MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Stabilizing Flow Problems using State-Dependent Riccati Equations

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Outline

- Starting point: consider optimized nonlinear process, operating at potentially unstable steady-state.

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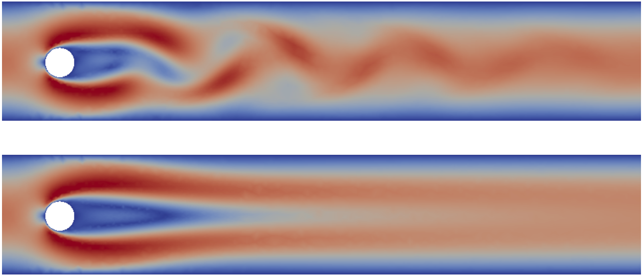

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- Here: apply this concept to semi-discretized unsteady PDE control problems with ultimate goal of stabilizing incompressible flows described by Navier-Stokes equations!

Set-point control of an autonomous system

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where $\xi := \zeta - z^*$, and, accordingly, $\tilde{f}(\xi) = f(\xi + z^*)$.

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If f is Lipshitz-continuous, then (since $\tilde{f}(0) = 0$) there is a matrix valued function A, such that

$$\tilde{f}(\xi) = A(\xi)\xi.$$

Subject:

- **Extended linearizations** or **state-dependent coefficient** systems

$$\dot{\xi}(t) = A[\xi(t)]\xi(t) + Bu(t), \quad \xi(0) = x_0 \in X_0 \subset \mathbb{R}^n,$$

where $A\colon \mathbb{R}^n \to \mathbb{R}^{n \times n}$.

Application:

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Related Work:

- Extends the approach of linearization based stabilization of *set points* as in [PB&JH '15, Breiten&Kunisch '14]
- [Banks&al. '07] – nonlinear feedback laws via state-dependent Riccati equations
- [Hill&Ilchmann '11] – stability of linear time-varying systems

Stability of SDC Systems

To describe exponential stability for the considered type of SDC systems (with $u \equiv 0$)

$$\dot{\xi}(t) = A[\xi(t)]\xi(t), \quad \xi(0) = x_0 \qquad \text{(SDC)}$$

we adapt a definition for time varying systems:

Definition

System (SDC) is called **exponentially stable** if there exist positive constants K and ω such that the solution of (SDC) – with $\xi(0) = x_0$ – satisfies

$$\|\xi(t)\| \leq Ke^{-\omega t}\|x_0\|, \quad \text{for } t \geq 0.$$

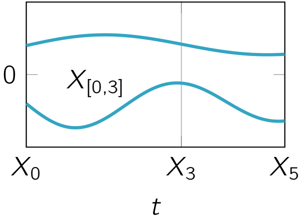
It is called **uniformly exponentially stable on X_0**, if, for some $X \subset \mathbb{R}^n$, stability is given for any $x_0 \in X_0$.

Key observation:

- In SDC systems, the coefficient A is stabilized together with the state.

Ideas:

- Consider the set $X_{[0,T]}$ in which the system evolves until a time T, and
- assume that $\|e^{A(x)t}\| \leq Ke^{-\omega t}$ for any $x \in X_{[0,T]}$ and $t \geq 0$.
- If there is a t^*, such that $X_{t^*} \subsetneq X_0$, then SDC is uniformly exponentially stable on X_0 [PB&JH '18].



