MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Gramian-based Model Reduction for Classes of Nonlinear Systems

Peter Benner Pawan K. Goyal Igor Pontes Duff

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MATHEMATISCHE
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Overview

Outline

Introduction

Model Reduction for Control Systems

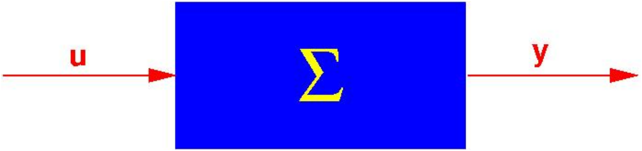
Nonlinear Control Systems

$$\Sigma : \begin{cases} E\dot{x}(t) &= f(t, x(t), u(t)), \quad Ex(t_0) = Ex_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- (generalized) states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.

If E singular $\rightsquigarrow$ descriptor system. Here, $E = I_n$ for simplicity.



Original System ($E = I_n$)

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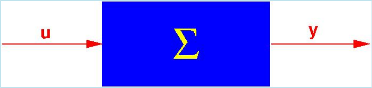

Goals:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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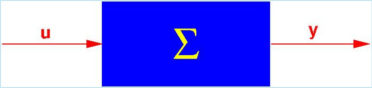

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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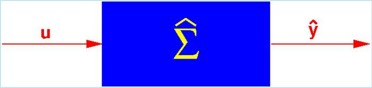

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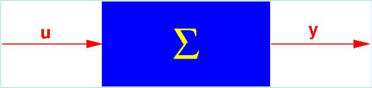

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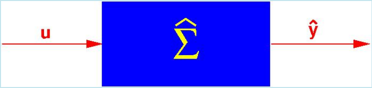

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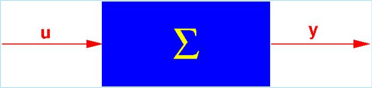

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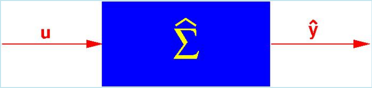

Goals:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

System Classes

Control-Affine (Autonomous) Systems

$$
\begin{aligned}
\dot{x}(t) &= f(t,x,u) &= \mathcal{A}(x(t)) + \mathcal{B}(x(t))u(t), \quad \mathcal{A}: \mathbb{R}^n \to \mathbb{R}^n,\ \mathcal{B}: \mathbb{R}^n \to \mathbb{R}^{n\times m},\\
y(t) &= g(t,x,u) &= \mathcal{C}(x(t)) + \mathcal{D}(x(t))u(t), \quad \mathcal{C}: \mathbb{R}^n \to \mathbb{R}^q,\ \mathcal{D}: \mathbb{R}^n \to \mathbb{R}^{q\times m}.
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Linear, Time-Invariant (LTI) Systems

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\dot{x}(t) &= f(t,x,u) &= Ax(t) + Bu(t), \quad & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\
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Bilinear Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + \sum_{i=1}^{m} u_i(t)A_i x(t) + Bu(t), \quad A, A_i \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

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Quadratic-Bilinear (QB) Systems

$$\begin{aligned} \dot{x}(t) &= f(t,x,u) = Ax(t) + H\left(x(t) \otimes x(t)\right) + \sum_{i=1}^{m} u_i(t)A_i x(t) + Bu(t), \\ &\qquad A, A_i \in \mathbb{R}^{n\times n},\ H \in \mathbb{R}^{n\times n^2},\ B \in \mathbb{R}^{n\times m}, \\ y(t) &= g(t,x,u) = Cx(t) + Du(t), \quad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}. \end{aligned}$$

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Written in control-affine form:

$$\begin{aligned}\mathcal{A}(x) &:= Ax + H\left(x\otimes x\right), & \mathcal{B}(x) &:= [A_1,\ldots,A_m]\left(I_m \otimes x\right) + B\\ \mathcal{C}(x) &:= Cx, & \mathcal{D}(x) &:= D.\end{aligned}$$

Quadratic-Bilinearization

QB systems can be obtained as approximation (by truncating Taylor expansion) to weakly nonlinear systems [Phillips '03].

C. Gu. QLMOR: A Projection-Based Nonlinear Model Order Reduction Approach Using Quadratic-Linear Representation of Nonlinear Systems. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 30(9):1307–1320, 2011.

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But **exact representation** of smooth nonlinear systems possible:

Theorem [Gu '09/'11]

Assume that the state equation of a nonlinear system is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking derivatives and adding algebraic equations, respectively, the nonlinear system can be transformed into a QB(DAE) system.

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McCormick Relaxation

Idea borrowed from non-convex optimization:

- **Lift to higher dimensions using $const. \cdot n$ additional variables,**
- convex relaxation.

