Colloquium
School of Mechatronic Engineering
and Automation
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Parametric Model Order Reduction of Dynamical Systems: Survey and Recent Advances

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Overview

Introduction to Parametric Model Order Reduction

Parametric Dynamical Systems

Dynamical Systems

$$\Sigma(p) : \begin{cases} E(p)\dot{x}(t;p) = f(t, x(t;p), u(t), p), \quad x(t_0) = x_0, & \text{(a)} \\ y(t;p) = g(t, x(t;p), u(t), p) & \text{(b)} \end{cases}$$

with

- (generalized) states $x(t;p) \in \mathbb{R}^n$ ($E \in \mathbb{R}^{n \times n}$),
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t;p) \in \mathbb{R}^q$, (b) is called output equation,
- $p \in \Omega \subset \mathbb{R}^d$ is a parameter vector, Ω is bounded.

Applications:

- Repeated simulation for varying material or geometry parameters, boundary conditions,
- control, optimization and design,
- of models, often generated by FE software (e.g., ANSYS, NASTRAN,...) or automatic tools (e.g., Modelica).

Introduction to Parametric Model Order Reduction

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- $p \in \Omega \subset \mathbb{R}^d$ is a parameter vector, Ω is bounded.

PDE and boundary conditions often not accessible!

Introduction to Parametric Model Order Reduction

Linear Parametric Systems

Linear, time-invariant (parametric) systems

$$
\begin{aligned}
E(p)\dot{x}(t;p) &= A(p)x(t;p) + B(p)u(t), \quad && A(p), E(p) \in \mathbb{R}^{n\times n},\\
y(t;p) &= C(p)x(t;p), && B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}.
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Laplace Transformation / Frequency Domain

Application of Laplace transformation $(x(t;p) \mapsto x(s;p),\ \dot{x}(t;p) \mapsto sx(s;p))$ to linear system with $x(0;p) \equiv 0$:

$$sE(p)x(s;p) = A(p)x(s;p) + B(p)u(s), \quad y(s;p) = C(p)x(s;p),$$

yields I/O-relation in frequency domain:

$$y(s;p) = \Big(\underbrace{C(p)(sE(p) - A(p))^{-1}B(p)}_{=:G(s,p)}\Big)u(s).$$

$G(s,p)$ is the parameter-dependent transfer function of $\Sigma(p)$.

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Goal: Fast evaluation of mapping $(u,p) \ \rightarrow \ y(s;p)$.

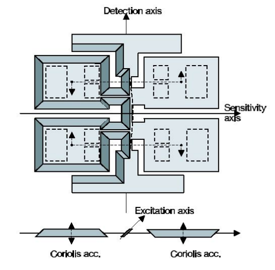
Introduction to Parametric Model Order Reduction

Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

- Applications:
  - inertial navigation,
  - electronic stability control (ESP).

- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order: $N = 17.361 \rightsquigarrow n = 34.722,\ m = 1,\ q = 12.$
- Sensor for position control based on acceleration and rotation.



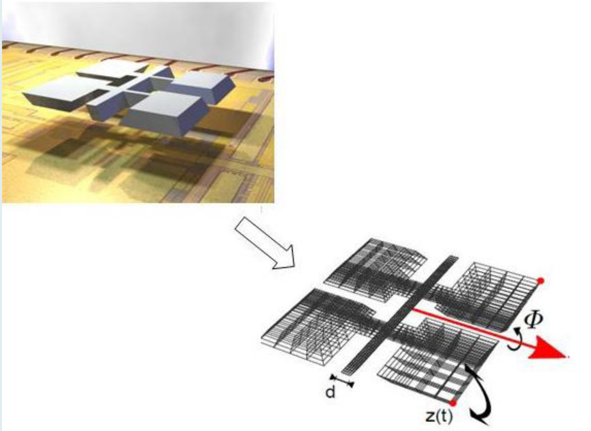
Source: MOR Wiki http://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Gyroscope

Introduction to Parametric Model Order Reduction

Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t).$



Introduction to Parametric Model Order Reduction

Motivating Example: Microsystems/MEMS Design

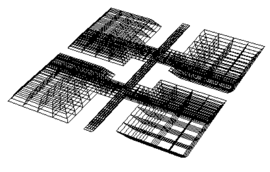
Microgyroscope (butterfly gyro)

Parametric FE model:

$$M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t),$$

where

$$\begin{aligned} M(d) &= M_1 + dM_2, \\ D(\theta, d, \alpha, \beta) &= \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d), \\ T(d) &= T_1 + \frac{1}{d} T_2 + dT_3, \end{aligned}$$



with

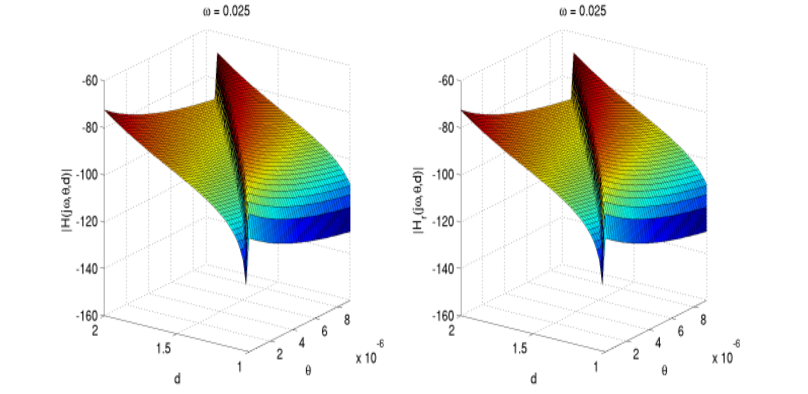
- width of bearing: d,
- angular velocity: θ,
- Rayleigh damping parameters: α, β.

