MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

DATA-ENHANCED REDUCED-ORDER MODELING OF DYNAMICAL SYSTEMS

Peter Benner Igor Pontes Duff Pawan Goyal

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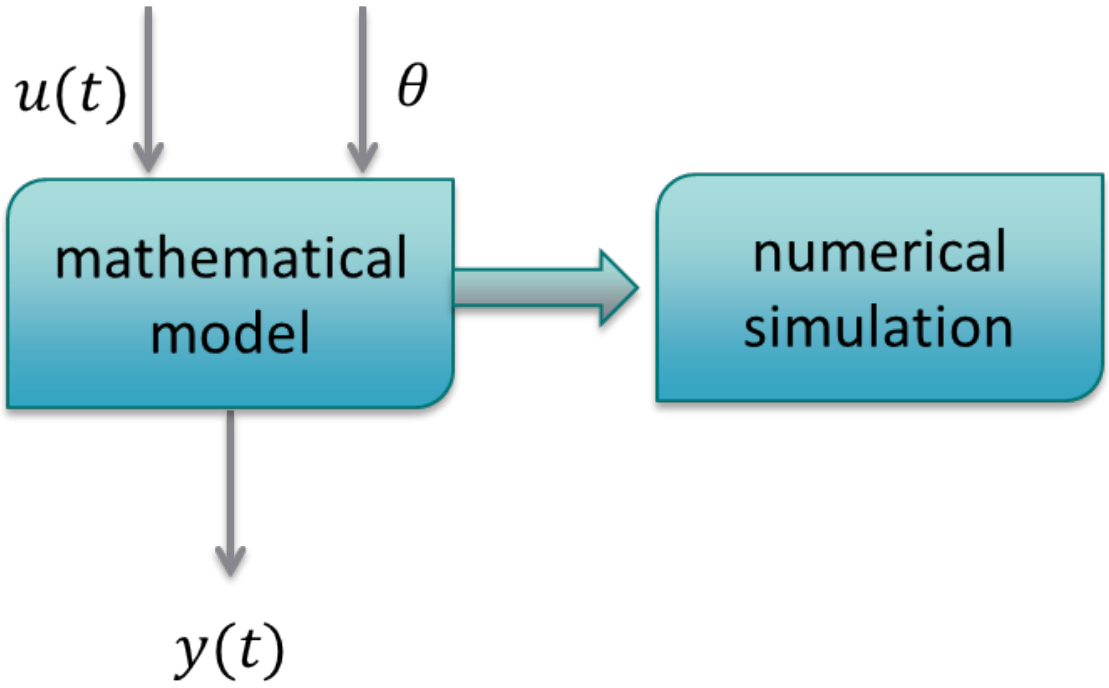
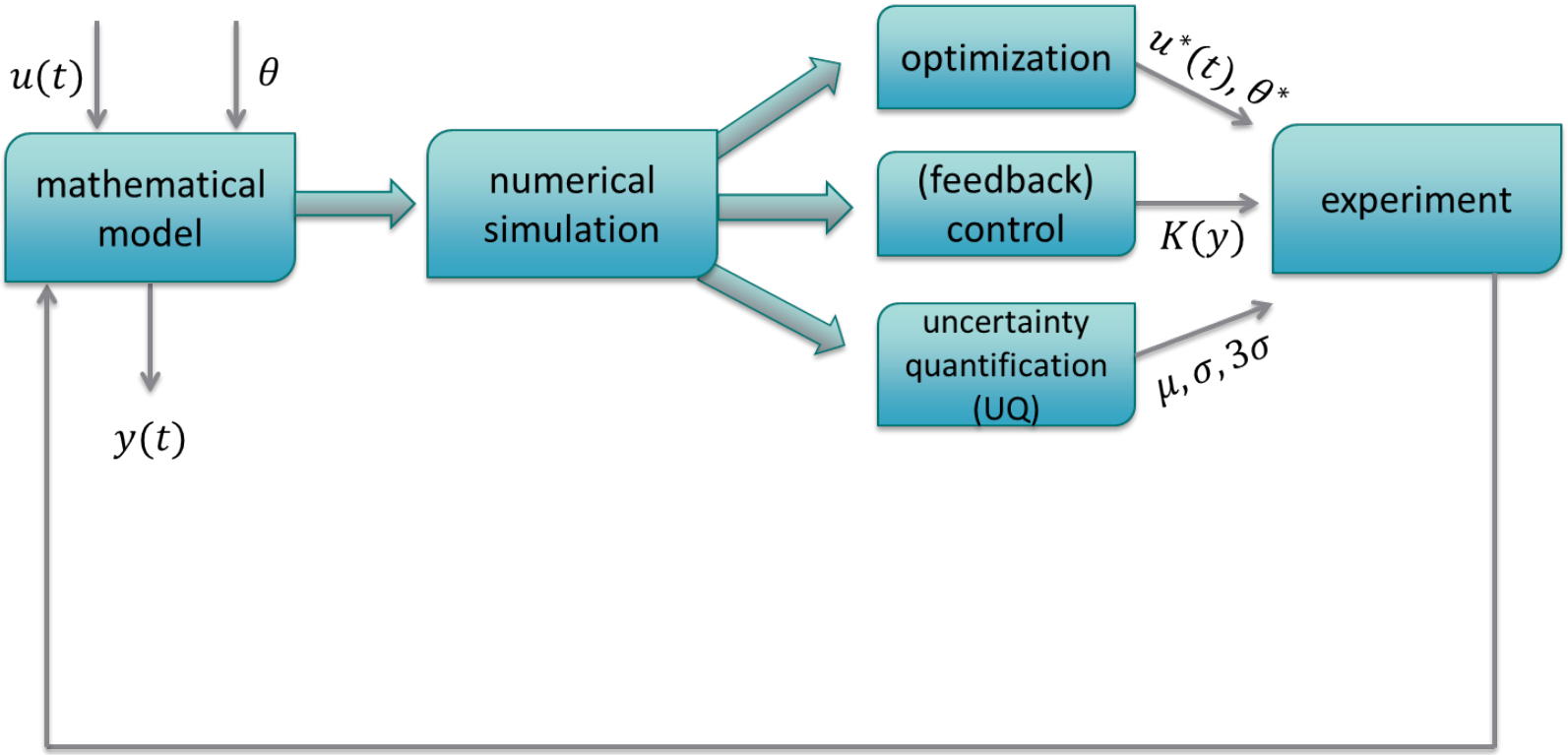
DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION


$u(t)$
θ
mathematical model
numerical simulation
$y(t)$

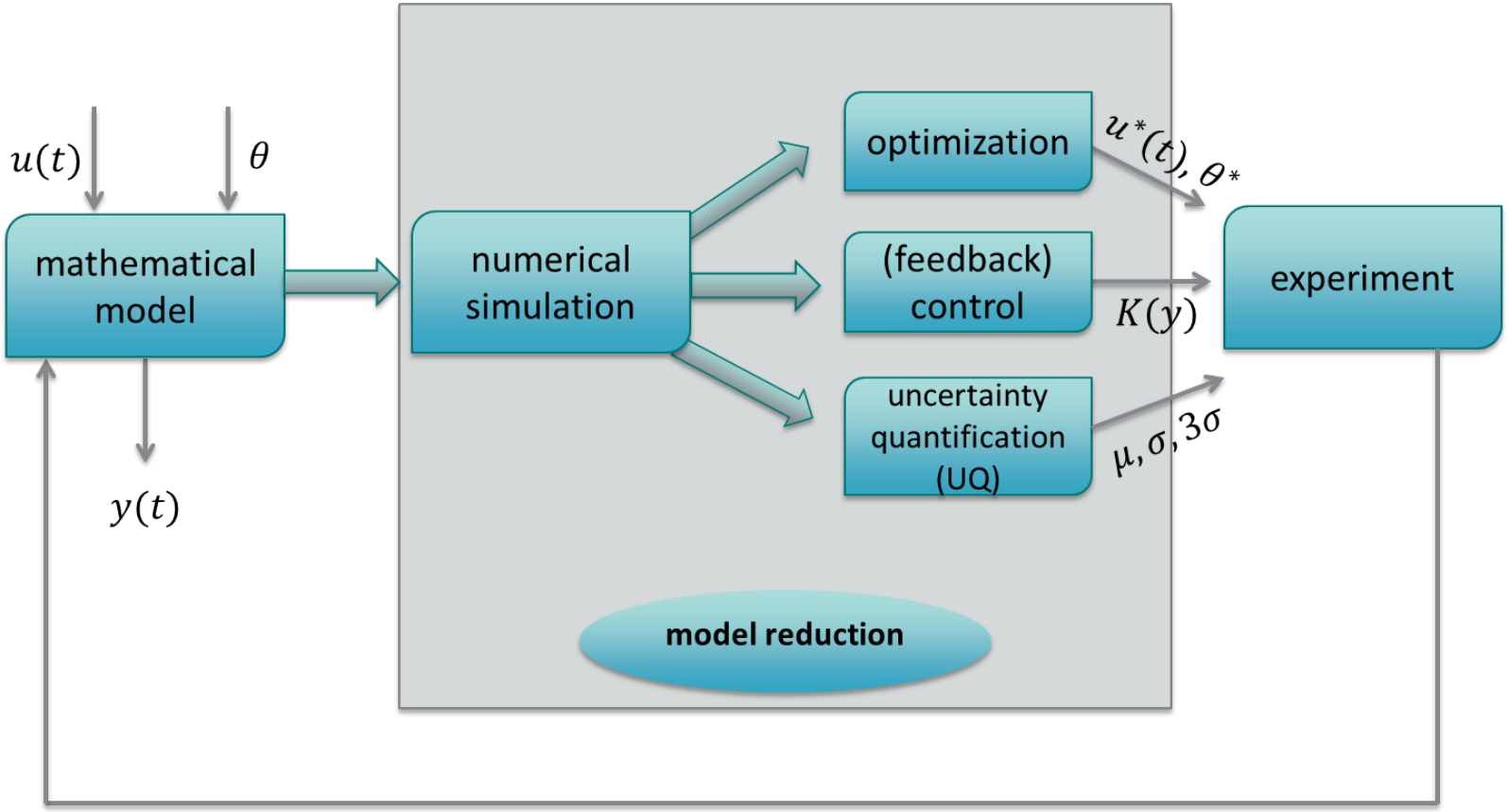
A Process Chain in Computational Sciences and Engineering (CSE)