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Example

$$\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \qquad \dot{x}_2 = -x_2 + u.$$

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Alternatively, polynomial-bilinear system can be obtained using iterated Lie brackets [Gu '11].

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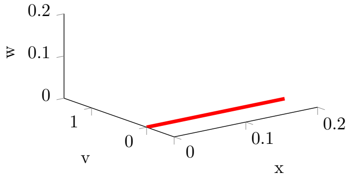
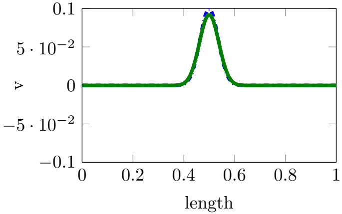
Some QB-transformable Systems

FitzHugh-Nagumo model



- Model describes activation and de-activation of neurons.
- Contains a cubic nonlinearity, which can be transformed to QB form.

Sine-Gordon equation



- Applications in biomedical studies, mechanical transmission lines, etc.
- Contains sin function, which can also be rewritten into QB form.

The Laplace transform

Definition

The Laplace transform of a time domain function $f \in L_{1,\mathrm{loc}}$ with $\mathrm{dom}\,(f) = \mathbb{R}_0^+$ is

$$\mathcal{L} : f \mapsto F, \quad F(s) := \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s \in \mathbb{C}.$$

F is a function in the (Laplace or) frequency domain.

Note: With $\Re s = 0$ and $\Im s \geq 0$, $\omega := \Im s$ takes the role of a frequency (in [rad/s], i.e., $\omega = 2\pi\nu$ with ν measured in [Hz]).

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Lemma

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Note: for ease of notation, in the following we will use lower-case letters for both, a function and its Laplace transform!

Transfer functions of linear systems

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s) - x(0))$ to linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(0) = 0$ yields:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),$$

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$\Longrightarrow$ I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=: G(s)} \Big) u(s).$$

$G(s)$ is the **transfer function** of Σ.

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Model reduction in frequency domain: Fast evaluation of mapping $u \to y$.

Linear Systems and their Transfer Functions

Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{q\times r},\ \hat{D} \in \mathbb{R}^{q\times m} \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

Linear Systems and their Transfer Functions

Formulating model reduction in frequency domain

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of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G}) \leq r} \|G - \hat{G}\|.$

Outline

Balanced Truncation for Linear Systems

Basic concept

- System Σ : $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

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Balanced Truncation for Linear Systems

Basic concept

- System Σ : $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

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$$\begin{aligned} \mathcal{T} : (A, B, C) &\mapsto (TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.

Balanced Truncation for Linear Systems

Motivation:

HSV are system invariants: they are preserved under $\mathcal{T}$ and determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

"functional analyst's point of view"

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Minimum energy to reach x_0 in balanced coordinates:

$$\inf_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \int_{-\infty}^{0} u(t)^T u(t)\, dt = x_0^T P^{-1} x_0 = \sum_{j=1}^{n} \frac{1}{\sigma_j} x_{0,j}^2$$

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Energy contained in the system if $x(0) = x_0$ and $u(t) \equiv 0$ in balanced coordinates:

$$\|y\|_2^2 = \int_0^{\infty} y(t)^T y(t)\, dt = x_0^T Q x_0 = \sum_{j=1}^{n} \sigma_j x_{0,j}^2$$

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$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int\limits_0^\infty y(t)^T y(t)\, dt}{\int\limits_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2.$$

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"engineer's point of view"

$\Longrightarrow$ Truncate states corresponding to "small" HSVs

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Properties

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

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Practical implementation

- Rather than solving Lyapunov equations for P, Q (n^2 unknowns!), find $S, R \in \mathbb{R}^{n \times s}$ with $s \ll n$ such that $P \approx SS^T$, $Q \approx RR^T$.
- Reduced-order model directly obtained via small-scale ($s \times s$) SVD of $R^T S$!
- No $\mathcal{O}(n^3)$ or $\mathcal{O}(n^2)$ computations necessary!

Outline

Approaches

- Nonlinear balancing based on energy functionals [Scherpen '93, Gray/Mesko '96].

Definition [Scherpen '93, Gray/Mesko '96]

The reachability energy functional, $L_c(x_0)$, and observability energy functional, $L_o(x_0)$ of a system are given as:

$$L_c(x_0) = \inf_{\substack{u \in L_2(-\infty,0] \\ x(-\infty)=0,\ x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt, \qquad L_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \|y(t)\|^2 dt.$$

Disadvantage: energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.