Introduction to Parametric Model Order Reduction

Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Original. . . and reduced-order model.



The Parametric Model Order Reduction (PMOR) Problem

Problem

Approximate the dynamical system

$$\begin{aligned} E(p)\dot{x} &= A(p)x + B(p)u, && E(p), A(p) \in \mathbb{R}^{n\times n}, \\ y &= C(p)x, && B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \hat{E}(p)\dot{\hat{x}} &= \hat{A}(p)\hat{x} + \hat{B}(p)u, && \hat{E}(p), \hat{A}(p) \in \mathbb{R}^{r\times r}, \\ \hat{y} &= \hat{C}(p)\hat{x}, && \hat{B}(p) \in \mathbb{R}^{r\times m}, \hat{C}(p) \in \mathbb{R}^{q\times r}, \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\| \quad \forall\ p \in \Omega.$$

The Parametric Model Order Reduction (PMOR) Problem

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of order $r \ll n$, such that

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$\Longrightarrow$ Approximation problem: $\displaystyle \min_{\text{order}\,(\hat{G}) \leq r} \|G - \hat{G}\|.$

PMOR $\longleftrightarrow$ Multivariate Function Approximation

- Approximate (for fast evaluation) function G, defined on $\mathbb{C} \times \Omega$.

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- But:

$$\begin{aligned} G : \mathbb{C} \times \Omega &\rightarrow \mathbb{C}^{q \times m}, \quad \Omega = [\alpha_1, \beta_1] \times \ldots \times [\alpha_d, \beta_d], \\ G(s; p_1, \ldots, p_d) &\in \mathbb{C}^{q \times m}. \end{aligned}$$

$\rightsquigarrow$ Variables s and p_j have different "meaning" for G.
Dynamical system is in the background!

$\rightsquigarrow$ Matrix-valued function, require matrix- not entry-wise approximation!

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- G is rational in s, $n \sim$ degree of denominator polynomial.
  $\rightsquigarrow$ Require approximation to be rational in s.

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- G is rational in s, $n \sim$ degree of denominator polynomial.
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- Require structure-preserving approximation, e.g., for control design.
  $\rightsquigarrow$ Need realization as linear parametric system!

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- But:

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- Require structure-preserving approximation, e.g., for control design.
  $\rightsquigarrow$ Need realization as linear parametric system!
- Also would like to be able to reproduce system dynamics (stability, passivity).

PMOR Methods — a Survey

Model Reduction for Linear Parametric Systems

Parametric System

$$\Sigma(p) : \begin{cases} E(p)\dot{x}(t;p) & = & A(p)x(t;p) + B(p)u(t), \\ y(t;p) & = & C(p)x(t;p). \end{cases}$$

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Appropriate parameter-affine representation:

$$\begin{aligned} E(p) &= E_0 + e_1(p)E_1 + \ldots + e_{q_E}(p)E_{q_E}, \\ A(p) &= A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \\ B(p) &= B_0 + b_1(p)B_1 + \ldots + b_{q_B}(p)B_{q_B}, \\ C(p) &= C_0 + c_1(p)C_1 + \ldots + c_{q_C}(p)C_{q_C}, \end{aligned}$$

allows easy parameter preservation for projection based model reduction.

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Appropriate parameter-affine representation:

$$A(p) = A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \quad \ldots$$

allows easy parameter preservation for projection based model reduction.

W.l.o.g. may assume this affine representation:

- Any system can be written in this affine form for some $q_X \leq n^2$, but for efficiency, need $q_X \ll n$! $(X \in \{E, A, B, C\})$
- Empirical (operator) interpolation yields this structure for "smooth enough" nonlinearities [Barrault/Maday/Nguyen/Patera 2004].

PMOR Methods — a Survey
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Parametric System

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allows easy parameter preservation for projection based model reduction.

Parametric model reduction goal:

preserve parameters as symbolic quantities in reduced-order model:

$$\widehat{\Sigma}(p): \begin{cases} \hat{E}(p)\dot{\hat{x}}(t;p) &= \hat{A}(p)\hat{x}(t;p) + \hat{B}(p)u(t), \\ \hat{y}(t;p) &= \hat{C}(p)\hat{x}(t;p) \end{cases}$$

with states $\hat{x}(t;p) \in \mathbb{R}^r$ and $r \ll n$.