Optimization, Control, and Uncertainty Quantification



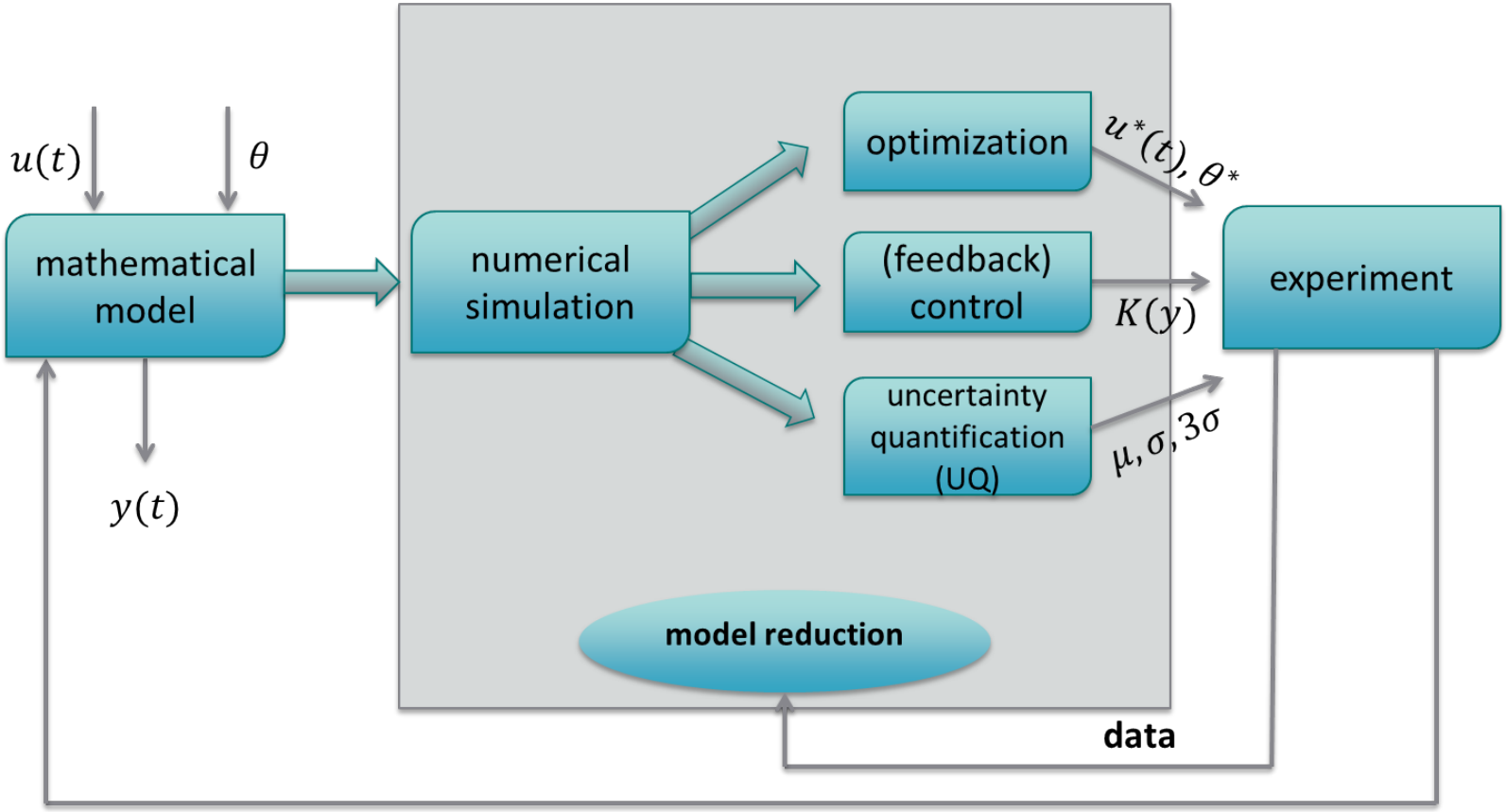
A Process Chain in Computational Sciences and Engineering (CSE)

Model Order Reduction (MOR)



A Process Chain in Computational Sciences and Engineering (CSE)