Balanced Truncation for Nonlinear Systems

Approaches

- Nonlinear balancing based on energy functionals [Scherpen '93, Gray/Mesko '96].
  Disadvantage: energy functionals are the solutions of nonlinear Hamilton-Jacobi equations which are hardly solvable for large-scale systems.
- Empirical Gramians/frequency-domain POD [Lall et al '99, Willcox/Peraire '02].

Example: controllability Gramian from time domain data (snapshots)

1. Define reachability Gramian of the system

$$P = \int_0^\infty x(t)x(t)^T dt, \quad \text{where } x(t) \text{ solves } \dot{x} = f(x, \delta),\ x(0) = x_0.$$

2. Use time-domain integrator to produce snapshots $x_k \approx x(t_k)$, $k = 1, \ldots, K$.
3. Approximate $P \approx \sum_{k=0}^{K} w_k x_k x_k^T$ with positive weights w_k.
4. Analogously for observability Gramian.
5. Compute balancing transformation and apply it to nonlinear system.

Disadvantage: Depends on chosen training input (e.g., $\delta(t_0)$) like other POD approaches.

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- For recent developments on empirical Gramians: next talk by C. Himpe!

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Balanced Truncation for QB Systems

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- For bilinear systems, such local bounds were derived in [B./Damm '11] using the solutions to the Lyapunov-plus-positive equations:

$$AP + PA^T + \sum_{i=1}^m A_i P A_i^T + BB^T = 0,$$
$$A^T Q + QA^T + \sum_{i=1}^m A_i^T Q A_i + C^T C = 0.$$

  (If their solutions exist, they define reachability and observability Gramians of BIBO stable bilinear system.)
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- Efficient solution methods for Lyapunov-plus-positive equations are derived in [B./Breiten '13, Shank/Simoncini/Szyld '16].
- **Here we aim at determining algebraic Gramians for QB (and polynomial) systems, which**
  - **provide bounds for the energy functionals of QB systems,**
  - **generalize the Gramians of linear and bilinear systems, and**
  - **allow us to find the states that are hard to control as well as hard to observe in an efficient and reliable way.**

Controllability Gramians

- Consider input $\to$ state map of QB system ($m = 1,\ N \equiv A_1$):

$$\dot{x}(t) = Ax(t) + Hx(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \qquad x(0) = 0.$$

- Integration yields

$$x(t) = \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Nx(t-\sigma_1)u(t-\sigma_1)d\sigma_1 + \int_0^t e^{A\sigma_1} Hx(t-\sigma_1) \otimes x(t-\sigma_1)d\sigma_1$$

[Rugh '81]

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$$= \int_0^t e^{A\sigma_1} Bu(t-\sigma_1)d\sigma_1 + \int_0^t \int_0^{t-\sigma_1} e^{A\sigma_1} Ne^{A\sigma_2} Bu(t-\sigma_1)u(t-\sigma_1-\sigma_2)d\sigma_1 d\sigma_2$$

$$+ \int_0^t \int_0^{t-\sigma_1} \int_0^{t-\sigma_1} e^{A\sigma_1} H(e^{A\sigma_2} B \otimes e^{A\sigma_3} B)u(t-\sigma_1-\sigma_2)u(t-\sigma_1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3 + \ldots$$

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- By iteratively inserting expressions for $x(t - \bullet)$, we obtain the **Volterra series expansion** for the QB system. [Rugh '81]

Gramians for QB Systems

Controllability Gramians

Using the *Volterra kernels*, we can define the controllability mappings

$$\Pi_1(t_1) := e^{At_1}B, \qquad \Pi_2(t_1,t_2) := e^{At_1}N\Pi_1(t_2),$$

$$\Pi_3(t_1,t_2,t_3) := e^{At_1}[H(\Pi_1(t_2)\otimes\Pi_1(t_3)), N\Pi_2(t_1,t_2)], \ldots$$

and a candidate for a new Gramian:

$$P := \sum_{k=1}^{\infty} P_k, \qquad \text{where} \quad P_k = \int_0^\infty \cdots \int_0^\infty \Pi_k(t_1,\ldots,t_k)\Pi_k(t_1,\ldots,t_k)^T \, dt_1 \ldots dt_k.$$

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Theorem [B./Goyal '16]

If it exists, the new controllability Gramian P for QB (MIMO) systems with stable A solves the **quadratic Lyapunov equation**

$$AP + PA^T + \sum_{k=1}^{m} A_k P A_k^T + H(P\otimes P)H^T + BB^T = 0.$$

Note: $H = 0 \rightsquigarrow$ "bilinear reachability Gramian"; if additionally, all $A_k = 0 \rightsquigarrow$ linear one.