Model Reduction for Linear Parametric Systems

Structure-Preservation

Petrov-Galerkin-type projection

For given projection matrices $V, W \in \mathbb{R}^{n\times r}$ with $W^T V = I_r$
($\rightsquigarrow (VW^T)^2 = VW^T$ is projector), compute

$$
\begin{aligned}
\hat{E}(p) &= W^T E_0 V + e_1(p) W^T E_1 V + \ldots + e_{q_E}(p) W^T E_{q_E} V,\\
&= \hat{E}_0 + e_1(p)\hat{E}_1 + \ldots + e_{q_E}(p)\hat{E}_{q_E},\\
\hat{A}(p) &= W^T A_0 V + a_1(p) W^T A_1 V + \ldots + a_{q_A}(p) W^T A_{q_A} V,\\
&= \hat{A}_0 + a_1(p)\hat{A}_1 + \ldots + a_{q_A}(p)\hat{A}_{q_A},\\
\hat{B}(p) &= W^T B_0 + b_1(p) W^T B_1 + \ldots + b_{q_B}(p) W^T B_{q_B},\\
&= \hat{B}_0 + b_1(p)\hat{B}_1 + \ldots + b_{q_B}(p)\hat{B}_{q_B},\\
\hat{C}(p) &= C_0 V + c_1(p) C_1 V + \ldots + c_{q_C}(p) C_{q_C} V,\\
&= \hat{C}_0 + c_1(p)\hat{C}_1 + \ldots + c_{q_C}(p)\hat{C}_{q_C}.
\end{aligned}
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\hat{A}(p) &= W^T A_0 V + a_1(p) W^T A_1 V + \ldots + a_{q_A}(p) W^T A_{q_A} V, \\
&= \hat{A}_0 + a_1(p)\hat{A}_1 + \ldots + a_{q_A}(p)\hat{A}_{q_A}, \\
\hat{B}(p) &= W^T B_0 + b_1(p) W^T B_1 + \ldots + b_{q_B}(p) W^T B_{q_B}, \\
&= \hat{B}_0 + b_1(p)\hat{B}_1 + \ldots + b_{q_B}(p)\hat{B}_{q_B}, \\
\hat{C}(p) &= C_0 V + c_1(p) C_1 V + \ldots + c_{q_C}(p) C_{q_C} V, \\
&= \hat{C}_0 + c_1(p)\hat{C}_1 + \ldots + c_{q_C}(p)\hat{C}_{q_C}.
\end{aligned}
$$

PMOR Methods — a Survey

A Short Introduction to Interpolatory Model Reduction

Computation of reduced-order model by projection

Given a linear (descriptor) system $E\dot{x} = Ax + Bu,\ y = Cx$ with transfer function $G(s) = C(sE - A)^{-1}B,$ a reduced-order model is obtained using truncation matrices $V, W \in \mathbb{R}^{n\times r}$ with $W^TV = I_r$ ($\rightsquigarrow (VW^T)^2 = VW^T$ is projector) by computing

$$\hat{E} = W^TEV,\ \hat{A} = W^TAV,\ \hat{B} = W^TB,\ \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

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($\rightsquigarrow (VW^T)^2 = VW^T$ is projector) by computing

$$\hat{E} = W^T EV,\ \hat{A} = W^T AV,\ \hat{B} = W^T B,\ \hat{C} = CV.$$

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Rational Interpolation/Moment-Matching

Choose V, W such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i} G(s_j) = \frac{d^i}{ds^i} \hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$

PMOR Methods — a Survey

A Short Introduction to Interpolatory Model Reduction

Theorem (simplified) [Grimme '97, Villemagne/Skelton '87]

If

$$\begin{aligned} \mathrm{span}\left\{(s_1E-A)^{-1}B,\dots,(s_kE-A)^{-1}B\right\} &\subset \mathrm{Ran}(V), \\ \mathrm{span}\left\{(s_1E-A)^{-T}C^T,\dots,(s_kE-A)^{-T}C^T\right\} &\subset \mathrm{Ran}(W), \end{aligned}$$

then

$$G(s_j)=\hat{G}(s_j), \quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j), \quad \text{for } j=1,\dots,k.$$

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Remarks:

computation of V, W from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme '97],
- Iter. Rational Krylov-Alg. (IRKA) [Antoulas/Beattie/Gugercin '06/'08].

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then

$$G(s_j)=\hat{G}(s_j),\quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j),\quad \text{for } j=1,\dots,k.$$

Remarks:

using Galerkin/one-sided projection ($W \equiv V$) yields $G(s_j)=\hat{G}(s_j)$, but in general

$$\frac{d}{ds}G(s_j)\neq\frac{d}{ds}\hat{G}(s_j).$$

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$$G(s_j)=\hat{G}(s_j), \quad \frac{d}{ds}G(s_j)=\frac{d}{ds}\hat{G}(s_j), \quad \text{for } j=1,\ldots,k.$$

Remarks:

$k=1$, standard Krylov subspace(s) of dimension K:

$$\operatorname{range}(V)=\mathcal{K}_K((s_1I-A)^{-1},(s_1I-A)^{-1}B).$$

$\rightsquigarrow$ moment-matching methods/Padé approximation,

$$\frac{d^i}{ds^i}G(s_1)=\frac{d^i}{ds^i}\hat{G}(s_1), \quad i=0,\ldots,K-1(+K).$$

