MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

LQG and $\mathcal{H}_\infty$ Balanced Truncation for Active Flow Control

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Outline

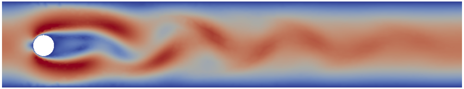
1. Introduction

2. Robust Control

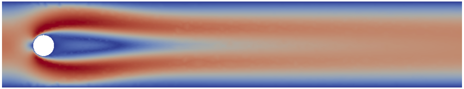
3. Reduced-Order Controllers

4. Numerical Realization

5. Conclusions



- Consider the cylinder wake at moderate *Reynolds* numbers.
- The steady state is a solution, but unstable $\rightsquigarrow$ transition to turbulent flow due to unavoidable perturbations, if they are not attenuated.
- Goal: Stabilizing feedback controller that works in experiments.
- Thus, the simulation needs to cope with:
  - ➙ limited measurements,
    - ➙ short evaluation times,
      - ➙ system uncertainties.



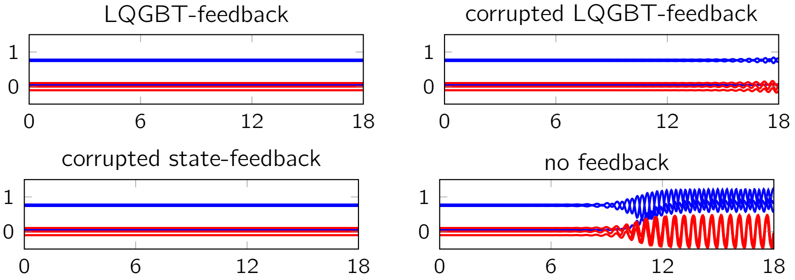
Idea: Linearization based feedback to stabilize a steady-state:

$$\dot{v} + (v \cdot \nabla)v - \frac{1}{Re}\Delta v + \nabla p = Bu \qquad \xrightarrow{\text{linearization and (semi-) discretization}} \qquad \dot{v} + Av + J^T p = Bu$$
$$\nabla \cdot v = 0 \qquad\qquad Jv = 0$$

- see [Raymond'05, '06] for theory concerning flows;
- see [PB&JH'15] for low-rank output-based feedback;
- see also [Breiten&Kunisch'14].

✗ Fragility of Observer-Based Controllers

LQG controllers have no guaranteed robustness margins and will likely fail in the presence of system uncertainties.



- corrupted linearization – about the not quite converged steady state
  - visually undistinguishable from the *exact* linearization point
  - relative difference in norm: 5%

Uncertainty A_Δ in the linearization A...

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned} \quad \leftarrow \quad \begin{aligned}\dot{x} &= [A + A_\Delta]x + Bu\\ y &= Cx\end{aligned}$$

or

$$G_0(s) = C(sI - A)^{-1}B \quad \leftarrow \quad \begin{aligned}G(s) &= C(sI - A - A_\Delta)^{-1}B\\ &= G_0(s) + G_\Delta(s)\end{aligned}$$

$$\text{with } G_\Delta(s) = CA_\Delta(sI - A)^{-1}(sI - A - A_\Delta)^{-1}B$$

... is an additive uncertainty in the transfer function.

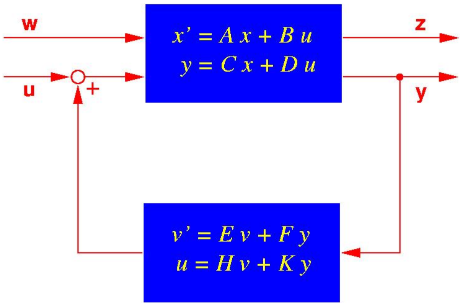
→ Robust $\mathcal{H}_\infty$controllers can compensate for that.

Linear Time-Invariant Systems (finite or infinite)

$$\Sigma : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w + D_{22} u, \end{cases}$$

where A, B_j, C_i, and D_{ij} are matrices of suitable sizes, $j, i \in \{1, 2\}$.

x – states of the system,
w – exogenous inputs
u – control inputs,
z – performance outputs
y – measured outputs



Laplace transform $\Longrightarrow$ transfer function (in frequency domain)

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \equiv \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right],$$

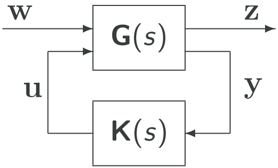
where for $x(0) = 0$, G_{ij} are the transfer functions

$$G_{ij}(s) = C_i(sI - A)^{-1}B_j + D_{ij}$$

with $i, j \in \{1, 2\}$, describing the transfer from inputs to outputs of Σ via

$$\begin{aligned} z(s) &= G_{11}(s)w(s) + G_{12}(s)u(s), \\ y(s) &= G_{21}(s)w(s) + G_{22}(s)u(s). \end{aligned}$$

Consider closed-loop system, where $K(s)$ is an internally stabilizing controller, i.e., K stabilizes G for $w \equiv 0$.



