MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

20 YEARS
1998-2018

New Gramians for Switched Linear Systems: Reachability, Observability, and Model Reduction

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GAMM FA Dynamik und Regelungstheorie
April 19–20, 2018, Berlin, Germany

Introduction

A Switched Linear Sytem (SLS) is a control system of the form

$$\Sigma_{SLS} : \begin{cases} \dot{x}(t) & = & A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad x(0) = x_0, \\ y(t) & = & C_{\sigma(t)}x(t). \end{cases}$$

- $x(t) \in \mathbb{R}^n$ state, $u(t) \in \mathbb{R}^m$ input, $y(t) \in \mathbb{R}^p$ output.
- $\Omega = \{1, 2, \ldots, M\}$ is the set of modes.
- $\sigma : \mathbb{R} \to \Omega$ is the switching signal, a piecewise constant function, *i.e.*,

$$\sigma(t) = \begin{cases} q_1, & t \in [0, T_1), \\ q_2, & t \in [T_1, T_2), \\ \vdots & \end{cases}$$

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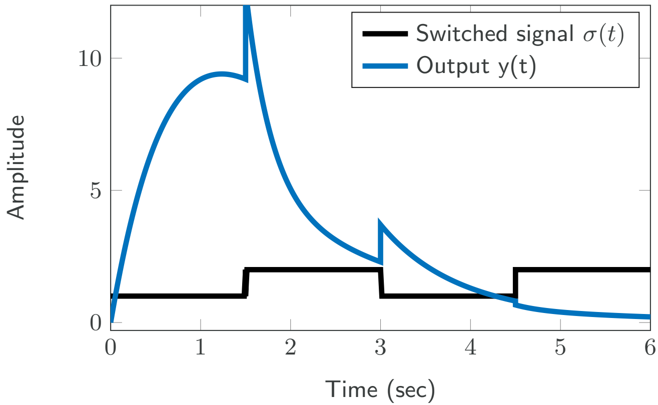
Assumptions:

- Zero initial condition, *i.e.*, $x(0) = 0$.
- The matrices A_j are Hurwitz for $j \in \Omega$.

$$\Sigma_1 : \begin{cases} \dot{x}(t) &= A_1 x(t) + B_1 u(t) \\ y(t) &= C_1 x(t) \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{x}(t) &= A_2 x(t) + B_2 u(t) \\ y(t) &= C_2 x(t) \end{cases}$$

Time-domain simulation with $u(t) = e^{-t}$



Projection-based framework

Given a large-scale SLS

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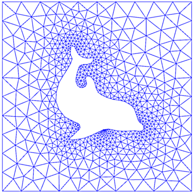

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find global projection matrices

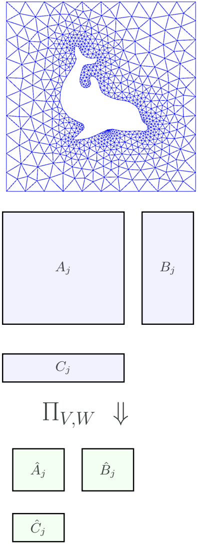
$$V, W \in \mathbb{R}^{n \times r}, \qquad W^T V = I_r,$$

(with $r \ll n$), such that

$$\hat{\Sigma}_{SLS} : \begin{cases} \dot{\hat{x}}(t) & = & \hat{A}_{\sigma(t)}\hat{x}(t) + \hat{B}_{\sigma(t)}u(t), \\ \hat{y}(t) & = & \hat{C}_{\sigma(t)}\hat{x}(t), \quad \hat{x}(0) = 0, \end{cases}$$

where

$$\boxed{\begin{array}{l} \hat{A}_j = W^T A_j V, \quad \hat{B}_j = W^T B_j, \\ \text{and} \quad \hat{C}_j = C_j V \quad \text{for } j \in \Omega. \end{array}}$$



Existing common approaches for SLS

Moment matching methods

- Model reduction by nice selections for linear switched systems.
  [Bastug/Petreczky/Wisniewski '16]

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Balancing-type methods involving LMIs

- Generalised Gramian framework involving LMIs. [Shaker/Wisniewski '11]
  [Petreczky/Wisniewski/Leth '13]

$$A_j\mathcal{P} + \mathcal{P}A_j^T + B_jB_j^T < 0 \quad \text{and} \quad A_j^T\mathcal{Q} + \mathcal{Q}A_j + C_j^TC_j < 0.$$

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Balancing-type methods involving matrix equations

- A simultaneous balanced truncation approach.
  [Monshizadeh/Trentelman/Camlibel '12]

$$\mathcal{P}_{\text{avg}} = \sum_{j=1}^{M} \mathcal{P}_j, \quad \text{where} \quad A_j \mathcal{P}_j + \mathcal{P}_j A_j^T + B_j B_j^T = 0.$$

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- Balanced truncation based on realization with coupling matrices.
  [Gosea/Petreczky/Antoulas/Fiter '17]

Balanced truncation for LTI systems

Consider an LTI system:

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$$\mathcal{P} = \int_0^{+\infty} e^{At} B (e^{At} B)^T dt,$$
$$\mathcal{Q} = \int_0^{+\infty} (C e^{At})^T C e^{At} dt,$$

which satisfy Lyapunov equations

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0,$$
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- $\mathcal{P}$ encodes the states that are important in the transfer $u \mapsto x$.

$$\forall u \in L_2, x(0) \in \text{range}\,(\mathcal{P}),$$
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- System is balanced if $\mathcal{P} = \mathcal{Q} = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$.
- Truncation of states leading to error bound

$$\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_\infty} \leq 2 \sum_{j=k+1}^{n} \sigma_j.$$

Reachable and observable sets

For linear systems (A,B,C), A Hurwitz matrix, and $\mathcal{P}, \mathcal{Q}$ matrices such that $A\mathcal{P} + \mathcal{P}A^T + BB^T = 0, \quad A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0$. Then,

$$\begin{aligned} \operatorname{range}(\mathcal{P}) &= \sum_{j=0}^{\infty} A^j \operatorname{range}(B) &= \mathcal{R} := \text{Reachable space},\\ \operatorname{range}(\mathcal{Q}) &= \sum_{j=0}^{\infty} (A^T)^j \operatorname{range}\left(C^T\right) &= \mathcal{O} := \text{Observable space}. \end{aligned}$$

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For a SLS, [Sun/Ge/Lee '02]

$$\sum_{k=0}^{\infty} \left(\sum_{i_0,\ldots,i_{k-1}\in\Omega}^{j_1,\ldots,j_{k-1}\in\mathbb{N}} A_{i_{k-1}}^{j_{k-1}} \ldots A_{i_1}^{j_1} \operatorname{range}(B_{i_0}) \right) = \mathcal{R} := \text{Reachable space},$$

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Goal: Find matrix equations encoding the (span of) reachabillity and observaility sets.

SLS recast as a bilinear system

Given

$$\Sigma_{SLS} : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \\ y(t) = C_{\sigma(t)}x(t), \end{cases}$$

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$$A = A_1, \quad D_j = A_j - A_1, \text{ for } j \in \Omega.$$

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In addition, let us introduce switching indicators $\{\sigma_1(t), \ldots, \sigma_M(t)\}$ taking $\{0, 1\}$ values, $\sum_{k=1}^{M} \sigma_k(t) = 1$.

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Therefore,

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{M} \sigma_j(t) D_j x(t) + \sum_{j=1}^{M} \sigma_j(t) B_j u(t),$$

$$y(t) = \sum_{j=1}^{M} \sigma_j(t) C_j x(t).$$

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Very similar to a bilinear system realization as

$$\Sigma_{\text{Bilinear}} = \begin{cases} \dot{x}(t) = Ax(t) + \sum_{j=1}^{M} u_j(t) N_j x(t) + Bu(t), \\ y(t) = Cx(t). \end{cases}.$$

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which is associated with generalized Gramians $\mathcal{P}$, $\mathcal{Q}$, satisfying

$$A\mathcal{P} + \mathcal{P}A^T + \sum_{j=1}^{M} N_j \mathcal{P} N_j^T + BB^T = 0,$$

$$A^T\mathcal{Q} + \mathcal{Q}A + \sum_{j=1}^{M} N_j^T \mathcal{Q} N_j + CC^T = 0.$$

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Inspired by the bilinear Gramians, we propose the following generalize Lyapunov equations to be associated to SLS:

$$\text{SLS Gramians} : \begin{cases} A\mathcal{P} + \mathcal{P}A^T + \sum_{j=1}^{M} \left(D_j \mathcal{P} D_j^T + B_j B_j^T \right) & = & 0, \\ A^T \mathcal{Q} + \mathcal{Q}A + \sum_{j=1}^{M} \left(D_j^T \mathcal{Q} D_j + C_j^T C_j \right) & = & 0, \end{cases}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are supposed to be the reachability and observability Gramians.

Gramian-based control sets

The proposed Gramians indeed encode the reachable and observable sets.

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Theorem: Gramian-base characterization of reachable and observable sets

Let $\mathcal{P}, \mathcal{Q}$ be the solutions of the generalized Lyapunov equations. Then, the reachable and observable spaces $\mathcal{R}$, $\mathcal{O}$ are given by

$$\mathcal{R} = \operatorname{range}(\mathcal{P}) \quad \text{and} \quad \mathcal{O} = \operatorname{range}(\mathcal{Q}).$$

The proof follows by showing that

$$\operatorname{range}(\mathcal{P}) = \sum_{k=0}^{\infty}\left(\sum_{i_0,\ldots,i_{k-1}\in\Omega}^{j_1,\ldots,j_{k-1}\in\mathbb{N}} A_{i_{k-1}}^{j_{k-1}}\ldots A_{i_1}^{j_1}\operatorname{range}(B_{i_0})\right).$$

Reachability and observability criteria

Proposition (reachability and observability criteria)

