MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Riccati-Based Feedback Control of Nonlinear Unsteady PDEs

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International Workshop on Optimal Control of Dynamical Systems and Applications
Osijek, Croatia
June 20–22, 2018

Optimal Control

is used for the optimization of **dynamical processes**,

described by ordinary or partial differential equations.

This is achieved by minimizing a **cost functional**

(penalizing, e.g. energy consumption, deviation from reference trajectory),

such that a prescribed target
is reached in given or **minimal time**

whilst complying with given control and state constraints.

Let (x_*, u_*) solve $\min_{u \in \mathcal{U}_{ad}} J(x,u)$ **s.t.** $\dot{x}(t) = f(x(t), u(t))$.

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Fundamental observation

Optimized trajectory $x_*(t; u_*)$ and precomputed optimal control $u_*(t)$ will not be attainable in practice due to

- modeling errors and/or unmodeled dynamics,
- model uncertainties,
- external perturbations,
- measurement errors.

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Consequence: need **feedback control**

$$u(t) = u_*(t) + U(t, x(t) - x_*(t))$$

in order to attenuate perturbations/errors!

Example: Optimal control of a simple transport model

Burgers' equation:

$$\begin{aligned}
\partial_t x(t,\xi) &= \nu\,\partial_{\xi\xi} x(t,\xi) - x(t,\xi)\,\partial_\xi x(t,\xi) + B(\xi)u(t),\\
x(t,0) &= x(t,1) = 0, \quad x(0,\xi) = x_0(\xi), \quad \xi \in (0,1),\\
y(t,\xi) &= C\,x(t,\xi).
\end{aligned}$$

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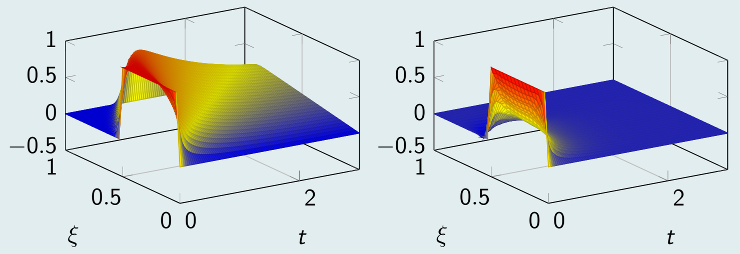

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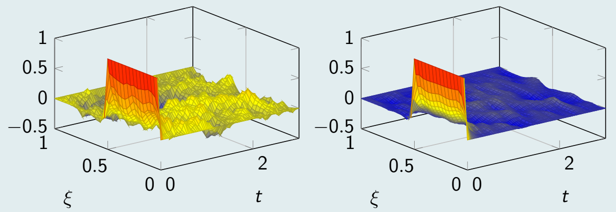
$$\begin{aligned}\partial_t x(t,\xi) &= \nu\,\partial_{\xi\xi}x(t,\xi) - x(t,\xi)\,\partial_\xi x(t,\xi) + B(\xi)u(t) + F(\xi)v(t),\\ x(t,0) &= x(t,1) = 0, \quad x(0,\xi) = x_0(\xi) + \eta(\xi), \quad \xi \in (0,1),\\ y(t,\xi) &= C\,x(t,\xi) + w(t,\xi).\end{aligned}$$

Nonlinear control (here: MPC-LQG):



Reduction of tracking error $\int_0^T \|x(t) - x_*(t)\|_2^2\,dt$ by factor > 10.

[Benner/Görner, PAMM 2006]; [Benner/Görner/Saak, Springer LNCSE 2006].

The Linear-Quadratic Regulator (LQR) Problem

Minimize $\mathcal{J}(u) = \frac{1}{2}\int\limits_0^\infty \left(y^T Q y + u^T R u\right) dt$ for $u \in \mathcal{L}_2(0,\infty;\mathbb{R}^m)$,

subject to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned}$$

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Solution of finite-dimensional LQR problem: **feedback control**

$$u_*(t) = -B^T X_* x(t) =: -K_* x(t),$$

where $X_* = X_*^T \geq 0$ is unique **stabilizing**[1] solution of **algebraic Riccati equation (ARE)**

$$0 = \mathcal{R}(X) := C^T QC + A^T X + XA - XBR^{-1}B^T X.$$

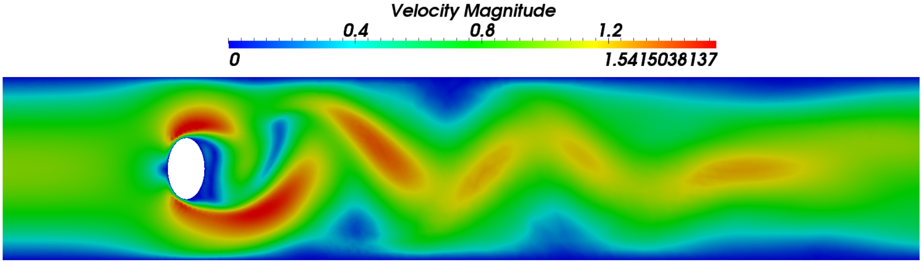
[1] X is stabilizing $\Leftrightarrow \Lambda\left(A - BB^T X\right) \subset \mathbb{C}^-$.