Gramians for QB Systems

Dual systems and observability Gramians [Fujimoto et al. '02]

- Controllability energy functional (Gramian) of the dual system $\Leftrightarrow$ observability energy functional (Gramian) of the original system.

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- Controllability energy functional (Gramian) of the dual system $\Leftrightarrow$ observability energy functional (Gramian) of the original system.
- Employ close relation between port-Hamiltonian systems and dual systems of nonlinear systems.
- This allows to define dual systems for QB systems:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Hx(t) \otimes x(t) + \sum\nolimits_{k=1}^{m} A_k x(t) u_k(t) + Bu(t), & x(0) &= 0,\\
\dot{x}_d(t) &= -A^T x_d(t) - H^{(2)} x(t) \otimes x_d(t) - \sum\nolimits_{k=1}^{m} A_k^T x_d(t) u_k(t) - C^T u_d(t), & x_d(\infty) &= 0,\\
y_d(t) &= B^T x_d(t),
\end{aligned}
$$

where $H^{(2)}$ is the mode-2 matricization of the QB Hessian.

Gramians for QB Systems

Dual systems and observability Gramians for QB systems [B./Goyal '16]

- Writing down the Volterra series for the dual system $\rightsquigarrow$ observability mapping.

- This provides the observability Gramian Q for the QB system. It solves

$$A^T Q + QA + \sum_{k=1}^{m} A_k^T Q A_k + H^{(2)}(P \otimes Q) \left(H^{(2)}\right)^T + C^T C = 0.$$

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Remarks:

- Observability Gramian depends on controllability Gramian!
- For $H = 0$, obtain "bilinear observability Gramian", and if also all $A_k = 0$, the linear one.

Gramians and Energy Functionals

Bounding the energy functionals:

Lemma [B./Goyal '16]

In a neighborhood of the stable equilibrium, $B_\varepsilon(0)$,

$$L_c(x_0) \geq \frac{1}{2} x_0^T P^{-1} x_0, \qquad L_o(x_0) \leq \frac{1}{2} x_0^T Q x_0, \qquad x_0 \in B_\varepsilon(0),$$

for "small signals" and x_0 pointing in unit directions.

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for "small signals" and x_0 pointing in unit directions.

Another interpretation of Gramians in terms of energy functionals

1. If the system is to be steered from 0 to x_0, where $x_0 \notin \mathrm{range}\,(P)$, then $L_c(x_0) = \infty$ for all feasible input functions u.
2. If the system is (locally) controllable and $x_0 \in \ker\,(Q)$, then $L_o(x_0) = 0$.

Gramians and Energy Functionals

Illustration using a scalar system

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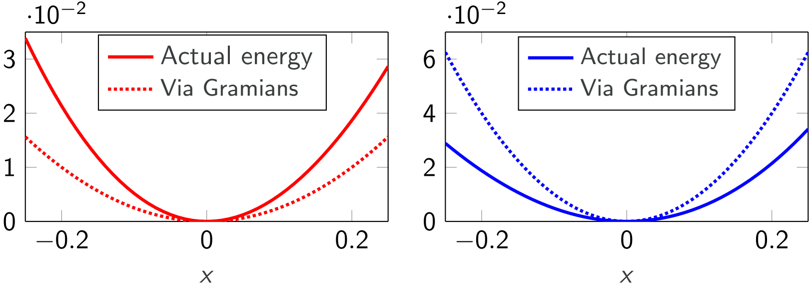

(a) Input energy lower bound. (b) Output energy upper bound.

Figure: Comparison of energy functionals for $-a = b = c = 2, h = 1, n = 0$.

Truncated Gramians

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Definition (Truncated Gramians) [B./Goyal '16]

The truncated Gramians $P_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ for QB systems satisfy

$$AP_{\mathcal{T}} + P_{\mathcal{T}}A^T = -BB^T - \sum_{k=1}^{m} A_k P_l A_k^T - H(P_l \otimes P_l)H^T,$$

$$A^T Q_{\mathcal{T}} + Q_{\mathcal{T}}A = -C^T C - \sum_{k=1}^{m} A_k^T Q_l A_k - H^{(2)}(P_l \otimes Q_l)(H^{(2)})^T,$$

where

$$AP_l + P_l A^T = -BB^T \quad \text{and} \quad A^T Q_l + Q_l A = -C^T C.$$

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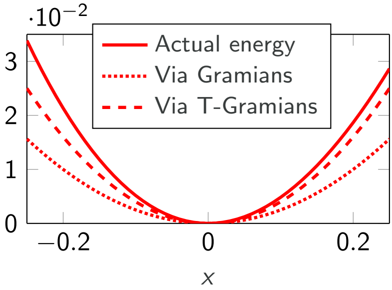

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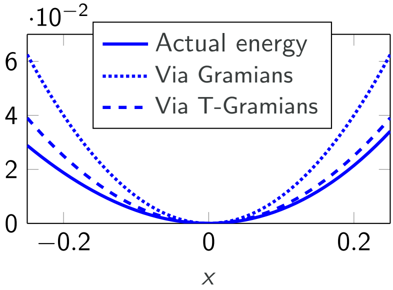

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Figure: Comparison of energy functionals for $-a = b = c = 2, h = 1, n = 0$.

