear systems, a lot of successful concepts can be extended, e.g. transfer functions, Gramians, Lyapunov equations, . . .
- Linear stochastic control systems possess an equivalent structure and can be treated alike [B./DAMM 2011].

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Balanced Truncation for Bilinear Systems



$$AP + PA^{T} + \sum_{i=1}^{m} A_{i}PA_{i}^{T} + BB^{T} = 0,$$
$$A^{T}Q + QA^{T} + \sum_{i=1}^{m} A_{i}^{T}QA_{i} + C^{T}C = 0.$$

- These equations also appear for stochastic control systems, see [B./DAMM 2011].
- "Twice-the-trail-of-the-HSVs" error bound does not hold [B./DAMM 2014], stability preservation not yet proved.



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Conclusions and Outlool

Balanced Truncation for Bilinear Systems

Some basic facts and assumptions

$$AX + XA^{T} + \sum_{i=1}^{m} A_{i}XA_{i}^{T} + BB^{T} = 0.$$
 (1)

• Need a positive semi-definite symmetric solution X.



 \mathcal{H}_2 -Model Reduction for Bilinear Systems

Balanced Truncation for Bilinear Systems

Some basic facts and assumptions

$$AX + XA^{T} + \sum_{i=1}^{m} A_{i}XA_{i}^{T} + BB^{T} = 0.$$
 (1)

- Need a positive semi-definite symmetric solution X.
- In standard Lyapunov case, existence and uniqueness guaranteed if A stable (Λ(A) ⊂ C[−]); this is not sufficient here: (1) is equivalent to

$$\left(I_n\otimes A + A\otimes I_n + \sum_{i=1}^m A_i\otimes A_i\right)\operatorname{vec}(X) = -\operatorname{vec}(BB^T).$$

One sufficient condition for stable A is smallness of A_i (related to stability radius of A)

 \rightsquigarrow bounded-input bounded-output (BIBO) stability of bilinear systems.

This will be assumed from here on, hence: existence and uniqueness of positive semi-definite solution $X = X^{T}$.

H₂-Model Reduction for Bilinear Systems

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• Want: solution methods for large scale problems, i.e., only matrix-matrix multiplication with *A*, *A_j*, solves with (shifted) *A* allowed!

H₂-Model Reduction for Bilinear Systems

Balanced Truncation for Bilinear Systems

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→ bounded-input bounded-output (BIBO) stability of bilinear systems.

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- Want: solution methods for large scale problems, i.e., only matrix-matrix multiplication with *A*, *A_j*, solves with (shifted) *A* allowed!
- Requires to compute data-sparse approximation to generally dense X; here: X ≈ ZZ^T with Z ∈ ℝ^{n×nz}, n_Z ≪ n!

H₂-Model Reduction for Bilinear Systems

Conclusions and Outlool

Balanced Truncation for Bilinear Systems

Existence of low-rank approximations

Q: Can we expect low-rank approximations $ZZ^T \approx X$ to the solution of

$$AX + XA^{T} + \sum_{j=1}^{m} A_j XA_j^{T} + BB^{T} = 0 ?$$



H₂-Model Reduction for Bilinear Systems

Conclusions and Outlool

Balanced Truncation for Bilinear Systems

Existence of low-rank approximations

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$$AX + XA^{T} + \sum_{j=1}^{m} A_{j}XA_{j}^{T} + BB^{T} = 0$$
?

Theorem

Assume existence and uniqueness assumption with stable A and $A_j = U_j V_j^T$, with $U_j, V_j \in \mathbb{R}^{n \times r_j}$. Set $r = \sum_{j=1}^m r_j$. Then the solution X of

$$AX + XA^T + \sum_{j=1}^m A_j XA_j^T + BB^T = 0$$

can be approximated by X_k of rank (2k+1)(m+r), with an error satisfying

$$\|X-X_k\|_2 \lesssim \exp(-\sqrt{k}).$$

Max Planck Institute Magdeburg



[B./BREITEN 2013]

H₂-Model Reduction for Bilinear Systems

Balanced Truncation for Bilinear Systems



- Generalized Alternating Directions Iteration (ADI) method.
 - Computing square solution matrix ($\sim n^2$ unknowns) [DAMM 2008].
 - Computing low-rank factors of solutions (~ n unknowns) [B./BREITEN 2013].
- Generalized Extended (or rational) Krylov Subspace Method (EKSM) [B./BREITEN 2013].
- Tensorized versions of standard Krylov subspace methods, e.g., PCG, PBiCGStab [Kressner/Tobler 2011, B./Breiten 2013].