- **Physical transport** is one of the most fundamental dynamical processes in nature.
- **Prediction** and **manipulation** of transport processes are important research topics, e.g., to
  - avoid stall — for stable and safe flight;
  - save energy (or increase attainable speed) by minimizing drag coefficient;
  - use fluid flow for optimal transport (e.g., in blood veins).
- **Open-loop** controllers are widely used in various engineering fields.
  $\rightarrow$ **Not robust** regarding perturbation
- Dynamical systems are often influenced via so called **distributed control**.
  $\rightarrow$ **Unfeasible** in many real-world areas

$\Rightarrow$ **Boundary feedback stabilization (closed-loop)**
should be used to increase robustness and feasibility.

Overview

- Consider 2D flow problems described by **incompressible Navier–Stokes equations**.
- Riccati feedback approach requires the solution of an **algebraic Riccati equation**.
- Conservation of mass introduces a **divergence-freeness** condition $\rightsquigarrow$ problems with mathematical basis of control design schemes.



Kármán vortex street

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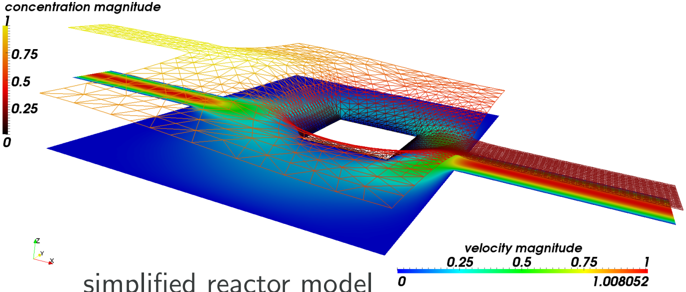
- Conservation of mass introduces a **divergence-freeness** condition $\rightsquigarrow$ problems with mathematical basis of control design schemes.
- **Coupling** flow problems with a **scalar reaction-advection-diffusion equation**.



simplified reactor model

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   Develop techniques to deal with complex-shifted multi-field flow systems.

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   - implicitly project on "hidden manifold"

   ⇒ Nested iteration: solve large-scale sparse saddle point system

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   - adapt various ideas from [Elman/Silvester/Wathen '05]
   - ⇒ develop suitable preconditioner to be used with GMRES
   - ⇒ efficient preconditioner uses various approximation methods
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- combine [Kürschner '16], [B./Byers '98], and [Feizinger/Hylla/Sachs '09]

⇒ extended ideas in [B./Heinkenschloss/Saak/Weichelt '16/'18]

⇒ develop a highly compatible method to solve Riccati equations

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- include feedback into forward simulation within NAVIER

⇒ closed-loop forward flow simulation

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Navier–Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} - \frac{1}{\text{Re}} \Delta \vec{v} + (\vec{v} \cdot \nabla) \vec{v} + \nabla p = \vec{f}$$

$$\operatorname{div} \vec{v} = 0$$

- defined for time $t \in (0, \infty)$ and space $\vec{x} \in \Omega \subset \mathbb{R}^2$ bounded with $\Gamma = \partial\Omega$
- $+$ boundary and initial conditions
- initial boundary value problem with additional algebraic constraints

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Linearize + Discretize → Index-2 DAE

$$M\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = A\mathbf{v}(t) + \widehat{G}\mathbf{p}(t) + B\mathbf{u}(t)$$

$$0 = \widehat{G}^T\mathbf{v}(t)$$

$$\mathbf{y}(t) = C\mathbf{v}(t)$$

$$A,\ M \in \mathbb{R}^{n\times n},\ \widehat{G} \in \mathbb{R}^{n\times n_{\mathbf{p}}}$$

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$$M\Pi^T = \Pi M \quad \wedge \quad \Pi^T\mathbf{v} = \mathbf{v}_{\mathrm{div},0}$$

[**Bänsch**/B./Saak/**Stoll**/Weichelt **'13**,**'15**]

Feedback Stabilization for Index-2 DAE Systems
— Physics of Multi-Field Flow —

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$$\frac{\partial c^{(\vec{v})}}{\partial t} - \frac{1}{\mathrm{Re}\,\mathrm{Sc}}\Delta c^{(\vec{v})} + (\vec{v}\cdot\nabla)c^{(\vec{v})} = 0$$

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Extension to coupled flow case, i.e.,

$$\hat{\Pi} := \begin{bmatrix}\Pi & 0\\ 0 & I\end{bmatrix} \quad \wedge \quad \begin{bmatrix}\Pi^T & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{c}\end{bmatrix} = \begin{bmatrix}\mathbf{v}_{\mathrm{div},0}\\ \mathbf{c}\end{bmatrix}.$$

[Bänsch/B./Saak/Weichelt '14]