Interpolatory Model Reduction

$\mathcal{H}_2$-Model Reduction for Linear Systems

Consider stable (i.e. $\Lambda(A) \subset \mathbb{C}^-$) linear systems Σ,

$$\dot{x}(t) = Ax(t) + Bu(t), \; y(t) = Cx(t) \quad \simeq \quad Y(s) = \underbrace{C(sI - A)^{-1}B}_{=:G(s)}\, U(s)$$

System norms

Two common system norms for measuring approximation quality:

- $\mathcal{H}_2$-norm, $\|\Sigma\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\left(\left(G^T(-\jmath\omega)G(\jmath\omega)\right)\right) d\omega\right)^{\frac{1}{2}}$,
- $\mathcal{H}_\infty$-norm, $\|\Sigma\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(G(\jmath\omega)\right)$,

where

$$G(s) = C\,(sI - A)^{-1}\,B.$$

Note: $\mathcal{H}_\infty$-*norm approximation* $\rightsquigarrow$ *balanced truncation, Hankel norm approximation.*

Interpolatory Model Reduction

Error system and $\mathcal{H}_2$-Optimality **[Meier/Luenberger 1967]**

In order to find an $\mathcal{H}_2$-optimal reduced system, consider the error system $G(s) - \hat{G}(s)$ which can be realized by

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

Interpolatory Model Reduction

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Assuming a coordinate system in which $\hat{A}$ is diagonal and taking derivatives of

$$\| G(\,.\,) - \hat{G}(\,.\,) \|^2_{\mathcal{H}_2}$$

with respect to free parameters in $\Lambda(\hat{A}), \hat{B}, \hat{C} \rightsquigarrow$ **first-order necessary $\mathcal{H}_2$-optimality conditions (SISO)**

$$G(-\hat{\lambda}_i) = \hat{G}(-\hat{\lambda}_i),$$
$$G'(-\hat{\lambda}_i) = \hat{G}'(-\hat{\lambda}_i),$$

where $\hat{\lambda}_i$ are the poles of the reduced system $\hat{\Sigma}$.

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First-order necessary $\mathcal{H}_2$-optimality conditions (MIMO):

$$\begin{aligned} G(-\hat{\lambda}_i)\tilde{B}_i &= \hat{G}(-\hat{\lambda}_i)\tilde{B}_i, && \text{for } i = 1, \ldots, \hat{n}, \\ \tilde{C}_i^T G(-\hat{\lambda}_i) &= \tilde{C}_i^T \hat{G}(-\hat{\lambda}_i), && \text{for } i = 1, \ldots, \hat{n}, \\ \tilde{C}_i^T H'(-\hat{\lambda}_i)\tilde{B}_i &= \tilde{C}_i^T \hat{G}'(-\hat{\lambda}_i)\tilde{B}_i && \text{for } i = 1, \ldots, \hat{n}, \end{aligned}$$

where $\hat{A} = R\hat{\Lambda}R^{-T}$ is the spectral decomposition of the reduced system and $\tilde{B} = \hat{B}^T R^{-T}$, $\tilde{C} = \hat{C}R$.

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$$\begin{aligned} \Leftrightarrow \quad & \operatorname{vec}(I_q)^T \left(e_j e_i^T \otimes C\right) \left(-\hat{\Lambda} \otimes I_n - I_{\hat{n}} \otimes A\right)^{-1} \left(\tilde{B}^T \otimes B\right) \operatorname{vec}(I_m) \\ &= \operatorname{vec}(I_q)^T \left(e_j e_i^T \otimes \hat{C}\right) \left(-\hat{\Lambda} \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A}\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \operatorname{vec}(I_m), \\ & \text{for } i = 1, \ldots, \hat{n} \text{ and } j = 1, \ldots, q. \end{aligned}$$

Interpolatory Model Reduction

Interpolation of the Transfer Function [Grimme 1997]

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{G}(s) = CV\left(sI - W^T AV\right)^{-1} W^T B,$$

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$$V = \left[(-\mu_1 I - A)^{-1}B, \ldots, (-\mu_r I - A)^{-1}B\right],$$
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Then

$$G(-\mu_i) = \hat{G}(-\mu_i) \quad \text{and} \quad G'(-\mu_i) = \hat{G}'(-\mu_i),$$

for $i = 1, \ldots, r$.

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for $i = 1, \ldots, r$.
Starting with an initial guess for $\hat{\Lambda}$ and setting $\mu_i \equiv \hat{\lambda}_i \rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. 2006/08], [Bunse-Gerstner et al. 2007], [Van Dooren et al. 2008]

Interpolatory Model Reduction

The Basic IRKA Algorithm

Algorithm 1 IRKA (MIMO version/MIRIAm)

Input: A stable, B, C, $\hat{A}$ stable, $\hat{B}$, $\hat{C}$, $\delta > 0$.
Output: A^{opt}, B^{opt}, C^{opt}