Data-Driven Sciences



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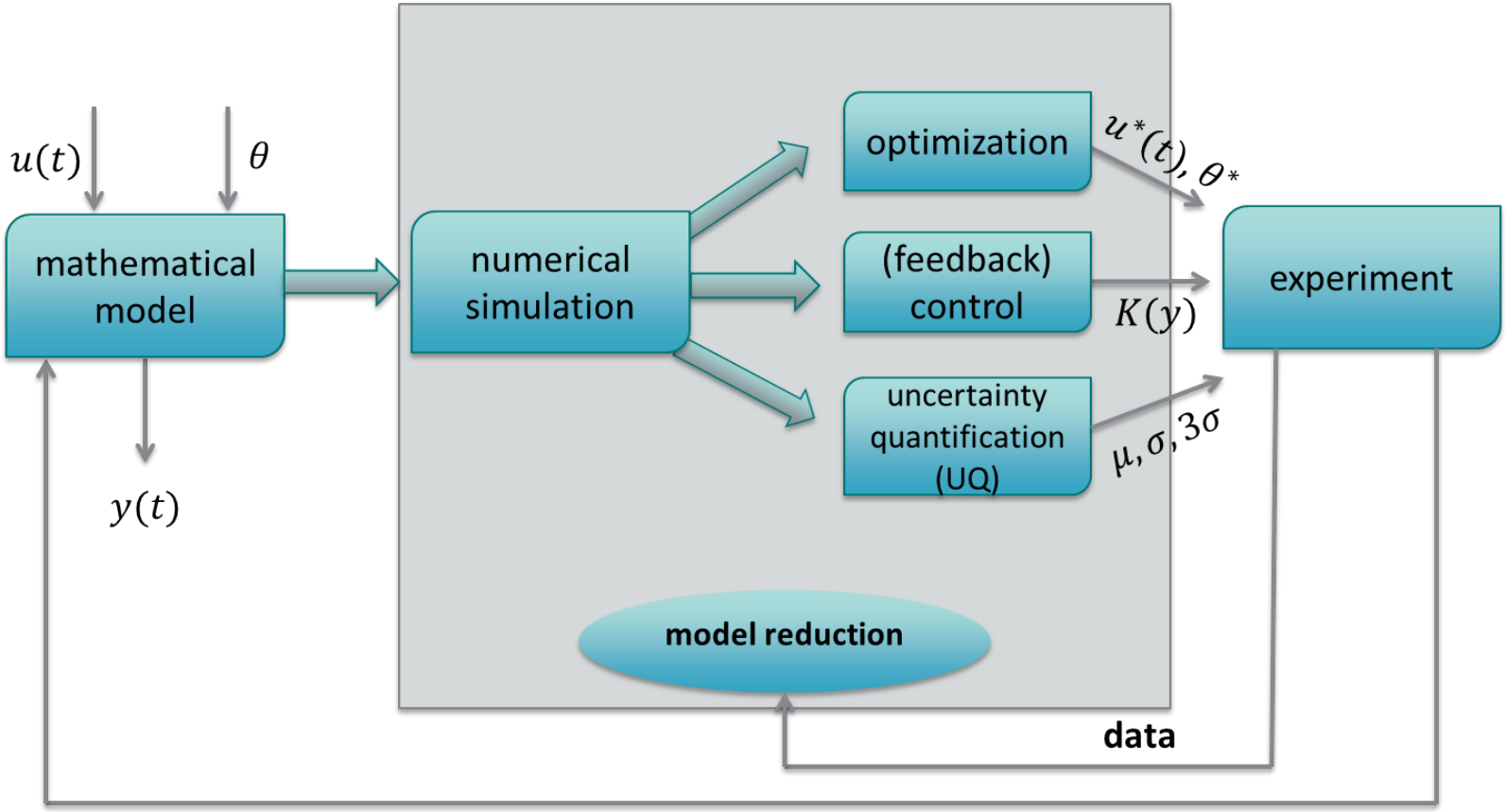