Feedback Stabilization for Index-2 DAE Systems
— Physics of Multi-Field Flow —

Navier–Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta \vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{f}$$

$$\operatorname{div} \vec{v} = 0$$

Concentration Equation

$$\frac{\partial c^{(\vec{v})}}{\partial t} - \frac{1}{\mathrm{Re}\,\mathrm{Sc}}\Delta c^{(\vec{v})} + (\vec{v}\cdot\nabla)c^{(\vec{v})} = 0$$

Linearize + Discretize → Index-2 DAE

$$M\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = A\mathbf{x}(t) + \widehat{G}\mathbf{p}(t) + B\mathbf{u}(t)$$
$$0 = \widehat{G}^T\mathbf{x}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Show that projection in [Hei/Sor/Sun '08] is discretized version of Leray projector in [Ray '06].

$$M\Pi^T = \Pi M \quad \wedge \quad \Pi^T\mathbf{v} = \mathbf{v}_{\mathrm{div},0}$$

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[Bänsch/B./Saak/Weichelt '14]

Helmholtz Decomposition [Girault/Raviart '86]

- Splitting:

$$(L^2(\Omega))^2 = \mathbf{H}(\mathrm{div}, 0) \perp \mathbf{H}(\mathrm{div}, 0)^{\perp}$$

Divergence-free: $\mathbf{H}(\mathrm{div}, 0) := \{\vec{v} \in (L^2(\Omega))^2 \mid \mathrm{div}\, \vec{v} = 0, \vec{v} \cdot \vec{n}_{|\Gamma} = 0\}$

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Leray Projector P

This splitting is equivalent to $\vec{v} = \vec{v}_{\mathrm{div},0} + \nabla p$, where $\vec{v}_{\mathrm{div},0}$ and p fulfill

$$\begin{aligned} \vec{v}_{\mathrm{div},0} + \nabla p &= \vec{v} \quad \text{in } \Omega, \\ \mathrm{div}\, \vec{v}_{\mathrm{div},0} &= 0 \quad \text{in } \Omega, \\ \vec{v}_{\mathrm{div},0} \cdot \vec{n} &= 0 \quad \text{on } \Gamma. \end{aligned}$$

$P : (L^2(\Omega))^2 \to \mathbf{H}(\mathrm{div}, 0)$ with $P : \vec{v} \mapsto \vec{v}_{\mathrm{div},0}$.

Projection Method

[Heinkenschloss/Sorensen/Sun '08]

- Index reduction for Lyapunov-solver.
- Projector:

$$\Pi^T := I_{n_v} - M_v^{-1} G (G^T M_v^{-1} G)^{-1} G^T.$$

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Recall $P : \vec{v} \mapsto \vec{w}$:

$$\begin{aligned} \vec{w} + \nabla p &= \vec{v}, \\ \operatorname{div} \vec{w} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} M_v & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M_v \mathbf{v} \\ 0 \end{bmatrix}$$

$$\mathbf{p} = (G^T M_v^{-1} G)^{-1} G^T \mathbf{v}$$

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Leray vs. Π^T

$$\begin{aligned} \vec{w} &= P(\vec{v}), \\ 0 &= \operatorname{div} \vec{w} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{w} &= \Pi^T \mathbf{v}, \\ 0 &= G^T \mathbf{w} \end{aligned}$$

Minimize

$$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \lambda^2 ||\mathbf{y}||^2 + ||\mathbf{u}||^2 \,\mathrm{d}t$$

subject to

$$\widehat{\Theta}_r^T M \widehat{\Theta}_r \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbf{x}}(t) = \widehat{\Theta}_r^T A \widehat{\Theta}_r \widetilde{\mathbf{x}}(t) + \widehat{\Theta}_r^T B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\widehat{\Theta}_r \widetilde{\mathbf{x}}(t)$$

with $\widehat{\Pi} = \widehat{\Theta}_l \widehat{\Theta}_r^T$ such that $\widehat{\Theta}_r^T \widehat{\Theta}_l = I \in \mathbb{R}^{(n-n_\mathbf{p})\times(n-n_\mathbf{p})}$ and $\widetilde{\mathbf{x}} = \widehat{\Theta}_l^T \mathbf{x}$.

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with $\mathcal{M} = \mathcal{M}^T \succ 0$.

Minimize

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Riccati Based Feedback Approach

- Optimal control: $\mathbf{u}(t) = -\mathcal{K}\widetilde{\mathbf{x}}(t)$, with feedback: $\mathcal{K} = \mathcal{B}^T\mathcal{X}\mathcal{M}$,

  where $\mathcal{X}$ is the solution of the generalized continuous-time algebraic Riccati equation (GCARE)

$$\mathcal{R}(\mathcal{X}) = \lambda^2\mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$$

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

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Kleinman–Newton method

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

Kleinman–Newton method

Step $\mathbf{m+1}$: Solve the Lyapunov equation

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)})^T\mathcal{X}^{(m+1)}\mathcal{M} + \mathcal{M}\mathcal{X}^{(m+1)}(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)}) = -(\mathcal{W}^{(m)})^T\mathcal{W}^{(m)} \quad (1)$$

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Kleinman–Newton method

low-rank ADI method

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low-rank ADI method

Step ℓ: Solve the projected and shifted linear system

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)} + q_\ell\mathcal{M})^T\mathcal{V}_\ell = \mathcal{Y} \tag{2}$$

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linear solver

Feedback Stabilization for Index-2 DAE Systems
— Nested Iteration without Projection —

Determine $\mathcal{X} = \mathcal{X}^T \succeq 0$ such that $\mathcal{R}(\mathcal{X}) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$

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Replace (2) and **solve instead** the saddle point system (SPS)

$$\begin{bmatrix} A^T - (K^{(m)})^T B^T + q_\ell M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix} \begin{bmatrix} V_\ell \\ * \end{bmatrix} = \begin{bmatrix} Y \\ 0 \end{bmatrix}$$

for different ADI shifts $q_\ell \in \mathbb{C}^-$ for a couple of rhs Y.