Theorem ([PB&JH '18])

For a given $T > 0$, let A be differentiable and uniformly stable with constants K and ω on $X_{[0,T]}$ and for $0 \leq t \leq T$, let

$$m_t := \inf_{\rho \in \mathbb{R}_{\geq 0}} \sup_{\xi \in \Xi_{[0,t]}} \frac{\int_0^t e^{-\omega(t-s)} \|A[\xi(s)] - A[\xi(\rho)]\| \|\xi(s)\| \, ds}{\int_0^t e^{-\omega(t-s)} |s - \rho| \|\xi(s)\| \, ds}.$$

If for some t^*, with $0 < t^* \leq T$,

$$-\omega_{t^*} := \sqrt{K m_{t^*} \ln 2} - \omega \quad \text{and} \quad -\omega^* := \frac{\ln K}{t^*} - \omega_{t^*}$$

are negative, then the snapshots $\xi(t)$ of any solution ξ to (SDC) with $\xi(0) = x_0 \in X_0$ on the discrete grid $\mathcal{T}^* := \{t \colon t = N \cdot t^*, \; N = 0, 1, \dots\}$ decay exponentially in the sense that

$$\|\xi(t)\| \leq \|x_0\| e^{-\omega^* t}, \quad \text{for all } t \in \mathcal{T}^*.$$

Now:

- Feedback stabilization of (SDC)

$$\dot{\xi}(t) = A[\xi(t)]\xi(t) + Bu(t) :$$

find a (state-dependent) feedback gain F, such, that the closed loop system

$$\dot{\xi}(t) = \big[A[\xi(t)] - BF[\xi(t)]\big]\xi(t) \quad \text{is stable.}$$

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- By the preceding theory, we need bounds on decay and transient behavior uniformly in $x \in X$:

Definition

We say that $A[\cdot]$ with $A[\xi(t)] \in \mathbb{R}^{n,n}$ is in class $\mathcal{S}_{K,\omega}$ for given constants K and ω, if

$$\|e^{At}\| \leq Ke^{-\omega t}, \quad \text{for } t > 0.$$

Regression:

- If $A(x) - BF(x) \in \mathcal{S}_{K,\omega}$, where $F(x) = R^{-1}B^{\mathsf{T}}P$ and where $P = P(x)$ solves the Riccati equation

$$PA(x) + A(x)^{\mathsf{T}}P - PBR^{-1}B^{\mathsf{T}}P + Q = 0,$$

  for given $R \succ 0$ and $Q \succcurlyeq 0$, Regression:

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- then

$$\begin{bmatrix} A(x) & -BR^{-1}B^\mathsf{T} \\ -Q & -A(x)^\mathsf{T} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} Z,$$

  and $Z = A(x) - BR^{-1}B^\mathsf{T}P \in \mathcal{S}_{K,\omega}$.

Lemma (Updating Riccati Feedback [PB&JH '18])

Consider the Riccati based feedback so that

$$Z := A(x) - BR^{-1}B^{\mathsf{T}}P \in \mathcal{S}_{K,\omega}.$$

If for an $A_\Delta \in \mathbb{R}^{n,n}$, *there exists* $E \in \mathbb{R}^{n,n}$ *such that*

$$\begin{bmatrix} A(x) + A_\Delta & -BR^{-1}B^{\mathsf{T}} \\ -Q & -A(x)^{\mathsf{T}} - A_\Delta^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} I + E \\ P \end{bmatrix} = \begin{bmatrix} I + E \\ P \end{bmatrix} Z,$$

and if $\|E\| < 1$, *then* $(I + E)$ *is invertible and*

$$A(x) + A_\Delta - BR^{-1}B^{\mathsf{T}}P(I + E)^{-1} \in \mathcal{S}_{\tilde{K},\omega}, \quad \text{with} \quad \tilde{K} = \tfrac{1+\|E\|}{1-\|E\|}K.$$

- As time evolves, the coefficients change:

$$A(\xi(t + t_\Delta)) = A(x + x_\Delta) =: A(x) + A_\Delta$$

Numerical Realization

- As time evolves, the coefficients change:

  $$A(\xi(t + t_\Delta)) = A(x + x_\Delta) =: A(x) + A_\Delta$$

- **The p-update-Algorithm:**

  (1) Start with a Riccati feedback P_0 based on $A(x)$

  (2) At every time instance, solve a Sylvester equation for an update E:

  $$(A(x) + A_\Delta)E + E(A(x) - BR^{-1}B^{\mathsf{T}}P_0) = -A_\Delta$$

  (2a) If $\|E\|$ exceeds a threshhold ϵ, reset the current feedback with a new Riccati feedback $P_0 \leftarrow P_{\text{new}}$.