1: **while** $(\max_{j=1,\dots,r} \left\{ \frac{|\mu_j - \mu_j^{\text{old}}|}{|\mu_j|} \right\} > \delta)$ **do**
2: $\quad$ $\operatorname{diag}\{\mu_1, \dots, \mu_r\} := T^{-1}\hat{A}T =$ spectral decomposition, $\tilde{B} = \hat{B}^H T^{-T}$, $\tilde{C} = \hat{C}T$.
3: $\quad$ $V = \left[(-\mu_1 I - A)^{-1} B\tilde{b}_1, \dots, (-\mu_r I - A)^{-1} B\tilde{b}_r\right]$
4: $\quad$ $W = \left[(-\mu_1 I - A^T)^{-1} C^T \tilde{c}_1, \dots, (-\mu_r I - A^T)^{-1} C^T \tilde{c}_r\right]$
5: $\quad$ $V = \operatorname{orth}(V)$, $W = \operatorname{orth}(W)$, $W = W(V^H W)^{-1}$
6: $\quad$ $\hat{A} = W^H A V$, $\hat{B} = W^H B$, $\hat{C} = CV$
7: **end while**
8: $A^{opt} = \hat{A}$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$

PMOR based on Multi-Moment Matching

Idea: choose appropriate frequency parameter $\hat{s}$ and parameter vector $\hat{p}$, expand into multivariate power series about $(\hat{s}, \hat{p})$ and compute reduced-order model, so that

$$G(s,p) = \hat{G}(s,p) + \mathcal{O}\left(|s-\hat{s}|^K + \|p-\hat{p}\|^L + |s-\hat{s}|^k \|p-\hat{p}\|^\ell\right),$$

i.e., first $K, L, k+\ell$ (mostly: $K = L = k+\ell$) coefficients (multi-moments) of Taylor/Laurent series coincide.

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Algorithms:

- [Daniel et al. 2004]: explicit computation of moments, numerically unstable.
- [Farle et al. 2006/07]: Krylov subspace approach, only polynomial parameter-dependance, numerical properties not clear, but appears to be robust.
- [Feng/B. 2007/14]: Arnoldi-MGS method, employ recursive dependance of multi-moments, numerically robust, r often larger as with [Farle et al.].

PMOR based on Multi-Moment Matching

Numerical Examples

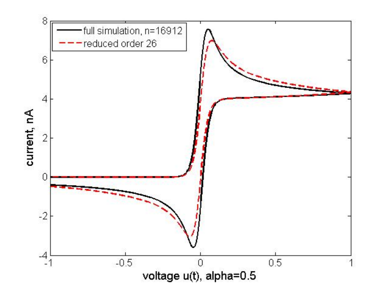
Electro-chemical SEM:

compute cyclic voltammogram based on FEM model

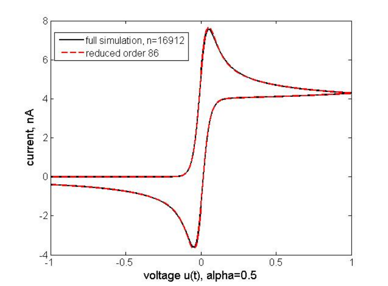
$$E\dot{x}(t) = (A_0 + p_1 A_1 + p_2 A_2)x(t) + Bu(t), \quad y(t) = c^T x(t),$$

where $n = 16,912$, $m = 3$, A_1, A_2 diagonal.

$K = L = k + \ell = 4 \;\Rightarrow\; r = 26$



$K = L = k + \ell = 9 \;\Rightarrow\; r = 86$



PMOR based on Rational Interpolation

Theory: Interpolation of the Transfer Function

Theorem 1 [Baur/Beattie/B./Gugercin 2007/2011]

Let
$$\begin{aligned}\hat{G}(s,p) &:= \hat{C}(p)(s\hat{E}(p) - \hat{A}(p))^{-1}\hat{B}(p) \\ &= C(p)V(sW^T E(p)V - W^T A(p)V)^{-1} W^T B(p).\end{aligned}$$

Suppose $\hat{p} = [\hat{p}_1, \ldots, \hat{p}_d]^T$ and $\hat{s} \in \mathbb{C}$ are chosen such that both $\hat{s}\,E(\hat{p}) - A(\hat{p})$ and $\hat{s}\,\hat{E}(\hat{p}) - \hat{A}(\hat{p})$ are invertible.

If
$$(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\,B(\hat{p}) \in \mathrm{Ran}(V)$$
or
$$\left(C(\hat{p})\,(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\right)^T \in \mathrm{Ran}(W),$$
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Note: result extends to MIMO case using tangential interpolation:
Let $0 \neq b \in \mathbb{R}^m$, $0 \neq c \in \mathbb{R}^q$ be arbitrary.

a) If $(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\,B(\hat{p})b \in \mathrm{Ran}(V)$, then $G(\hat{s},\hat{p})b = \hat{G}(\hat{s},\hat{p})b$;

b) If $\left(c^TC(\hat{p})\,(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\right)^T \in \mathrm{Ran}(W)$, then $c^TG(\hat{s},\hat{p}) = c^T\hat{G}(\hat{s},\hat{p})$.