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$$\begin{bmatrix} A^T + q_\ell M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix} \begin{bmatrix} V_\ell \\ * \end{bmatrix} = \begin{bmatrix} \tilde{Y} \\ 0 \end{bmatrix}$$

for different ADI shifts $q_\ell \in \mathbb{C}^-$ for a couple of rhs $\tilde{Y}$.

Theorem [B./Heinkenschloss/Saak/Weichelt '16]

- assume $(\mathbf{A}, \mathbf{B}; \mathbf{M})$ stabilizable, $(\mathbf{C}, \mathbf{A}; \mathbf{M})$ detectable
- $\Rightarrow \exists$ unique, symmetric solution $X^{(*)} = \widehat{\Theta}_r \mathcal{X}^{(*)} \widehat{\Theta}_r^T$ with $\mathcal{R}(\mathcal{X}^{(*)}) = 0$ that stabilizes
$$\left(\begin{bmatrix} A - BB^T X^{(*)} M & \widehat{G} \\ \widehat{G}^T & 0 \end{bmatrix}, \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \right)$$
- for $\{X^{(k)}\}_{k=0}^{\infty}$ defined by $X^{(k)} := \widehat{\Theta}_r \mathcal{X}^{(k)} \widehat{\Theta}_r^T$, (1), and $X^{(0)}$ symmetric with $\left(\mathbf{A} - \mathbf{B}\left(\mathbf{K}^{(0)}\right)^T, \mathbf{M}\right)$ stable, it holds that, for $k \geq 1$,
$$X^{(1)} \succeq X^{(2)} \succeq \cdots \succeq X^{(k)} \succeq 0 \quad \text{and} \quad \lim_{k \to \infty} X^{(k)} = X^{(*)}$$
- $\exists\, 0 < \widetilde{\kappa} < \infty$ such that, for $k \geq 1$,
$$||X^{(k+1)} - X^{(*)}||_F \leq \widetilde{\kappa} ||X^{(k)} - X^{(*)}||_F^2$$

Additional Contributions [Bänsch/B./Saak/Weichelt '15,'16]

- Suitable approximation framework for Raymond's projected boundary control input.
- Proposed method directly iterates on the feedback matrix $K \in \mathbb{R}^{n \times n_r}$.
- Initial feedback for index-2 DAE systems using a special eigenvalue shifting technique.
- Improved ADI shift computation for index-2 DAE systems (Penzl- and projection shifts).

Feedback Stabilization for Index-2 DAE Systems — Remarks/Open Problems —

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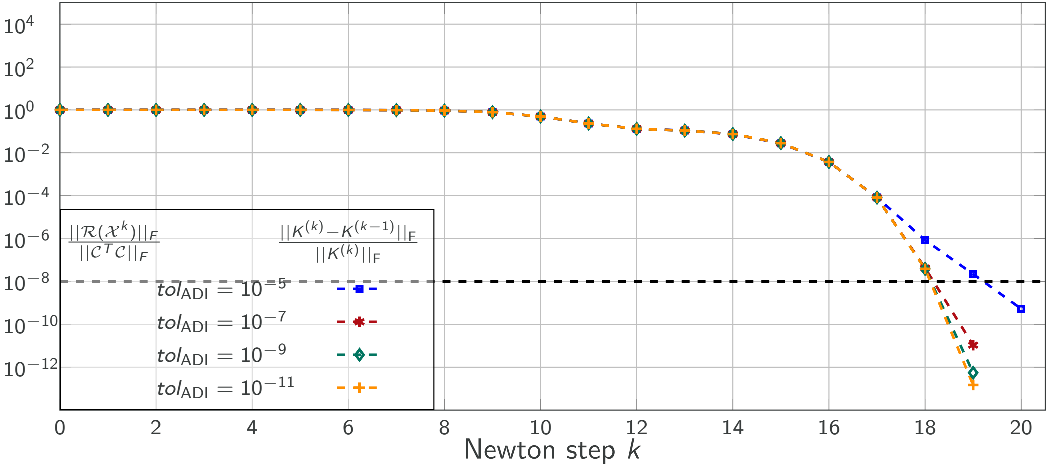
Problems up to here

- Determination of suitable stopping criteria/tolerances.
- Computation of projected residuals is very costly ($\approx$ 10x ADI step).
  $\Rightarrow$ use relative change of feedback matrix [B./Li/Penzl '08]