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Advantages of truncated Gramians (T-Gramians)

- T-Gramians approximate energy functionals better than the actual Gramians.

- $\sigma_i(P \cdot Q) > \sigma_i(P_{\mathcal{T}} \cdot Q_{\mathcal{T}}) \Rightarrow$ obtain smaller order of reduced system if truncation is done at the same cutoff threshold.

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- Most importantly, we need solutions of only four standard Lyapunov equations.
- Interpretation of controllability/observability of the system via T-Gramians:
  - If the system is to be steered from 0 to x_0, where $x_0 \notin \mathrm{range}\,(P_{\mathcal{T}})$, then $L_c(x_0) = \infty$.
  - If the system is controllable and $x_0 \in \ker\,(Q_{\mathcal{T}})$, then $L_o(x_0) = 0$.

Balanced Truncation Algorithm

Algorithm 1 Balanced Truncation MOR for QB Systems (Truncated Gramians).

1: **Input:** A, H, A_k, B, C.

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5: **Output:** reduced-order matrices:
$$\hat{A} = \mathcal{W}^T A\mathcal{V}, \quad \hat{H} = \mathcal{W}^T H(\mathcal{V}\otimes\mathcal{V}), \quad \hat{A}_k = \mathcal{W}^T A_k\mathcal{V},$$
$$\hat{B} = \mathcal{W}^T B, \quad \hat{C} = C\mathcal{V}.$$

Remark: There are efficient ways to compute $\hat{H}$, avoiding the explicit computation of $\mathcal{V}\otimes\mathcal{V}$. [B./Breiten '15, B./Goyal/Gugercin. '16]

Chafee-Infante equation

$$\begin{aligned} v_t + v^3 &= v_{xx} + v, && (0, L) \times (0, T), \\ v(0, .) &= u(t), && (0, T), \\ v_x(L, .) &= 0, && (0, T), \\ v(x, 0) &= v_0(x), && (0, L). \end{aligned}$$

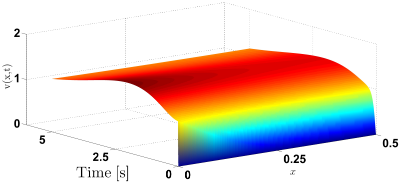

Figure: Chafee-Infante equation.

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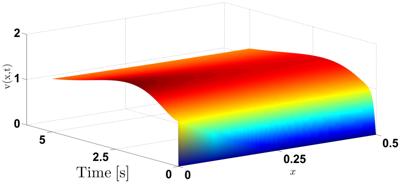

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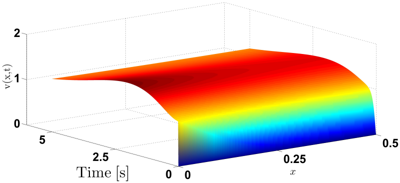

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- Cubic nonlinearity that can be rewritten into QB form. [B./Breiten '15']
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- We determine the reduced-order system of order $r = 10$.

Chafee-Infante equation

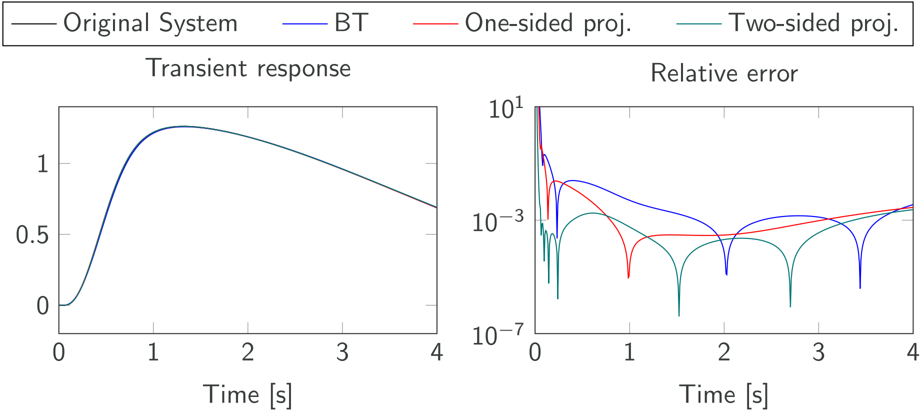

Figure: Boundary control for a control input $u(t) = 5t\ \exp(-t)$.