PMOR based on Rational Interpolation

Theory: Interpolation of the Parameter Gradient

Theorem 2 [Baur/Beattie/B./Gugercin '07/'09]

Suppose that $E(p)$, $A(p)$, $B(p)$, $C(p)$ are C^1 in a neighborhood of $\hat{p} = [\hat{p}_1, \ldots, \hat{p}_d]^T$ and that both $\hat{s}\,E(\hat{p}) - A(\hat{p})$ and $\hat{s}\,\hat{E}(\hat{p}) - \hat{A}(\hat{p})$ are invertible. If

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then

$$\nabla_p G(\hat{s}, \hat{p}) = \nabla_p G_r(\hat{s}, \hat{p}), \qquad \frac{\partial}{\partial s} G(\hat{s}, \hat{p}) = \frac{\partial}{\partial s} \hat{G}(\hat{s}, \hat{p}).$$

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Let $0 \neq b \in \mathbb{R}^m$, $0 \neq c \in \mathbb{R}^q$ be arbitrary. If $(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\,B(\hat{p})b \in \mathrm{Ran}(V)$ and $\left(c^T C(\hat{p})\,(\hat{s}\,E(\hat{p}) - A(\hat{p}))^{-1}\right)^T \in \mathrm{Ran}(W)$, then $\nabla_p c^T G(\hat{s},\hat{p}) b = \nabla_p c^T \hat{G}(\hat{s},\hat{p}) b$.

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1. Assertion of theorem satisfies necessary conditions for surrogate models in trust region methods [Alexandrov/Dennis/Lewis/Torczon '98].
2. Approximation of gradient allows use of reduced-order model for sensitivity analysis.

PMOR based on Rational Interpolation

Algorithm

Generic implementation of interpolatory PMOR

Define $\mathcal{A}(s,p) := sE(p) - A(p)$.

1. Select "frequencies" $s_1, \ldots, s_k \in \mathbb{C}$ and parameter vectors $p^{(1)}, \ldots, p^{(\ell)} \in \mathbb{R}^d$.
2. Compute (orthonormal) basis of
$$\mathcal{V} = \operatorname{span}\left\{\mathcal{A}(s_1, p^{(1)})^{-1} B(p^{(1)}), \ldots, \mathcal{A}(s_k, p^{(\ell)})^{-1} B(p^{(\ell)})\right\}.$$
3. Compute (orthonormal) basis of
$$\mathcal{W} = \operatorname{span}\left\{\mathcal{A}(s_1, p^{(1)})^{-T} C(p^{(1)})^T, \ldots, \mathcal{A}(s_k, p^{(\ell)})^{-T} C(p^{(\ell)})^T\right\}.$$
4. Set $V := [v_1, \ldots, v_{k\ell}]$, $\tilde{W} := [w_1, \ldots, w_{k\ell}]$, and $W := \tilde{W}(\tilde{W}^T V)^{-1}$. (Note: $r = k\ell$).
5. Compute $\begin{cases} \hat{A}(p) := W^T A(p) V, & \hat{B}(p) := W^T B(p) V, \\ \hat{C}(p) := W^T C(p) V, & \hat{E}(p) := W^T E(p) V. \end{cases}$

PMOR based on Rational Interpolation

Remarks

- If directional derivatives w.r.t. p are included in $\mathrm{Ran}(V)$, $\mathrm{Ran}(W)$, then also the Hessian of $G(\hat{s}, \hat{p})$ is interpolated by the Hessian of $\hat{G}(\hat{s}, \hat{p})$.

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- For prescribed parameter vectors $p^{(k)}$, we can use corresponding $\mathcal{H}_2$-optimal frequencies $s_{k,\ell}$, $\ell = 1, \ldots, r_k$ computed by IRKA, i.e., reduced-order systems $\hat{G}_*^{(k)}$ so that

$$\|G(., p^{(k)}) - \hat{G}_*^{(k)}(.)\|_{\mathcal{H}_2} = \min_{\substack{\mathrm{order}(\hat{G})=r_k \\ \hat{G}\ \mathrm{stable}}} \|G(., p^{(k)}) - \hat{G}^{(k)}(.)\|_{\mathcal{H}_2},$$

where

$$\|G\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \|G(\jmath\omega)\|_{\mathrm{F}}^2 \, d\omega \right)^{1/2}.$$

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- Optimal choice of interpolation frequencies s_k and parameter vectors $p^{(k)}$ possible for special cases.

PMOR based on Rational Interpolation

Numerical Example: Thermal Conduction in a Semiconductor Chip

- Important requirement for a compact model of thermal conduction is boundary condition independence.
- The thermal problem is modeled by the heat equation, where heat exchange through device interfaces is modeled by convection boundary conditions containing film coefficients $\{p_i\}_{i=1}^{3}$, to describe the heat exchange at the ith interface.
- Spatial semi-discretization leads to

$$E\dot{x}(t) = (A_0 + \sum_{i=1}^{3} p_i A_i)x(t) + bu(t), \quad y(t) = c^T x(t),$$

where $n = 4,257$, A_i, $i = 1, 2, 3$, are diagonal.

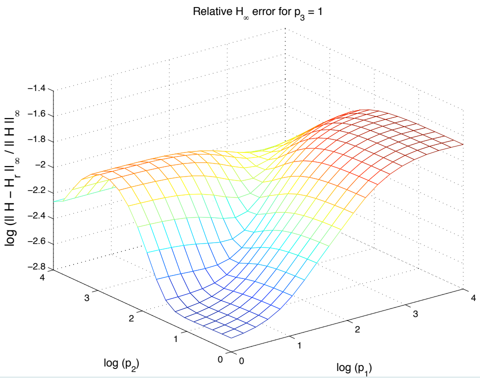
Source: C.J.M Lasance, *Two benchmarks to facilitate the study of compact thermal modeling phenomena*, IEEE. Trans. Components and Packaging Technologies, Vol. 24, No. 4, pp. 559–565, 2001.