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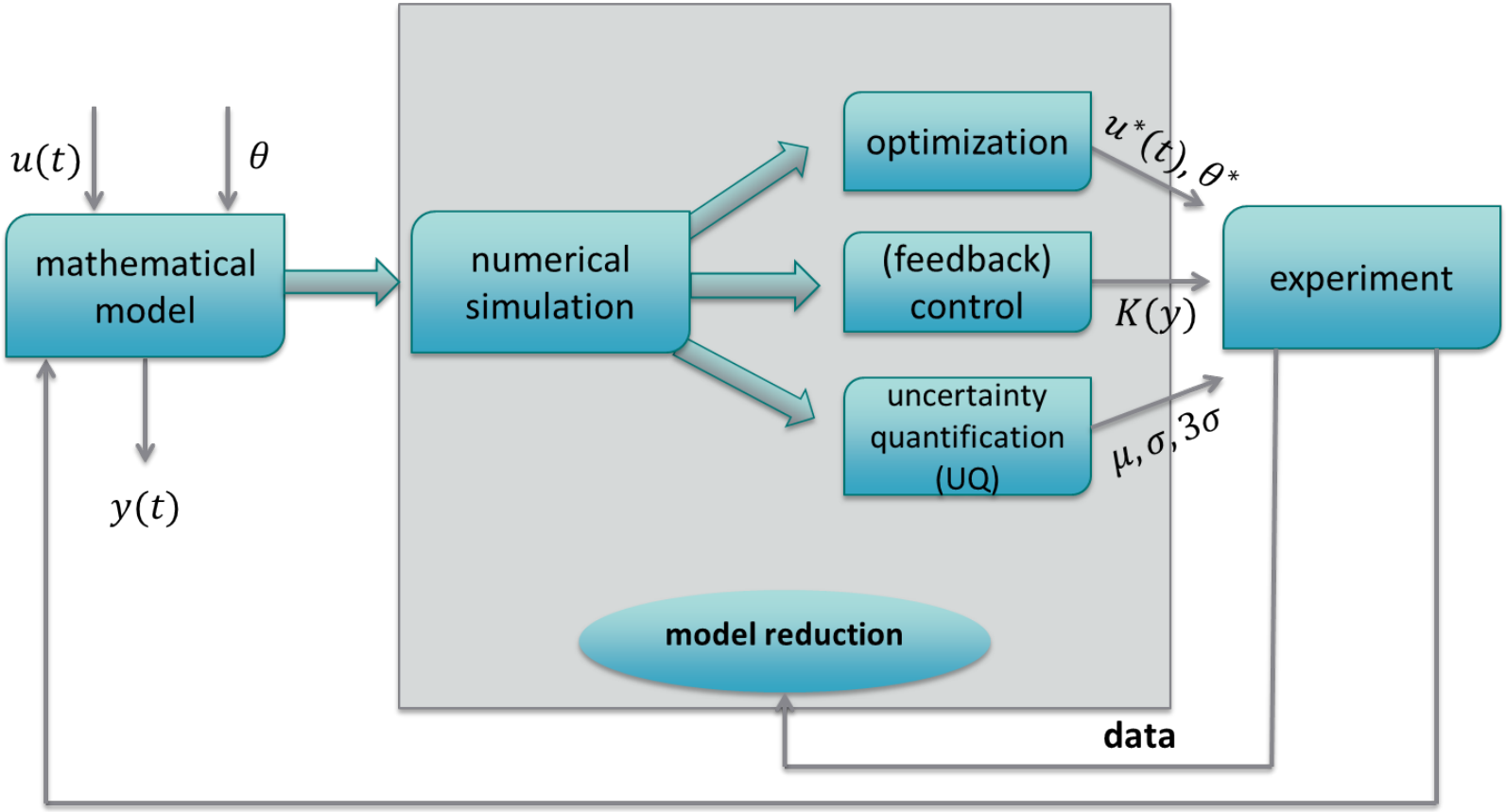

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$\rightsquigarrow$ *Data-enhanced model reduction methods.*

Overview

Outline

Original System ($E = I_n$)

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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Goals:

$||y - \hat{y}|| <$ tolerance $\cdot\, ||u||$ for all admissible input signals.

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Reduced-Order Model (ROM)

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- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sX(s) - x(0))$ to LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

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Model reduction in frequency domain: Fast evaluation of mapping $U \ \rightarrow \ Y$.

Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, & C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{p\times r},\ \hat{D} \in \mathbb{R}^{p\times m} \end{aligned}$$

of order $r \ll n$, such that

$$\begin{aligned} ||y - \hat{y}|| \simeq \left|\left|Y - \hat{Y}\right|\right| &= \left|\left|\mathbf{H}U - \hat{\mathbf{H}}U\right|\right| \\ &\leq \left|\left|\mathbf{H} - \hat{\mathbf{H}}\right|\right| \cdot ||U|| \simeq \left|\left|\mathbf{H} - \hat{\mathbf{H}}\right|\right| \cdot ||u|| \\ &\leq \text{tolerance} \cdot ||u|| \,. \end{aligned}$$

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Some methods:

- Koopman/Dynamic Mode Decomposition (DMD): time domain
  [Mezič 2005; Schmid 2008; Brunton, Kevrekidis, Kutz, Rowley, Noé, Schütte, ...],
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- System identification (incl. ERA, N4SID, MOESP): frequency and time domain
  [Ho/Kalmann 1966; Ljung 1987/1999; Van Overschee/De Moor 1994; Verhaegen 1994; De Wilde, Eykhoff, Moonen, Sima, ...]

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Grey box Σ: additionally, some structural assumptions on the full model are given.
(Approach considered here, in frequency domain.)

Consider **Structured Linear System (SLS)** in frequency domain, using general set-up:

$$\boxed{\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s),} \tag{1}$$

where

$$\mathcal{C}(s) = \sum_{i=1}^{\ell_\gamma} \gamma_i(s)\mathbf{C}_i, \quad \mathcal{K}(s) = s\mathbf{E} - \sum_{i=1}^{\ell_\alpha} \alpha_i(s)\mathbf{A}_i, \quad \mathcal{B}(s) = \sum_{i=1}^{\ell_\beta} \beta_i(s)\mathbf{B}_i,$$

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1) **First-order systems:** $\mathcal{C}(s) = C$, $\mathcal{B}(s) = B$, and $\mathcal{K}(s) = sE - A$.
2) **Second-order systems:** $\mathcal{C}(s) = C_p + sC_v$, $\mathcal{B}(s) = B$, and $\mathcal{K}(s) = s^2M + sL + K$.
3) **Time-delay systems:** $\mathcal{C}(s) = C$, $\mathcal{B}(s) = B$, and $\mathcal{K}(s) = sE - A_1 - A_2e^{-s\tau}$.
4) **EM w/ surface loss:** $\mathcal{C}(s) = sB$, $\mathcal{B}(s) = B$, and $\mathcal{K}(s) = s^2M + sL + K - \frac{1}{\sqrt{s}}N$.
5) **Integro-differential Volterra systems**, **input delays**, **fractional systems** ...

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$$\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n\times r}, \quad \mathbf{W}^T\mathbf{V} = \mathbf{I}_r,$$

(with $r \ll n$), such that

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- Note $\hat{\mathbf{A}}_i = \mathbf{W}^T\mathbf{A}_i\mathbf{V}$, $\hat{\mathbf{E}} = \mathbf{W}^T\mathbf{E}\mathbf{V}, \hat{\mathbf{C}}_i = \mathbf{C}_i\mathbf{V}$ and $\hat{\mathbf{B}}_i = \mathbf{W}^T\mathbf{B}_i$.
- The ROM preserves the $\alpha_i(s), \beta_i(s)$ and $\gamma_i(s)$ functions.

Interpolation-based methods

- Interpolatory projection methods for structure-preserving model reduction.