Balanced Truncation for Bilinear Systems

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Balanced Truncation for Bilinear Systems



Numerical Examples: Heat Equation with Boundary Control



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Conclusions and Outlool

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Numerical Examples: Fokker-Planck Equation



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Numerical Examples: RC Circuit Simulation



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Comparison of CPU times

	Heat equation	RC circuit	Fokker-Planck
Bilin. ADI 2 \mathcal{H}_2 shifts	-	-	1.733 (1.578)
Bilin. ADI 6 \mathcal{H}_2 shifts	144,065 (2,274)	20,900 (3091)	-
Bilin. ADI 8 \mathcal{H}_2 shifts	135,711 (3,177)	-	-
Bilin. ADI 10 \mathcal{H}_2 shifts	33,051 (4,652)	-	-
Bilin. ADI 2 Wachspress shifts	-	-	6.617 (4.562)
Bilin. ADI 4 Wachspress shifts	41,883 (2,500)	18,046 (308)	-
CG (Bilin. ADI precond.)	15,640	-	-
BiCG (Bilin. ADI precond.)	-	16,131	11.581
BiCG (Linear ADI precond.)	-	12,652	9.680
EKSM	7,093	19,778	8.555

Numbers in brackets: computation of shift parameters.



H₂-Model Reduction for Bilinear Systems

Application to Parametric MOR

Fast Simulation of Cyclic Voltammogramms [Feng/Koziol/Rudnyi/Korvink '06]

$$\begin{aligned} E\dot{x}(t) &= (A + p_1(t)A_1 + p_2(t)A_2)x(t) + B, \\ y(t) &= Cx(t), \quad x(0) = x_0 \neq 0, \end{aligned}$$

- Rewrite as system with zero initial condition,
- FE model: n = 16,912, m = 3, q = 1,
- *p_j* ∈ [0, 10⁹] time-varying voltage functions,
- transfer function $H(i\omega, p_1, p_2)$,
- reduced system dimension r = 67,

$$\bullet \max_{\substack{\omega \in \{\omega_{min}, \dots, \omega_{max}\}\\ p_j \in \{p_{min}, \dots, p_{max}\}}} \frac{||H - \hat{H}||_2}{||H||_2} < 6 \cdot 10^{-4},$$

evaluation times: FOM 4.5h, ROM 38s
 → speed-up factor ≈ 426.



Balanced Truncation for Bilin

H₂-Model Reduction for Bilinear Systems

Application to Parametric MOR

Fast Simulation of Cyclic Voltammogramms [Feng/Koziol/Rudnyi/Korvink '06]

Original...





Application to Parametric MOR 2D Model of an Anemometer [Baur et al. '10]





Figure : [BAUR ET AL. '10]

Consider an anemometer, a flow sensing device located on a membrane used in context of minimizing heat dissipation.

$$E\dot{x}(t) = (A + pA_1)x(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = 0,$$

- FE model: n = 29,008, m = 1, q = 3,
- $p_1 \in [0,1]$ fluid velocity,
- transfer function $H(i\omega, p_1)$, reduced system dimension r = 146,

$$\bullet \max_{\substack{\omega \in \{\omega_{min},\ldots,\omega_{max}\}\\ p_1 \in \{p_{min},\ldots,p_{max}\}}} \frac{\|H(\omega,p)-\hat{H}(\omega,p)\|_2}{\|H(\omega,p)\|_2} < 3 \cdot 10^{-5},$$

• evaluation times: FOM 51min, ROM 21sec.