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- The code for the example is available from gitlab.mpi-magdeburg.mpg.de/heiland/code-ext-lin-stab.

For the spatial coordinate $z \in (0,2)$ and time $t \in (0,3]$, consider the autonomous PDE

$$\begin{aligned} \dot{\xi} &= \partial_{zz}\xi + 5(1-\xi^2)\xi, \quad x_0 = 0.2\sin(0.5\pi z), \\ \partial_z \xi(t)\big|_{z=2} &= u(t) \\ \xi(t)\big|_{z=0} &= 0 \end{aligned}$$

which we turn into a SDRC system of order N via a spatial finite element discretization with N degrees of freedom.

The *Chafee-Infante* equation has one unstable set point at $\xi(t,z) = 0$ and two stable set points.

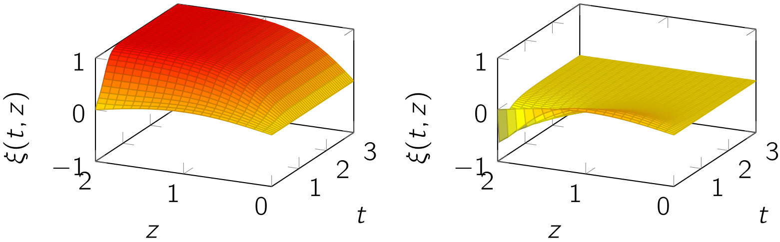

Figure: The uncontrolled (left) and the stabilized (right) evolution of the solution to the *Chafee-Infante* equation.

Task: force the system into the unstable set point.

	Scheme	ϵ	#fb-switches	#f-eva	comp-time
$N = 20$	sdre	0	—	442	$1.921s$
	p-update	0.5	2	838	$3.266s$
	p-update	0.9	0	451	$1.756s$
$N = 60$	sdre	0	—	1194	$15.426s$
	p-update	0.5	4	2240	$18.379s$
	p-update	0.9	2	1770	$14.140s$
$N = 100$	sdre	0	—	2106	$90.148s$
	p-update	0.5	7	3778	$68.080s$
	p-update	0.9	4	2423	$43.816s$

Table: Influence of ϵ on the number of switches #fb-switches in the feedback, on the number of function evaluations #f-eva in the time integrator, and on the overall computation time for the simulation of the stabilized *Chafee-Infante* equation with finite element discretizations with mesh size N.

We consider the incompressible Navier-Stokes equations

$$
\begin{aligned}
\dot{\mathrm{v}}(t) &= -(\mathrm{v}(t)\cdot\nabla)\mathrm{v}(t) + \tfrac{1}{Re}\Delta\mathrm{v}(t) - \nabla\mathrm{p}(t) + \mathcal{B}u(t), \quad \mathrm{v}(0) = \mathrm{v}_0\\
0 &= \nabla\cdot\mathrm{v}(t)
\end{aligned}
$$

and its semi-discretization (e.g. by mixed Finite Elements):

$$
\begin{aligned}
M\dot{v}(t) &= N(v(t))v(t) + Lv(t) + G^{\mathsf{T}}p(t) + Bu(t), \quad v(0) = v_0,\\
0 &= Gv(t),
\end{aligned}
$$

that models the evolution of the velocity v and pressure p in an incompressible flow with

- $M \in \mathbb{R}^{n,n}$, $M \succ 0$ – the mass matrix,
- $L \in \mathbb{R}^{n,n}$ – the discrete diffusion,
- $G^{\mathsf{T}} \in \mathbb{R}^{n,m}$, $n > m$, full rank – the discrete gradient
- $N\colon \mathbb{R}^n \to \mathbb{R}^{n,n}$, linear – the discrete convection