Chafee-Infante equation

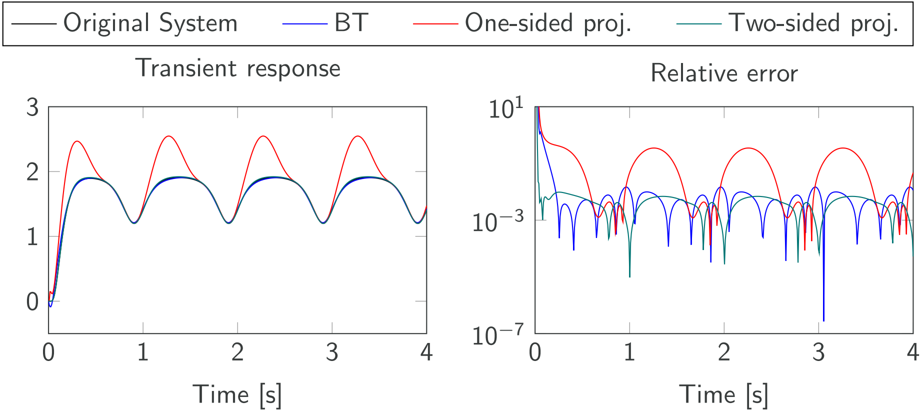

Figure: Boundary control for a control input $u(t) = 25(1 + \sin(2\pi t))/2$.

FitzHugh-Nagumo (F-N) model

$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + q,$$
$$w_t(x,t) = hv(x,t) - \gamma w(x,t) + q,$$

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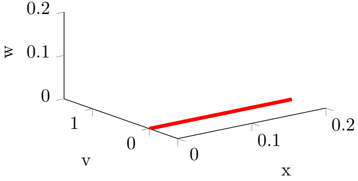
$$f(v(x,t)) = v(v-0.1)(1-v).$$



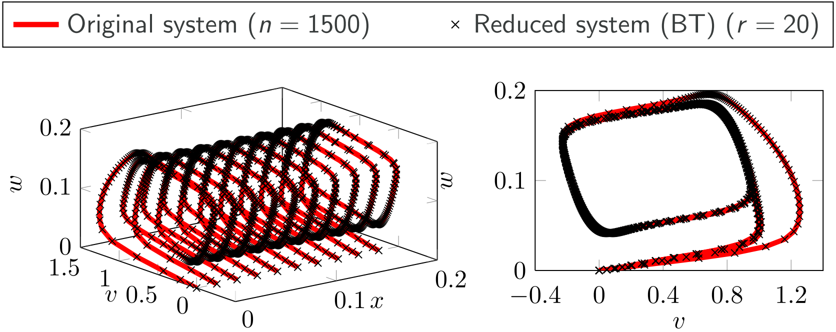
The boundary conditions are as follows:

$$v_x(0,t) = i_0(t), \quad v_x(L,t) = 0, \quad t \geq 0,$$

where $\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ q = 0.05,$
$L = 0.2.$

- Input $i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)$ serves as actuator.

FitzHugh-Nagumo (F-N) model



(a) Limit-cycles at various x. (b) Projection onto the $v-w$ plane.

Figure: Comparison of the limit-cycles obtained via the original and reduced-order (BT) systems. The reduced-order systems constructed by moment-matching methods were unstable.

Outline

Polynomial Control Systems

Now, consider the class of **polynomial control (PC) Systems**:

$$\dot{x}(t) = Ax(t) + \sum_{j=2}^{n_p} H_j \left(\otimes^j x(t)\right) + \sum_{j=2}^{n_p}\sum_{k=1}^{m} N_j^k \left(\otimes^j x(t)\right) u_k(t) + Bu(t),$$
$$y(t) = Cx(t), \quad x(0) = 0,$$

where

- n_p is the degree of the polynomial part of the system,
- $x(t) \in \mathbb{R}^n$, $\otimes^j x(t) = \underbrace{x(t) \otimes \cdots \otimes x(t)}_{j\text{-times}}$,
- $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$, $n \gg m, p$.
- $A \in \mathbb{R}^{n\times n}$, $H_j, N_j^k \in \mathbb{R}^{n\times n^j}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$.
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Examples: FitzHugh-Nagumo and Chafee-Infante equations lead to cubic control systems; cubic-quintic Allen-Cahn equation to quintic control system.