Balanced Truncation for Bilinear Systems

*H*₂-Model Reduction for Bilinear Systems ●00000000 Conclusions and Outlool

\mathcal{H}_2 -Model Reduction for Bilinear Systems \mathcal{H}_2 -Model Reduction for Linear Systems



First consider stable (i.e. $\Lambda(A) \subset \mathbb{C}^-$) linear systems,

 $\dot{x}(t) = Ax(t) + Bu(t), \ y(t) = Cx(t) \simeq Y(s) = C(sI - A)^{-1}BU(s)$

System norms

Two common system norms for measuring approximation quality:

•
$$\mathcal{H}_2$$
-norm, $\|\Sigma\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\left((H^*(-i\omega)H(i\omega))\right)d\omega\right)^{\frac{1}{2}}$,

•
$$\mathcal{H}_{\infty}$$
-norm, $\|\Sigma\|_{\mathcal{H}_{\infty}} = \sup_{\omega \in \mathbb{R}} \sigma_{\max} \left(\mathcal{H}(i\omega) \right)$,

where

$$H(s)=C\left(sI-A\right) ^{-1}B.$$

Note: \mathcal{H}_{∞} -norm approximation \rightsquigarrow balanced truncation.

Conclusions and Outlool

\mathcal{H}_2 -Model Reduction for Bilinear Systems Error system and \mathcal{H}_2 -Optimality [Meier/

Meier/Luenberger 1967]

In order to find an \mathcal{H}_2 -optimal reduced system, consider the error system $H(s) - \hat{H}(s)$ which can be realized by

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}$$

 \mathcal{H}_2 -Model Reduction for Bilinear Systems

Conclusions and Outlool





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Assuming a coordinate system in which \hat{A} is diagonal and taking derivatives of

$$\|H(\,.\,)-\hat{H}(\,.\,)\|^2_{\mathcal{H}_2}$$

with respect to free parameters in $\Lambda(\hat{A}), \hat{B}, \hat{C} \rightarrow$ first-order necessary \mathcal{H}_2 -optimality conditions (SISO)

$$H(-\hat{\lambda}_i) = \hat{H}(-\hat{\lambda}_i),$$

$$H'(-\hat{\lambda}_i) = \hat{H}'(-\hat{\lambda}_i),$$

where $\hat{\lambda}_i$ are the poles of the reduced system $\hat{\Sigma}$.

 \mathcal{H}_2 -Model Reduction for Bilinear Systems 000000000

\mathcal{H}_2 -Model Reduction for Bilinear Systems Error system and \mathcal{H}_2 -Optimality [Meier/



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First-order necessary \mathcal{H}_2 -optimality conditions (MIMO):

$$\begin{aligned} & \mathcal{H}(-\hat{\lambda}_i)\tilde{B}_i = \hat{\mathcal{H}}(-\hat{\lambda}_i)\tilde{B}_i, & \text{for } i = 1, \dots, \hat{n}, \\ & \tilde{C}_i^{\,\mathsf{T}}\mathcal{H}(-\hat{\lambda}_i) = \tilde{C}_i^{\,\mathsf{T}}\hat{\mathcal{H}}(-\hat{\lambda}_i), & \text{for } i = 1, \dots, \hat{n}, \\ & \tilde{c}_i^{\,\mathsf{T}}\mathcal{H}'(-\hat{\lambda}_i)\tilde{B}_i = \tilde{C}_i^{\,\mathsf{T}}\hat{\mathcal{H}}'(-\hat{\lambda}_i)\tilde{B}_i & \text{for } i = 1, \dots, \hat{n}, \end{aligned}$$

where $\hat{A} = R\hat{A}R^{-T}$ is the spectral decomposition of the reduced system and $\tilde{B} = \hat{B}^T R^{-T}$, $\tilde{C} = \hat{C}R$.