Given a SLS, and suppose that $\mathcal{P}, \mathcal{Q}$ are solution of the generalized Lyapunov equations. Then,

1. Σ is completely reachable if and only if
$$\operatorname{range}(\mathcal{P}) = \mathbb{R}^n.$$
2. Σ is completely observable if and only if
$$\operatorname{range}(\mathcal{Q}) = \mathbb{R}^n.$$

Example: reachability set

Let us consider Σ, a 2-modes SLS:

$$A_1 = -I_8, \quad A_2 = A_1 + D,$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and
$$B_1^T = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix},$$
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- The proposed reachability Gramian is

$$\mathcal{P} = \operatorname{diag}\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 0, 0, 0, \frac{1}{2}\right).$$

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- The average reachability Gramian ([Monshizadeh '12]) is

$$\mathcal{P}_{\text{avg}} = \frac{1}{2}(\mathcal{P}_1 + \mathcal{P}_2) = \operatorname{diag}\left(\frac{1}{2}, 0, 0, 0, 0, 0, 0, \frac{1}{2}\right).$$

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- The proposed reachability Gramian is

$$\mathcal{P} = \operatorname{diag}\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 0, 0, 0, \frac{1}{2}\right).$$

- Σ is not completely reachable $(\dim(\operatorname{range}(\mathcal{P})) \neq 8)$.

- The reachable set is

$$\mathcal{R} = \operatorname{range}(\mathcal{P}) = \operatorname{span}(e_1, e_2, e_3, e_4, e_8).$$

- The average reachability Gramian ([Monshizadeh '12]) is

$$\mathcal{P}_{\text{avg}} = \frac{1}{2}(\mathcal{P}_1 + \mathcal{P}_2) = \operatorname{diag}\left(\frac{1}{2}, 0, 0, 0, 0, 0, 0, \frac{1}{2}\right).$$

- $\operatorname{range}(\mathcal{P}_{\text{avg}})$ does not encode the reachable set.
- In general, $\operatorname{range}(\mathcal{P}_{\text{avg}}) \subseteq \operatorname{range}(\mathcal{P})$.

Balancing-type algorithm

Algorithm 1 BT using generalized Lyapunov equations

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5: Construct the projection matrices

$$V = SU_1\Sigma_1^{-\frac{1}{2}} \text{ and } W = RV_1\Sigma_1^{-\frac{1}{2}}.$$

Output: Reduced order matrices $\hat{A}_j, \hat{B}_j, \hat{C}_j$, for $j \in \Omega$.
6: **end procedure**

Numerical results

For the next experiment, let us consider a 2-modes SLS of order $n = 300$, whose matrices are given by

$$A_1 = \begin{bmatrix} -2 & 1 & & \\ 0.1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 0.1 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} -2 & 0.5 & & \\ 1 & -2 & 0.5 & \\ & \ddots & \ddots & \ddots \\ & & 0.1 & -2 \end{bmatrix},$$

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and compute averaged Gramians [Monshizadeh 2012]

$$\mathcal{P}_{\text{avg}} = \frac{1}{2}(\mathcal{P}_1 + \mathcal{P}_2) \quad \text{and} \quad \mathcal{Q}_{\text{avg}} = \frac{1}{2}(\mathcal{Q}_1 + \mathcal{Q}_2).$$

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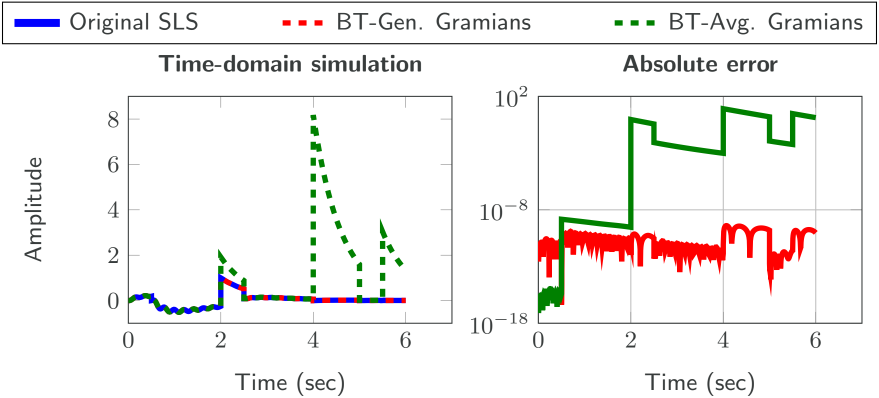

- Full order model $n = 300$. Reduced order models $r = 20$.
- Simulation with input $u(t) = 10 \cdot \sin(20t)e^{-t}$ and switching at $\{0.5, 2, 2.5, 4, 5, 5.5, 6\}$.

Outlook and conclusions

Contribution of this talk

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Open questions and future work

- Investigate general error bounds.
- Restricting switching and hybrid system model reduction.
- Extension to discrete-time SLS.
- Solution of large-scale Lyapunov-type equations.