Gramians for PC Systems

The reachability Gramian

Expanding the response of the PC system into a Volterra series representation and following the same ideas as in the QB case, we define the reachability Gramian as

$$P = \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \bar{P}_k(t_1, \ldots, t_k) \bar{P}_k(t_1, \ldots, t_k)^T dt_1 \ldots dt_k,$$

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where $\bar{P}_1(t_1) = e^{At_1} B, \quad \bar{P}_2(t_1, t_2) = \sum_{k=1}^{m} e^{At_1} N_1^k e^{At_2} B,$

$\bar{P}_3(t_1, t_2, t_3) = e^{At_1} H_2 e^{At_2} B \otimes e^{At_3} B, \ldots$ are the kernels of the Volterra series.

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Theorem

The reachability Gramian **P** of a PC system solves the polynomial Lyapunov equation

$$AP + PA^T + BB^T + \sum_{j=2}^{n_p} H_j \left(\otimes^j P \right) H_j^T + \sum_{j=2}^{n_p} \sum_{k=1}^{m} N_j^k \left(\otimes^j P \right) \left(N_j^k \right)^T = 0.$$

Dual system and observability Gramian

The Observability Gramian is defined as follows

Gramians for PC Systems

Dual system and observability Gramian

The Observability Gramian is defined as follows

- First, we write the adjoint system as [Fujimoto et. al. '02]

$$
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\dot{x}(t) &= Ax(t) + \sum_{j=2}^{n_p} H_j x_j^{\otimes}(t) + \sum_{j=1}^{n_p}\sum_{k=1}^{m} N_j^k x_j^{\otimes}(t) u_k(t) + Bu(t),\\
\dot{x_d}(t) &= -A^T x_d(t) - \sum_{j=2}^{n_p} H_j^{(2)} x_{d,j}^{\otimes}(t) - \sum_{j=1}^{n_p}\sum_{k=1}^{m} \left(N_j^{k,(2)}\right) x_{d,j}^{\otimes}(t) u_{d,k}(t) - C^T u_d(t), \quad x_d(\infty) = 0,\\
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y_d(t) &= B^T x_d(t).
\end{aligned}
$$

- Then, by taking the kernel of Volterra series, one has

Theorem

Let $\mathbf{P}$ be the reachability Gramian. Then, the observability Gramian $\mathbf{Q}$ of a PC system solves the polynomial Lyapunov equation

$$
A^T Q + QA + C^T C + \sum_{j=2}^{n_p} H_j^{(2)} \left(\otimes^{j-1} P \otimes Q\right) \left(H_j^{(2)}\right)^T + \sum_{j=2}^{n_p} \sum_{k=1}^{m} N_j^{k,(2)} \left(\otimes^{j-1} P \otimes Q\right) \left(N_j^{k,(2)}\right)^T = 0.
$$

Truncated Gramians

- Polynomial Lyapunov equations are very expensive to solve.
- As for QB systems, we thus propose truncated Gramians that only involve a finite number of kernels.

$$P_{\mathcal{T}} = \sum_{k=1}^{n_p+1} \int_0^\infty \cdots \int_0^\infty \bar{P}_k(t_1,\ldots,t_k)\bar{P}_k(t_1,\ldots,t_k)^T dt_1 \ldots dt_k,$$

Truncated Gramians

The reachability truncated Gramian solves

$$AP_{\mathcal{T}} + P_{\mathcal{T}}A^T + BB^T + \sum_{j=2}^{n_p} H_j \otimes^j P_l H_j^T + \sum_{j=2}^{n_p}\sum_{k=1}^{m} N_j^k \otimes^j P_l \left(N_j^k\right)^T = 0.$$

where $AP_l + P_l A^T + BB^T = 0$

- Advantage: Only need to solve a finite number of (linear) Lyapunov equations.

Numerical Example, the FitzHugh-Nagumo model, revisited

$$\epsilon v_t(x,t) = \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + q,$$
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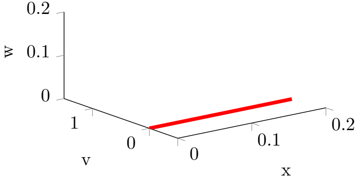
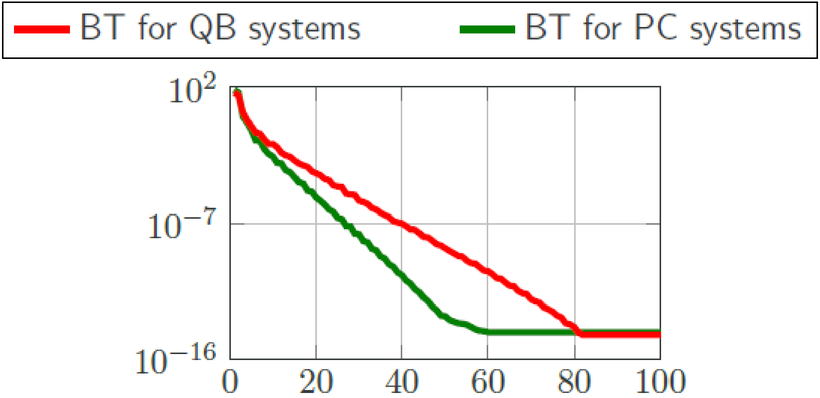