 \mathcal{H}_2 -Model Reduction for Bilinear Systems 000000000

H₂-Model Reduction for Bilinear Systems Error system and H₂-Optimality [Meier/Luenberger 1967]

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$$H(-\hat{\lambda}_{i})\tilde{B}_{i} = \hat{H}(-\hat{\lambda}_{i})\tilde{B}_{i}, \qquad \text{for } i = 1, \dots, \hat{n},$$

$$\tilde{C}_{i}^{T}H(-\hat{\lambda}_{i}) = \tilde{C}_{i}^{T}\hat{H}(-\hat{\lambda}_{i}), \qquad \text{for } i = 1, \dots, \hat{n},$$

$$\tilde{C}_{i}^{T}H'(-\hat{\lambda}_{i})\tilde{B}_{i} = \tilde{C}_{i}^{T}\hat{H}'(-\hat{\lambda}_{i})\tilde{B}_{i} \qquad \text{for } i = 1, \dots, \hat{n},$$

$$vec(I_{i})^{T}(e:e^{T} \otimes C)(-\hat{\lambda} \otimes I_{i} = h \otimes A)^{-1}(\tilde{B}^{T} \otimes B)vec(I_{i})$$

$$= \operatorname{vec}(I_q)^T \left(e_j e_i^T \otimes \hat{C} \right) \left(-\hat{\Lambda} \otimes I_n - I_n \otimes \hat{\Lambda} \right)^{-1} \left(\tilde{B}^T \otimes \hat{B} \right) \operatorname{vec}(I_m)$$
$$= \operatorname{vec}(I_q)^T \left(e_j e_i^T \otimes \hat{C} \right) \left(-\hat{\Lambda} \otimes I_n - I_n \otimes \hat{\Lambda} \right)^{-1} \left(\tilde{B}^T \otimes \hat{B} \right) \operatorname{vec}(I_m),$$
for $i = 1, \dots, \hat{n}$ and $j = 1, \dots,$

q.

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*H*₂-Model Reduction for Bilinear Systems ○○●○○○○○○ Conclusions and Outlook

H₂-Model Reduction for Bilinear Systems Interpolation of the Transfer Function [Grimme 1997]



Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{H}(s) = CV \left(sI - W^{\mathsf{T}} AV \right)^{-1} W^{\mathsf{T}} B,$$

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$$V = \left[(\sigma_1 I - A)^{-1} B, \dots, (\sigma_r I - A)^{-1} B \right],$$

$$W = \left[(\sigma_1 I - A^T)^{-1} C^T, \dots, (\sigma_r I - A^T)^{-1} C^T \right]$$

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Then

$$H(\sigma_i) = \hat{H}(\sigma_i)$$
 and $H'(\sigma_i) = \hat{H}'(\sigma_i)$,

for i = 1, ..., r.

Balanced Truncation for Bilinear Systems

*H*₂-Model Reduction for Bilinear Systems ○●●○○○○○ Conclusions and Outlool

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$$H(\sigma_i) = \hat{H}(\sigma_i)$$
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for $i = 1, \ldots, r$. Starting with an initial guess for $\hat{\Lambda}$ and setting $\sigma_i \equiv -\hat{\lambda}_i \rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield \mathcal{H}_2 -optimal models.

> [Gugercin et al. 2006/08], [Bunse-Gerstner et al. 2007], [Van Dooren et al. 2008]

alanced Truncation for Bilinear Systems

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Conclusions and Outlool

\mathcal{H}_2 -Model Reduction for Bilinear Systems



Consider bilinear system

$$\Sigma: \left\{ \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t) u_i(t) + Bu(t), \quad y(t) = Cx(t). \right.$$

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*H*₂-Model Reduction for Bilinear Systems 000●00000 Conclusions and Outlool

Some background

Consider bilinear system

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Output Characterization (SISO): Volterra series

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} K(t_1, \ldots, t_k) u(t-t_1-\ldots-t_k) \cdots u(t-t_k) dt_k \cdots dt_1,$$

with kernels $K(t_1, \ldots, t_k) = Ce^{At_k}A_1 \cdots e^{At_2}A_1e^{At_1}B$.