PMOR based on Rational Interpolation

Numerical Example: Thermal Conduction in a Semiconductor Chip

Choose 2 interpolation points for parameters ("important" configurations), 8/7 interpolation frequencies are picked H_2 optimal by IRKA. $\Longrightarrow$ $k = 2, \ell = 8, 7$, hence $r = 15$.

$p_3 = 1,\ p_1, p_2 \in [1, 10^4]$.



PMOR Methods — a Survey

Other Approaches

- Transfer function interpolation (= output interpolation in frequency domain) [B./Baur 2008]

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- Reduced basis method (RBM) [Haasdonk, Maday, Patera, Prud'homme, Rozza, Urban, ...]

Parametric Systems as Bilinear Systems

Linear Parametric Systems — An Alternative Interpretation

Consider bilinear control systems:

$$\Sigma: \quad \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t) u_i(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

where $A, A_i \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m},\ C \in \mathbb{R}^{q \times n}$.

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Key Observation [B./Breiten 2011]

Consider parameters as additional inputs, a linear parametric system

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m_p} a_i(p) A_i x(t) + B_0 u_0(t), \quad y(t) = Cx(t)$$

with $B_0 \in \mathbb{R}^{n\times m_0}$ can be interpreted as bilinear system:

$$u(t) := \begin{bmatrix} a_1(p) & \dots & a_{m_p}(p) & u_0(t) \end{bmatrix}^T,$$

$$B := \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & B_0 \end{bmatrix} \in \mathbb{R}^{n\times m}, \quad m = m_p + m_0.$$

Parametric Systems as Bilinear Systems

Linear Parametric Systems — An Alternative Interpretation

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Consequence

Model order reduction techniques for bilinear systems can be applied to linear parametric systems!

Here:

- Balanced truncation,
- $\mathcal{H}_2$ optimal model reduction.

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Some background

Consider bilinear system ($m = 1$, i.e. SISO)

$$\Sigma : \ \{\dot{x}(t) = Ax(t) + A_1 x(t)u(t) + Bu(t), \quad y(t) = Cx(t).$$

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Output Characterization (SISO): Volterra series

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} K(t_1, \dots, t_k) u(t - t_1 - \dots - t_k) \cdots u(t - t_k) dt_k \cdots dt_1,$$

with kernels $K(t_1, \dots, t_k) = Ce^{At_k} A_1 \cdots e^{At_2} A_1 e^{At_1} B$.

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Multivariate Laplace-transform:

$$G_k(s_1, \dots, s_k) = C(s_k I - A)^{-1} A_1 \cdots (s_2 I - A)^{-1} A_1 (s_1 I - A)^{-1} B.$$

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Bilinear $\mathcal{H}_2$-norm: [Zhang/Lam 2002]

$$||\Sigma||_{\mathcal{H}_2} := \left(\mathrm{tr} \left(\left(\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} \overline{G_k(i\omega_1, \dots, i\omega_k)} G_k^T(i\omega_1, \dots, i\omega_k) \right) \right) \right)^{\frac{1}{2}}.$$

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Measuring the Approximation Error

Lemma [B./Breiten 2012]

Let Σ denote a bilinear system. Then, the $\mathcal{H}_2$-norm is given as:

$$||\Sigma||^2_{\mathcal{H}_2} = (\mathrm{vec}(I_q))^T (C \otimes C) \left(-A \otimes I - I \otimes A - \sum_{i=1}^{m} A_i \otimes A_i \right)^{-1} (B \otimes B)\, \mathrm{vec}(I_m).$$

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Error System

In order to find an $\mathcal{H}_2$-optimal reduced system, define the error system $\Sigma^{err} := \Sigma - \hat{\Sigma}$ as follows:

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad A_i^{err} = \begin{bmatrix} A_i & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

$\mathcal{H}_2$-Model Reduction

$\mathcal{H}_2$-Optimality Conditions

Assume $\hat{\Sigma}$ is given in coordinate system induced by eigenvalue decomposition of $\hat{A}$:

$$\hat{A} = R\Lambda R^{-1}, \quad \tilde{A}_i = R^{-1}\hat{A}_i R, \quad \tilde{B} = R^{-1}\hat{B}, \quad \tilde{C} = \hat{C}R.$$

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Using Λ, $\tilde{A}_i$, $\tilde{B}$, $\tilde{C}$ as optimization parameters, we can derive necessary conditions for $\mathcal{H}_2$-optimality, e.g.:

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$$\begin{aligned}
&(\mathrm{vec}(I_q))^T \left(e_j e_\ell^T \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^{m} \tilde{A}_i \otimes A_i\right)^{-1} \left(\tilde{B} \otimes B\right) \mathrm{vec}(I_m) \\
&= (\mathrm{vec}(I_q))^T \left(e_j e_\ell^T \otimes \hat{C}\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A} - \sum_{i=1}^{m} \tilde{A}_i \otimes \hat{A}_i\right)^{-1} \left(\tilde{B} \otimes \hat{B}\right) \mathrm{vec}(I_m).
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Connection to interpolation of transfer functions?