NSE scenario: $\mathrm{Re} = 500$, $n = 5\,468$, $\lambda = 10^2$, $tol_{\mathrm{Newton}} = 10^{-8}$



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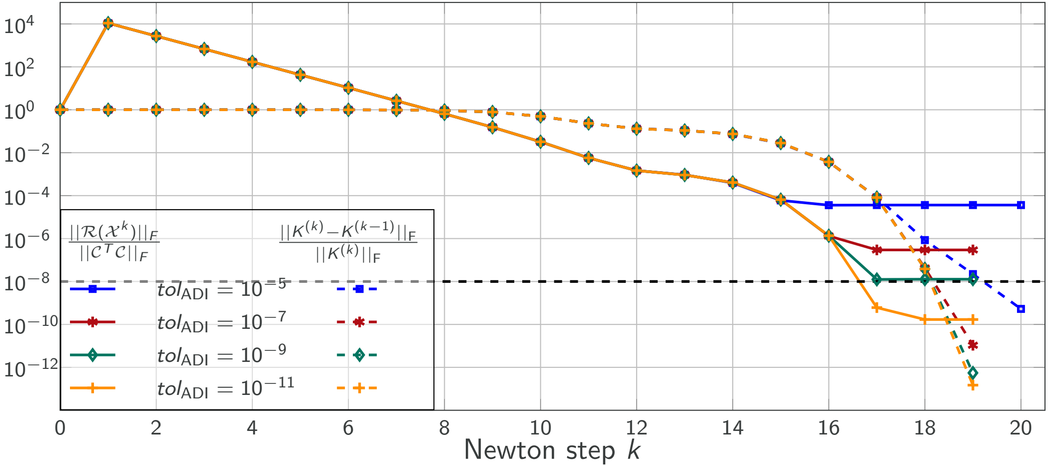

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- Quadratic system matrices A, $M = M^T \in \mathbb{R}^{n \times n}$ are sparse.

$$\mathcal{R}(X) = C^T C + A^T XM + MXA - MXBB^T XM$$

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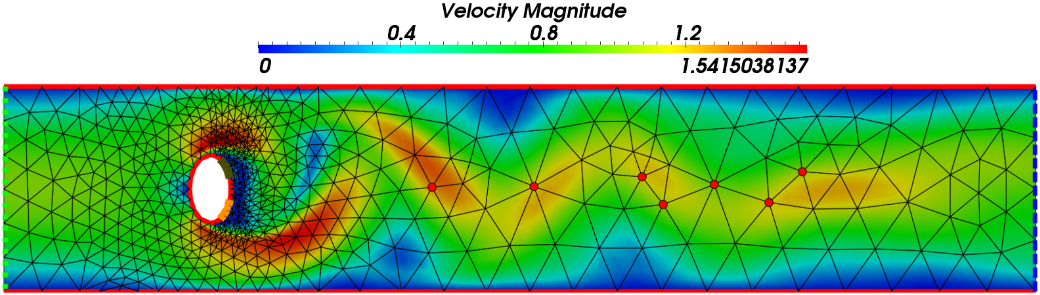

Kármán vortex street

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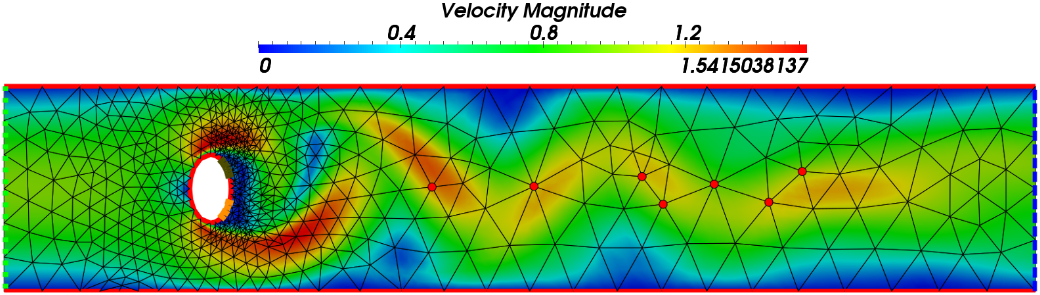

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Accelerated Solution of Riccati Equations
— Structure —

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  $\Rightarrow X = ZZ^T$ exists, with $Z \in \mathbb{R}^{n \times m}$, $n_r + n_a < m \ll n$.

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$$\mathcal{R}(ZZ^T) = C^T C + A^T ZZ^T M + MZZ^T A - MZZ^T BB^T ZZ^T M$$

[B./Kürschner/Saak '14/'15].