 \mathcal{H}_2 -Model Reduction for Bilinear Systems

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Conclusions and Outlool

\mathcal{H}_2 -Model Reduction for Bilinear Systems



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Multivariate Laplace-transform:

$$H_k(s_1,\ldots,s_k) = C(s_kI - A)^{-1}N_1\cdots(s_2I - A)^{-1}N_1(s_1I - A)^{-1}B.$$

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*H*₂-Model Reduction for Bilinear Systems 000●00000 Conclusions and Outlool

\mathcal{H}_2 -Model Reduction for Bilinear Systems



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Bilinear \mathcal{H}_2 -norm:

[Zhang/Lam 2002]

$$||\boldsymbol{\Sigma}||_{\mathcal{H}_2} := \left(\operatorname{tr} \left(\left(\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} \ \overline{H_k(i\omega_1, \dots, i\omega_k)} H_k^{\mathsf{T}}(i\omega_1, \dots, i\omega_k) \right) \right) \right)^{\frac{1}{2}}.$$

Max Planck Institute Magdeburg

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Conclusions and Outlool



\mathcal{H}_2 -Model Reduction for Bilinear Systems

Measuring the Approximation Error

Lemma

[B./BREITEN 2012]

Let Σ denote a bilinear system. Then, the $\mathcal{H}_2\text{-norm}$ is given as:

$$||\Sigma||_{\mathcal{H}_2}^2 = \left(\mathsf{vec}(I_q)\right)^T \left(C \otimes C\right) \left(-A \otimes I - I \otimes A - \sum_{i=1}^m A_i \otimes A_i\right)^{-1} \left(B \otimes B\right) \mathsf{vec}(I_m).$$

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Conclusions and Outlool



\mathcal{H}_2 -Model Reduction for Bilinear Systems

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Error System

In order to find an $\mathcal{H}_2\text{-optimal}$ reduced system, define the error system $\Sigma^{err}:=\Sigma-\hat{\Sigma}$ as follows:

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad A_i^{err} = \begin{bmatrix} A_i & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

 \mathcal{H}_2 -Model Reduction for Bilinear Systems

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\mathcal{H}_2 -Model Reduction \mathcal{H}_2 -Optimality Conditions

Let us assume $\hat{\Sigma}$ is given by its eigenvalue decomposition:

$$\hat{A} = R\Lambda R^{-1}, \quad \tilde{A}_i = R^{-1}\hat{A}_i R, \quad \tilde{B} = R^{-1}\hat{B}, \quad \tilde{C} = \hat{C}R.$$

 \mathcal{H}_2 -Model Reduction for Bilinear Systems 000000000

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Using Λ , \tilde{A}_i , \tilde{B} , \tilde{C} as optimization parameters, we can derive necessary conditions for \mathcal{H}_2 -optimality, e.g.:

\mathcal{H}_2 -Model Reduction \mathcal{H}_2 -Optimality Conditions



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$$(\operatorname{vec}(I_q))^T \left(e_j e_\ell^T \otimes C \right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^m \tilde{A}_i \otimes A_i \right)^{-1} \left(\tilde{B} \otimes B \right) \operatorname{vec}(I_m)$$

= $(\operatorname{vec}(I_q))^T \left(e_j e_\ell^T \otimes \hat{C} \right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A} - \sum_{i=1}^m \tilde{A}_i \otimes \hat{A}_i \right)^{-1} \left(\tilde{B} \otimes \hat{B} \right) \operatorname{vec}(I_m).$

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Connection to interpolation of transfer functions?

\mathcal{H}_2 -Model Reduction \mathcal{H}_2 -Optimality Conditions



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For $A_i \equiv 0$, this is equivalent to

$$H(-\lambda_\ell) ilde{B}_\ell^{\mathsf{T}} = \hat{H}(-\lambda_\ell) ilde{B}_\ell^{\mathsf{T}}$$

 \rightsquigarrow tangential interpolation at mirror images of reduced system poles!

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 \mathcal{H}_2 -Model Reduction \mathcal{H}_2 -Optimality Conditions

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 \rightsquigarrow tangential interpolation at mirror images of reduced system poles!

Note: [FLAGG 2011] shows equivalence to interpolating the Volterra series!