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For $A_i \equiv 0$, this is equivalent to

$$G(-\lambda_\ell)\tilde{B}_\ell^T = \hat{G}(-\lambda_\ell)\tilde{B}_\ell^T$$

$\rightsquigarrow$ tangential interpolation at mirror images of reduced system poles!

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Note: [Flagg 2011] shows equivalence to interpolating the Volterra series!

A First Iterative Approach

Algorithm 2 Bilinear IRKA

Input: A, A_i, B, C, $\hat{A}$, $\hat{A}_i$, $\hat{B}$, $\hat{C}$

Output: A^{opt}, A_i^{opt}, B^{opt}, C^{opt}

1: **while** (change in $\Lambda > \epsilon$) **do**

2: $R\Lambda R^{-1} = \hat{A}$, $\tilde{B} = R^{-1}\hat{B}$, $\tilde{C} = \hat{C}R$, $\tilde{A}_i = R^{-1}\hat{A}_i R$

3: $$\mathrm{vec}(V) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^{m} \tilde{A}_i \otimes A_i\right)^{-1} \left(\tilde{B} \otimes B\right) \mathrm{vec}(I_m)$$

4: $$\mathrm{vec}(W) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A^T - \sum_{i=1}^{m} \tilde{A}_i^T \otimes A_i^T\right)^{-1} \left(\tilde{C}^T \otimes C^T\right) \mathrm{vec}(I_q)$$

5: $V = \mathrm{orth}(V)$, $W = \mathrm{orth}(W)$

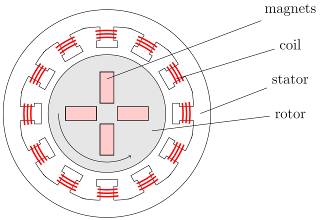
6: $\hat{A} = \left(W^T V\right)^{-1} W^T A V$, $\hat{A}_i = \left(W^T V\right)^{-1} W^T A_i V$,
$\hat{B} = \left(W^T V\right)^{-1} W^T B$, $\hat{C} = CV$

7: **end while**

8: $A^{opt} = \hat{A}$, $A_i^{opt} = \hat{A}_i$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$

$\mathcal{H}_2$-Model Reduction for Bilinear Systems
Industrial Case Study: Thermal Analysis of Electrical Motor

- Thermal simulations to detect whether temperature changes lead to fatigue or deterioration of employed materials.
- Main heat source: thermal losses resulting from current stator coil/rotor.
- Many different current profiles need to be considered to predict whether temperature on certain parts of the motor remans in feasible region.
- Finite element analysis on rather complicated geometries $\rightsquigarrow$ large-scale linear models with many (here: 7/13) parameters.



Schematic view of an electrical motor.

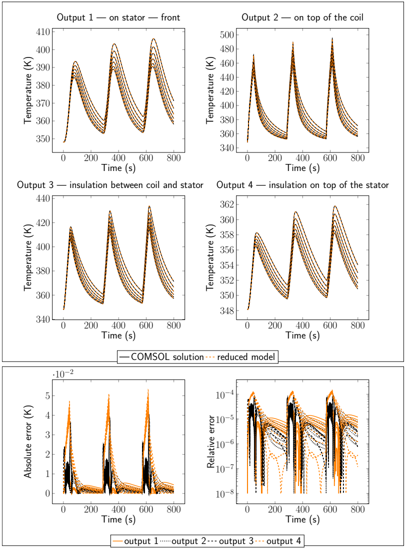
Bosch integrated motor generator used in hybrid variants of Porsche Cayenne, VW Touareg.

Pictures: BOSCH

$\mathcal{H}_2$-Model Reduction for Bilinear Systems

Industrial Case Study: Thermal Analysis of Electrical Motor

- FEM analysis of thermal model $\rightsquigarrow$ linear parametric systems with $n = 41,199$, $m = 4$ inputs, and $d = 13$ parameters,
- measurements taken at $q = 4$ heat sensors;
- time for 1 transient simulation in COMSOL® $\sim$ 90min;
- ROM order $\hat{n} = 300$, time for 1 transient simulation $\sim$ 15sec.
- Legend: Temperature curves for six different values (5, 25, 45, 65, 85, 100$[W/m^2K]$) of the heat transfer coefficient on the coil.



Conclusions and Outlook

- We have reviewed the most popular PMOR methods developed in the last decade, in particular those based on rational interpolation.
  Open problem in general: optimal interpolation points.

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- **Balanced truncation:**
  - Under certain assumptions, we can expect the existence of low-rank approximations to the solution of generalized Lyapunov equations.
  - Solutions strategies via extending the ADI iteration to bilinear systems and EKSM as well as using preconditioned iterative solvers like CG or BiCGstab up to dimensions $n \sim 500,000$ in MATLAB®.
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  - Existence of low-rank solutions in case of A_i being full rank?
- $\mathcal{H}_2$ **optimal model reduction:**
  - Yields competitive approach, proven in industrial context.
  - Still high offline cost (= time for generating reduced-order model).
  - May need to switch to one-sided projection ($W = V$) to preserve stability.

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