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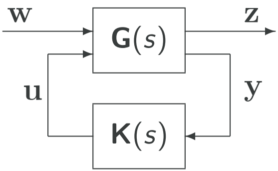

Goal:

Find robust controller, i.e., K that minimizes error outputs

$$z = \left(G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}\right) w =: \mathcal{F}(G, K)w,$$

where $\mathcal{F}(G, K)$ is the linear fractional transformation of G, K.

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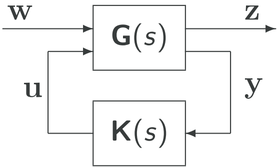

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$\mathcal{H}_\infty$-optimal Control Problem:

$$\min_{K \text{ stabilizing}} \|\mathcal{F}(G, K)\|_{\mathcal{H}_\infty}$$

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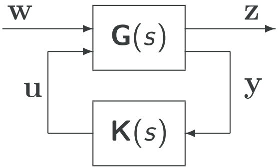

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$\mathcal{H}_\infty$-Suboptimal Control Problem:

For given constant $\gamma > 0$, find all internally stabilizing controllers satisfying

$$\|\mathcal{F}(G, K)\|_{\mathcal{H}_\infty} < \gamma.$$

Simplifying Assumptions

1. $D_{11} = 0$
2. $D_{22} = 0$
3. (A, B_1) stabilizable, (C_1, A) detectable
4. (A, B_2) stabilizable, (C_2, A) detectable ($\Longrightarrow \Sigma$ internally stabilizable)
5. $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$
6. $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$

Remark. 1.,2.,5.,6. only for notational convenience, 3. can be relaxed, but derivations get even more complicated.

Theorem [Doyle/Glover/Khargonekar/Francis '89, Van Keulen '93]

Given the Assumptions 1.–6., there exists an admissible controller $K(s)$ solving the $\mathcal{H}_\infty$-suboptimal control problem $\iff$

(i) There exists a stabilizing solution $X_\infty = X_\infty^T \geq 0$ to the Riccati equation

$$C_1^T C_1 + A^T X + XA + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X = 0.$$

(ii) There exists a stabilizing solution $Y_\infty = Y_\infty^* \geq 0$ to the Riccati equation

$$B_1 B_1^T + AY + YA^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2)Y = 0.$$

(iii) $\gamma^2 > \rho(X_\infty Y_\infty)$.

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$\mathcal{H}_\infty$-optimal Control

Find minimal γ for which (i)–(iii) are satisfied $\rightsquigarrow$ γ-iteration based on solving the Riccati equations above repeatedly for different γ.

$\mathcal{H}_\infty$-(sub-)optimal Controller

If (i)–(iii) hold, a suboptimal controller is given by

$$\widehat{K}(s) = \left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C} & 0 \end{array}\right] = \widehat{C}(sI - \widehat{A})^{-1}\widehat{B},$$

where for

$$Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1},$$

$$\begin{aligned} \widehat{A} &:= A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty - Z_\infty Y_\infty C_2^T C_2, \\ \widehat{B} &:= Z_\infty Y_\infty C_2^T, \\ \widehat{C} &:= -B_2^T X_\infty. \end{aligned}$$

$\widehat{K}(s)$ is the central or minimum entropy controller.

Balancing Related Methods

1. Solve the primal and dual matrix equations defining the characteristic matrices P and Q.
2. Balance the system with respect to P and Q.
3. Truncate states corresponding to small characteristic values of PQ.

- **LQG**:

$$A^T P_{LQG} + P_{LQG}A - P_{LQG}B_2B_2^T P_{LQG} + C_1^T C_1 = 0,$$
$$AQ_{LQG} + Q_{LQG}A^T - Q_{LQG}C_2^T C_2 Q_{LQG} + B_1 B_1^T = 0$$

- $\boldsymbol{\mathcal{H}_\infty}$:

$$A^T P_{\mathcal{H}_\infty} + P_{\mathcal{H}_\infty}A + P_{\mathcal{H}_\infty}(\gamma^{-2}B_1B_1^T - B_2B_2^T)P_{\mathcal{H}_\infty} + C_1^T C_1 = 0,$$
$$AQ_{\mathcal{H}_\infty} + Q_{\mathcal{H}_\infty}A^T + Q_{\mathcal{H}_\infty}(\gamma^{-2}C_1^T C_1 - C_2^T C_2)Q_{\mathcal{H}_\infty} + B_1B_1^T = 0$$

The $\mathcal{H}_\infty$ controller design for the stabilization of incompressible flows comes with two immediate numerical challenges:

1. High-dimensional model equations
   - ↢ direct approach not feasible because of memory constraints
   - ↣ low-rank Riccati iteration
2. Differential-algebraic structure
   - ↢ due to the incompressibility constraint
   - ↣ implicit realization of the discrete *Leray projector*