Application to Incompressible NSE

Let (v^*, p^*) be a set point of the NSE:

- define $v_\delta(t) := v(t) - v^*$ and $p_\delta(t) = p(t) - p^*$
- and consider the difference system[34]

$$
\begin{aligned}
M\dot{v}_\delta(t) &= A(v_\delta(t))v_\delta(t) + G^\mathsf{T} p_\delta(t) + Bu(t), \quad v_\delta(0) = v_0 - v^*,\\
0 &= Gv_\delta(t),
\end{aligned}
$$

- where $A(v_\delta) := N(v_\delta) + N(v^*) + N^*(v^*) + L \in \mathbb{R}^{n,n}$
- and N^* is defined via $N^*(v^*)v_\delta = N(v_\delta)v^*$.

[34] no approximation here

Lemma

If $G(v_0 - v^*) = 0$*, then* v_δ *satisfies the ODE*

$$M\dot{v}_\delta(t) = \Pi^\mathsf{T} A(v_\delta(t))\Pi v_\delta(t) + \Pi^\mathsf{T} Bu(t), \quad v_\delta(0) = v_0 - v^*, \qquad (*)$$

where $\Pi := I - M^{-1}G^\mathsf{T}(GM^{-1}G^\mathsf{T})^{-1}G$.

Lemma

If $G(v_0 - v^*) = 0$, then v_δ satisfies the ODE

$$M\dot{v}_\delta(t) = \Pi^\mathsf{T} A(v_\delta(t))\Pi v_\delta(t) + \Pi^\mathsf{T} Bu(t), \quad v_\delta(0) = v_0 - v^*, \qquad (*)$$

where $\Pi := I - M^{-1}G^\mathsf{T}(GM^{-1}G^\mathsf{T})^{-1}G$.

Since $(*)$ is in SDC form, the preceding results are readily extended:

Corollary

Let P_0 be the Riccati solution that stabilizes $(*)$ at the instance $v_\delta(t) = x$. The p-update that compensates $\Pi^\mathsf{T} A(x)\Pi \leftarrow \Pi^\mathsf{T}(A(x) + A_\Delta)\Pi$ is given through a solution E to

$$\Pi^\mathsf{T}(A(x) + A_\Delta)\Pi E + MEM^{-1}\Pi^\mathsf{T}(A(x)\Pi - BR^{-1}B^\mathsf{T}P_0) = -\Pi^\mathsf{T} A_\Delta \Pi$$

provided that $\|E\| < 1$.

Two major numerical linear algebra challenges:

1. the explicit use of $\Pi = I - M^{-1}G^\mathsf{T}(GM^{-1}G^\mathsf{T})^{-1}G$ must be avoided;
2. the p-update requires the solve of a large-scale unstructured Sylvester equation.

Two major numerical linear algebra challenges:

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An answer to the first:

Lemma 4 in [PB&JH '17]

The updated gains can equivalently be obtained via Z that is a solution to

$$(A_0 + A_\Delta)ZM - MZ(A_0 - BF_0) + MY_1G + G^{\mathsf{T}}Y_2^{\mathsf{T}}M = -A_\Delta,$$
$$GZM = 0, \quad \text{and} \quad MZG^{\mathsf{T}} = 0,$$

with $\|ZM\| < 1$ and for suitable $Y_1,\ Y_2 \in \mathbb{R}^{n,m}$.

Conclusion

- Sufficient conditions for stability of SDC systems:
  - Adaption of known results for linear time-varying systems with
  - localized estimates
  - based on integral mean values.
- The needed uniform bounds on the decay and the transient behavior
  - can be achieved by updating Riccati based feedback
  - using solutions of Sylvester equations.
- Proof of concept for a numerical example.
- Next steps:
  - Solvability of the Sylvester equation for the updates,
  - implementation in large scale settings.

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