- Coefficients of GCARE are large-scale matrices (resulting from FE discretization).
- Quadratic system matrices A, $M = M^T \in \mathbb{R}^{n \times n}$ are sparse.
- In-/output matrices are rectangular and dense: $B \in \mathbb{R}^{n \times n_r}$, $C \in \mathbb{R}^{n_a \times n}$ with $n_r + n_a \ll n$.
- Unique stabilizing solution $X \in \mathbb{R}^{n \times n}$ is symmetric, positive-semidefinite, but dense [Lancaster/Rodman '95], [B./Heinkenschloss/Saak/Weichelt '16].
- Singular values of X decay rapidly [Grasedyck '04], [B./Bujanović '16] $\Rightarrow X = ZZ^T$ exists, with $Z \in \mathbb{R}^{n \times m}$, $n_r + n_a < m \ll n$.
- Residual is of low rank; $R(ZZ^T) = WW^T$, $W \in \mathbb{R}^{n \times k}$, $k \leq 2n_r + n_a \ll n$

$$WW^T = C^T C + A^T ZZ^T M + MZZ^T A - MZZ^T BB^T ZZ^T M$$

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- **extension to index-2 DAE case "straight forward"**

Theorem **[B./Heinkenschloss/Saak/Weichelt '16]**

Set $\tau_k \in (0,1)$ and assume: $(\mathcal{A},\mathcal{B};\mathcal{M})$ stabilizable, $(\mathcal{C},\mathcal{A};\mathcal{M})$ detectable, and $\exists\, \widetilde{\mathcal{X}}^{(k+1)} \succeq 0\ \forall k$ that solves

$$(\mathcal{A}-\mathcal{B}\mathcal{K}^{(k)})^T \widetilde{\mathcal{X}}^{(k+1)}\mathcal{M} + \mathcal{M}\widetilde{\mathcal{X}}^{(k+1)}(\mathcal{A}-\mathcal{B}\mathcal{K}^{(k)}) = -\mathcal{C}^T\mathcal{C} - (\mathcal{K}^{(k)})^T\mathcal{K}^{(k)} + \mathcal{L}^{(k+1)}$$

such that

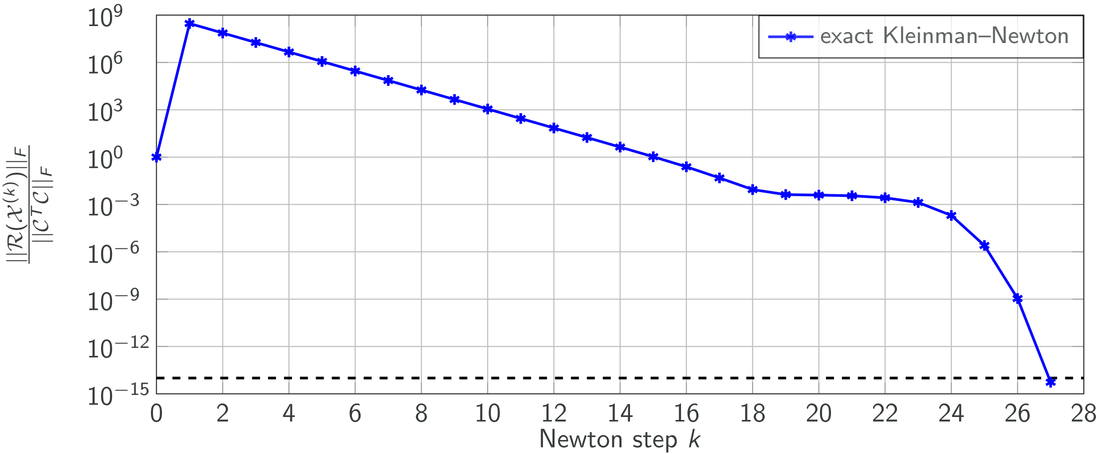
$$||\mathcal{L}^{(k+1)}||_F \leq \tau_k ||\mathcal{R}(\mathcal{X}^{(k)})||_F.$$

Find $\xi_k \in (0,1]$ such that $||\mathcal{R}(\mathcal{X}^{(k)} + \xi_k \mathcal{S}^{(k)})||_F \leq (1-\xi_k\alpha)||\mathcal{R}(\mathcal{X}^{(k)})||_F$ and set

$$\mathcal{X}^{(k+1)} = (1-\xi_k)\mathcal{X}^{(k)} + \xi_k \widetilde{\mathcal{X}}^{(k+1)}.$$

1. **IF** $\xi_k \geq \xi_{\min} > 0\ \forall k \ \Rightarrow\ \|\mathcal{R}(\mathcal{X}^{(k)})\|_F \to 0$.
2. **IF** $\mathcal{X}^{(k)} \succeq 0$, and $(\mathcal{A}-\mathcal{B}\mathcal{B}^T\mathcal{X}^{(k)}, \mathcal{M})$ stable for $k \geq K > 0 \ \Rightarrow\ \mathcal{X}^{(k)} \to \mathcal{X}^{(*)}$ ($\mathcal{X}^{(*)} \succeq 0$ the unique stabilizing solution).