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A First Iterative Approach

Algorithm 1 Bilinear IRKA

nput:
$$A, A_i, B, C, \hat{A}, \hat{A}_i, \hat{B}, \hat{C}$$

Dutput: $A^{opt}, A^{opt}_i, B^{opt}, C^{opt}$
1: while (change in $\Lambda > \epsilon$) do
2: $R\Lambda R^{-1} = \hat{A}, \tilde{B} = R^{-1}\hat{B}, \tilde{C} = \hat{C}R, \tilde{A}_i = R^{-1}\hat{A}_iR$
3: $\operatorname{vec}(V) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^m \tilde{A}_i \otimes A_i\right)^{-1} \left(\tilde{B} \otimes B\right) \operatorname{vec}(I_m)$
4: $\operatorname{vec}(W) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A^T - \sum_{i=1}^m \tilde{A}_i^T \otimes A_i^T\right)^{-1} \left(\tilde{C}^T \otimes C^T\right) \operatorname{vec}(I_q)$
5: $V = \operatorname{orth}(V), W = \operatorname{orth}(W)$
6: $\hat{A} = (W^T V)^{-1} W^T A V, \hat{A}_i = (W^T V)^{-1} W^T A_i V, \hat{B} = (W^T V)^{-1} W^T B, \hat{C} = C V$
7: end while
8: $A^{opt} = \hat{A}, A^{opt}_i = \hat{A}_i, B^{opt} = \hat{B}, C^{opt} = \hat{C}$

Balanced Truncation for Bilinear Systems

 \mathcal{H}_2 -Model Reduction for Bilinear Systems

\mathcal{H}_2 -Model Reduction for Bilinear Systems



Industrial Case Study: Thermal Analysis of Electrical Motor

- Thermal simulations to detect whether temperature changes lead to fatigue or deterioration of employed materials.
- Main heat source: thermal losses resulting from current stator coil/rotor.
- Many different current profiles need to be considered to predict whether temperature on certain parts of the motor remans in feasible region.
- Finite element analysis on rather complicated geometries → large-scale linear models with many (here: 7/13) parameters.



Schematic view of an electrical motor.



Bosch integrated motor generator used in hybrid variants of Porsche Cayenne, VW Touareg.



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\mathcal{H}_2 -Model Reduction for Bilinear Systems

Industrial Case Study: Thermal Analysis of Electrical Motor

- FEM analysis of thermal model →→ linear parametric systems with n = 41,199, m = 4 inputs, and d = 13 parameters,
- measurements taken at q = 4 heat sensors;
- time for 1 transient simulation in $\mbox{COMSOL}^{\mbox{$\mathbb{R}$}}\sim 90\mbox{min};$
- ROM order î = 300, time for 1 transient simulation ~ 15sec.
- Legend: Temperature curves for six different values (5, 25, 45, 65, 85, 100[W/m²K]) of the heat transfer coefficient on the coil.





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- **Conclusions and Outlook**
 - We have established a connection between special linear parametric and bilinear systems that automatically yields structure-preserving model reduction techniques for linear parametric systems.

Balanced Truncation for Bilinear Systems

H₂-Model Reduction for Bilinear Systems

Conclusions and Outlook



- We have established a connection between special linear parametric and bilinear systems that automatically yields structure-preserving model reduction techniques for linear parametric systems.
- Balanced truncation:
 - Under certain assumptions, we can expect the existence of low-rank approximations to the solution of generalized Lyapunov equations.
 - Solutions strategies via extending the ADI iteration to bilinear systems and EKSM as well as using preconditioned iterative solvers like CG or BiCGstab up to dimensions n ~ 500,000 in MATLAB[®].
 - Optimal choice of shift parameters for ADI is a nontrivial task.
 - Existence of low-rank solutions in case of A_i being full rank?

Balanced Truncation for Bilinear Systems

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Conclusions and Outlook



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 - Existence of low-rank solutions in case of A_i being full rank?
- $\bullet \ \mathcal{H}_2$ optimal model reduction:
 - Yields competitive approach, proven in industrial context.
 - Still high offline cost (= time for generating reduced-order model).
 - May need to switch to one-sided projection (W = V) to preserve stability.



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