- After discretization we obtain a PC system with cubic nonlinearity of order $n_{pc} = 600$. [B./Breiten '15]
- The transformed quadratic-bilinear (QB) system is of order $n_{qb} = 900$.
- The outputs of interest $v(0,t), w(0,t)$ are the responses at the left boundary at $x = 0$.
- We compare balanced truncation for PC and QB systems.

Numerical Example

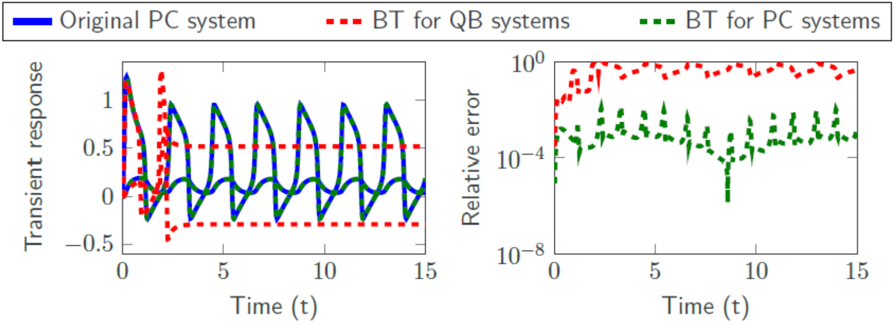
Singular values decay



- Decay singular values for PC systems is faster ⇒ smaller reduced order models.

Numerical Example

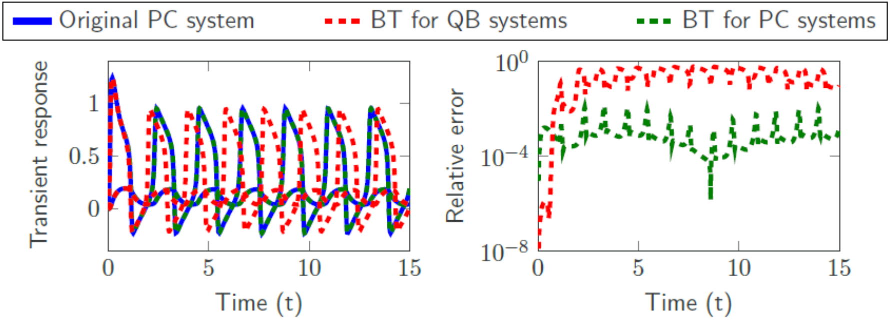
Time-domain simulations



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 10.

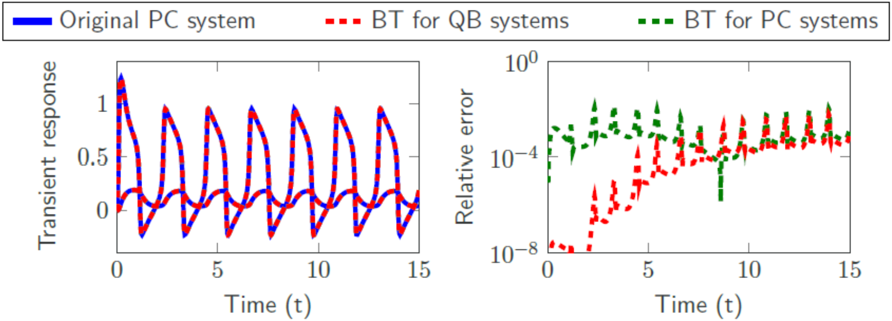
Numerical Example

Time-domain simulations



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 30.

Time-domain simulations



- Original PC system of order 600. Original QB system of order 900.
- Reduced PC system of order 10. Reduced QB system of order 43.

Conclusions

- BT extended to bilinear, QB, and polynomial systems.
- Local Lyapunov stability is preserved.
- As of yet, only weak motivation by bounding energy functionals.
- No error bounds in terms of "Hankel" singular values.
- Computationally efficient (as compared to nonlinear balancing), and input independent.
- **To do:**
  - improve efficiency of Lyapunov solvers with many right-hand sides further;
  - error bound;
  - conditions for existence of new QB Gramians;
  - extension to descriptor systems;
  - time-limited versions.

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