Low-Rank Riccati Iteration [Lanzon/Feng/Anderson '07, B. '08/'12]

1. Solve the ARE
$$C_1^T C_1 + A^T Z_0 + Z_0 A - Z_0 B_2 B_2^T Z_0 = 0$$
using Newton-ADI / RADI, yielding Y_0 with $Z_0 \approx Y_0 Y_0^T$.
2. Set $R_1 := Y_0$. $\{\% \ R_1 R_1^T \approx X_1\}$
3. FOR $k = 1, 2, \ldots$
   (i) Set $A_k = A + U_k V_k^T := A + \gamma^{-2} B_1 (B_1^T R_k) R_k^T - B_2 (B_2^T R_k) R_k^T$.
   (ii) Solve the ARE
   $$\gamma^{-2} Z_{k-1} B_1 B_1^T Z_{k-1} + A_k^T Z_k + Z_k A_k - Z_k B_2 B_2^T Z_k = 0$$
   using Newton-ADI / RADI, yielding Y_k with $Z_k \approx Y_k Y_k^T$.
   (iii) Set $R_{k+1} := \mathtt{rrqr}([\,R_k,\ Y_k\,], \tau)$. $\{\% \ R_{k+1} R_{k+1}^T \approx X_{k+1}\}$
   (iv) IF $\|(B_1^T Y_k) Y_k^T\|_2 <$ tol THEN Stop.

Under standard assumptions, the semi-discrete incompressible (Navier-)Stokes equations can be realized as an ODE:

$$\begin{aligned} \dot{v} &= Av + J^T p + Bu \\ 0 &= Jv \end{aligned} \quad \rightarrow \quad \dot{v} = \Pi A v + \Pi B u$$

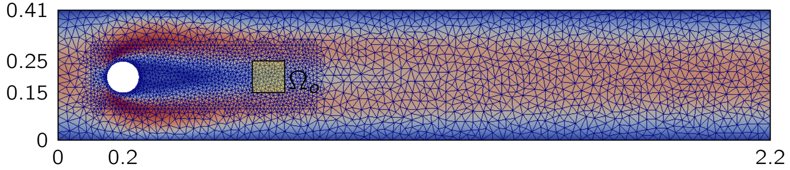
with $\Pi := I - J^T(JJ^T)^{-1}J$.

In turn, the feedback defined via the Π-based Riccati equations can be realized via constrained Riccati equations [PH&JH'17]:

$$\begin{aligned} A^T \Pi X + X \Pi A \\ - X \Pi B B^T \Pi X = -C^T C \end{aligned} \quad \leftarrow \quad \begin{aligned} A^T X + XA - XBB^T X \\ + J^T Y + Y^T J = -C^T C, \\ JX = 0, \quad XJ = 0, \end{aligned}$$

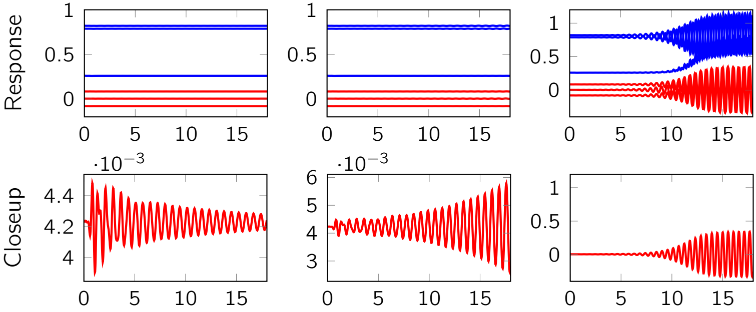
avoiding Π altogether.

Simulation Setup



- 2D cylinder wake
- Navier-Stokes equations
- $Re = 90$
- *Taylor-Hood* finite elements
- 19500 velocity nodes

- Boundary control at 2 outlets at the cylinder periphery
- Distributed observation:
  - 3 *sensors* in the wake
  - measuring both v-components each
- $\mathcal{H}_\infty$-BT reduced controller
- Target: stabilization of the steady-state solution



- Controller of dimension 14 (left) and 8 (middle)
- Based on an inexact linearization
  - only 3 *Picard* iteration on the *Stokes* steady-state
  - relative difference to the *exact* linearization point: 8%
- Random perturbation of the initial value to trigger instabilities

Conclusions

Summary

- $\mathcal{H}_\infty$-BT reduced controllers are
  - output based and of low dimensions,
  - robust against system uncertainties – as opposed to LQG.
- The application to flow stabilization becomes feasible with
  - low-rank Riccati iterations,
  - implicit realization of the incompressibility constraint.

Code Availability:

- The Riccati iteration will be available in the M-M.E.S.S. library version 2.0.
- $\mathcal{H}_\infty$-BT and LQGBT implementations can be found in the MORLAB toolbox.

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