NSE scenario: Re $= 500$, Level 1, $\lambda = 10^4$, $tol_{\text{Newton}} = 10^{-14}$



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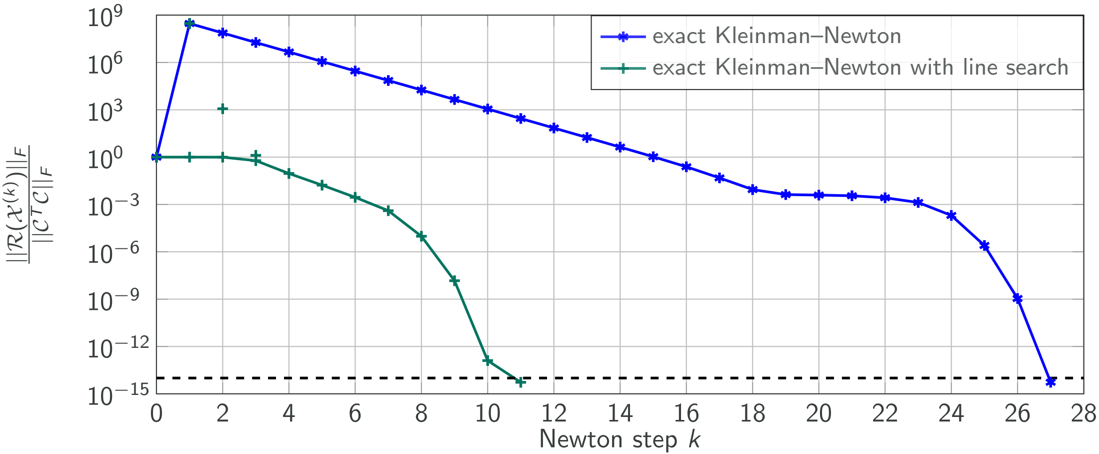

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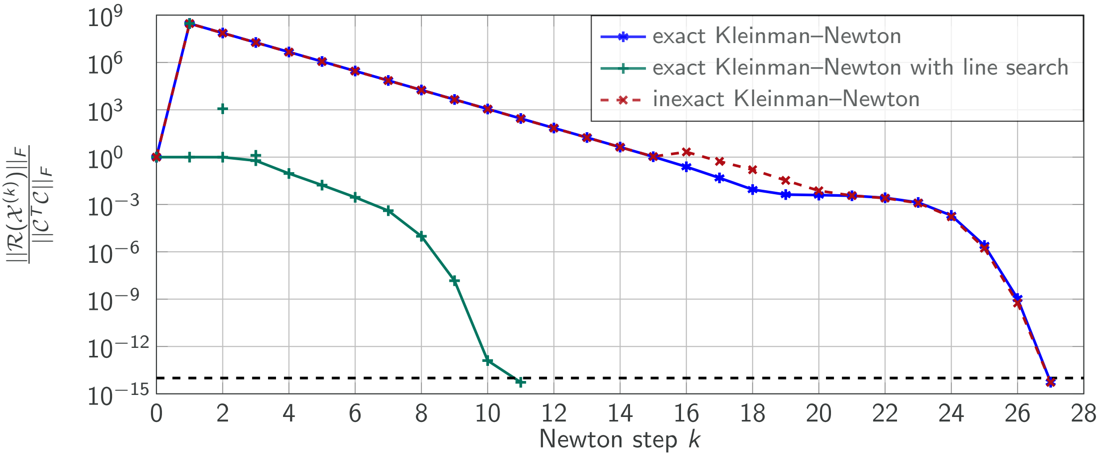

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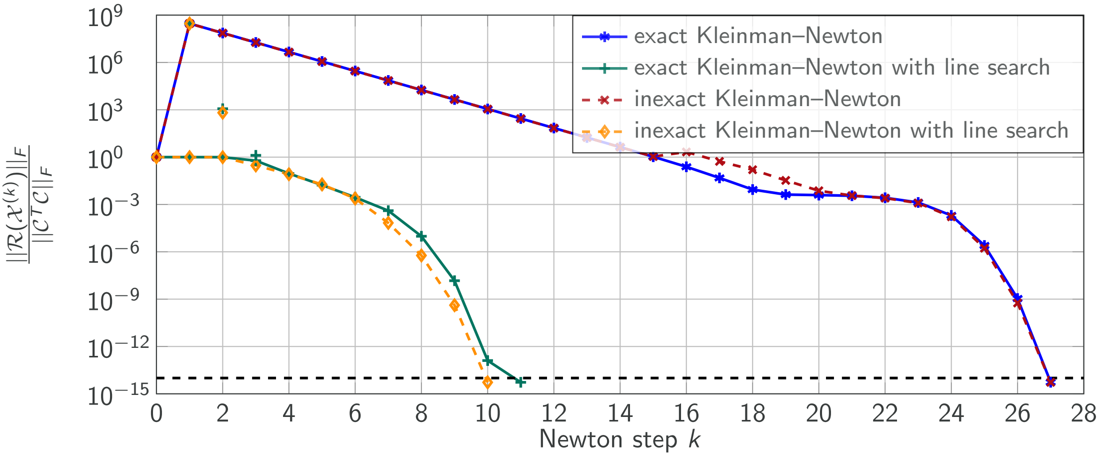

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$\#Newt$	27	11	27	10
$\#ADI$	3185	1351	852	549
$t_{\text{Newt-ADI}}$	1304.769	540.984	331.871	222.295
t_{shift}	29.998	12.568	7.370	5.507
t_{LS}	–		–	
t_{total}	**1334.767**	**553.581**	**339.241**	**227.824**

Table: Numbers of steps and timings in seconds.

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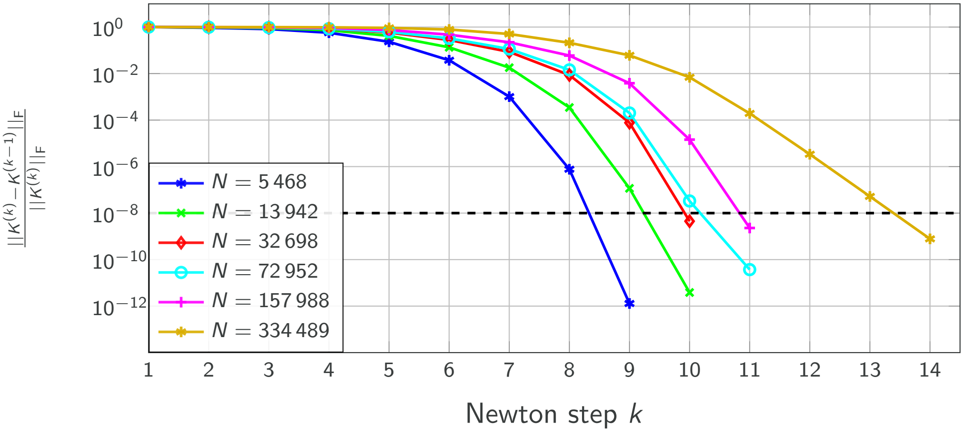
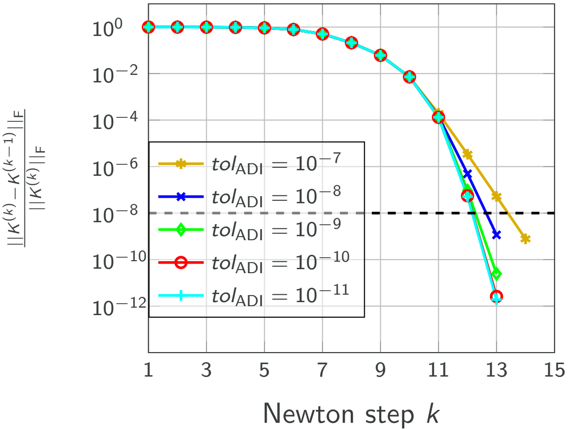
Accelerated Solution of Riccati Equations
Feedback Stabilization for Index-2 DAE systems

NSE scenario: Re $= 500$, $tol_{\text{ADI}} = 10^{-7}$, $tol_{\text{Newton}} = 10^{-8}$



NSE scenario: NSE scenario: $\mathrm{Re} = 500$, $tol_{\mathrm{Newton}} = 10^{-8}$, $N = 334\,489$



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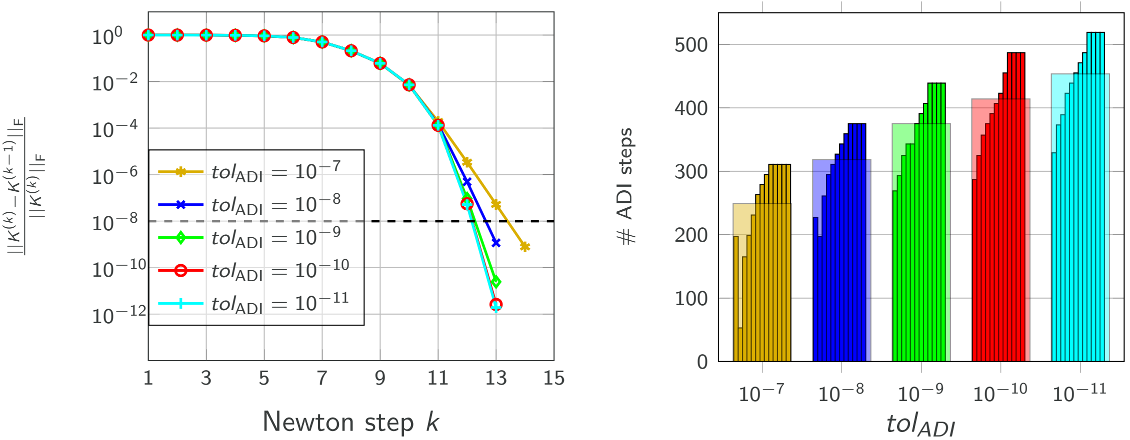

Conclusions

Main Contributions

- Analyzed **Riccati-based feedback** for **scalar** and **vector-valued transport** problems.
- Wide-spread usability tailored for standard **inf-sup stable finite element** discretizations.
- Established **specially tailored Kleinman–Newton-ADI** that **avoids explicit projections**.
- **Suitable preconditioners** for multi-field flow problems have been developed.
- **Ongoing research** in similar areas has been **incorporated**.
- Major run time improvements due to combination of **inexact Newton** and **line search**.
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$\Rightarrow$ Showed **overall usability** of new approach by a closed-loop forward simulation.

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