[Beattie/Gugercin '09]

Interpolation points σ_k, μ_j $\Rightarrow$ $\boxed{\begin{array}{l} \mathcal{K}^{-1}(\sigma_k)\mathcal{B}(\sigma_k) \in \operatorname{range}(\mathbf{V}) \text{ and} \\ \mathcal{K}^{-T}(\mu_k)\mathcal{C}^T(\mu_j) \in \operatorname{range}(\mathbf{W}). \end{array}}$

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- Interpolatory projection methods for structure-preserving model reduction.
  [Beattie/Gugercin '09]

Balancing truncation methods

- Structure-preserving model reduction for integro-differential equations.
  [Breiten '16]

$$\boxed{\begin{aligned}\mathbf{P} &= \frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathcal{K}_s(s)^{-1}\mathcal{B}(s)\mathcal{B}(s)^T\mathcal{K}(s)^{-T}ds,\\ \mathbf{Q} &= \frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathcal{K}_s(s)^{-T}\mathcal{C}(s)^T\mathcal{C}(s)\mathcal{K}(s)^{-1}ds.\end{aligned}} \Rightarrow \text{Find } \mathbf{V}, \mathbf{W} \text{ from } T^{-1}PQT = \Sigma.$$

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- Interpolatory projection methods for structure-preserving model reduction. [Beattie/Gugercin '09]

Balancing truncation methods

- Structure-preserving model reduction for integro-differential equations. [Breiten '16]

Data-driven methods

- Data-driven structured realization. [Schulze/Unger/Beattie/Gugercin '18]

Outline

Let us consider the first order system

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \text{ with } \mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and } \mathbf{C}^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

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Note that $\mathbf{H}(s) = \dfrac{1}{s+2} = \hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}$, with $\hat{\mathbf{A}} = -2, \hat{\mathbf{B}} = 1$ and $\hat{\mathbf{C}} = 1$.

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Find order r and matrices $\mathbf{V}$ and $\mathbf{W}$ such that the reduced-order model obtained by projection satisfies

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Solutions:

- Kalman reachability/observability criteria,
- Hankel matrix (Silverman method),
- reachability and observability Gramians,
- **Loewner matrix.** [Mayo/Antoulas '07]

Minimal Realization
. . . of Structured Linear Systems

For illustration, consider the time-delay systems

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Is there a way to find the order r and matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$ such that the system $\hat{\mathbf{H}}(s)$ obtained by projection is "minimal", *i.e*

$$\mathbf{H}(s) = \hat{\mathbf{H}}(s), \forall s?$$

Minimal Realization
Some Results

Given a first order system

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, \text{ with } \mathbf{E} \in \mathbb{R}^{n \times n} \text{ invertible.}$$

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Reachability characterization [Anderson/Antoulas '90]

If $(\mathbf{E}, \mathbf{A}, \mathbf{B})$ is $\mathbb{R}^n$-reachable, $t \geq n$, $\sigma_i \neq \sigma_j$ for $i \neq j$, and

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Rank encodes minimality [Anderson/Antoulas '90]

$$\operatorname{rank}\left(\mathbf{O}^T\mathbf{E}\mathbf{R}\right) = \text{order of minimal realization} = \mathsf{r}.$$

Outline

For **SLS**, we use the notion of $\mathbb{R}^n$-reachability and -observability. Let us consider the SLS

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s) \text{ of order n.}$$

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Let's go back to the time-delay example

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Reachability and Observability for SLS

An Illustrative Example

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Input: SLS $\mathcal{K}(s)$, $\mathcal{B}(s)$, $\mathcal{C}(s)$ and reduced order r.

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$$\mathbf{V} = \mathbf{R}\mathbf{X}(:, 1:r) \text{ and } \mathbf{W} = \mathbf{O}\mathbf{Y}(:, 1:r).$$

Output: Reduced-order model is given by

$$\hat{\mathcal{K}}(s) = \mathbf{W}^T \mathcal{K}(s) \mathbf{V}, \;\; \hat{\mathcal{B}}(s) = \mathbf{W}^T \mathcal{B}(s) \text{ and } \hat{\mathcal{C}}(s) = \mathcal{C}(s)\mathbf{V}.$$

Outline

Numerical Results
Second-order System

Let us consider the second order system

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t)$$
$$y(t) = Cx(t).$$

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- Full order model with $n = 301$.
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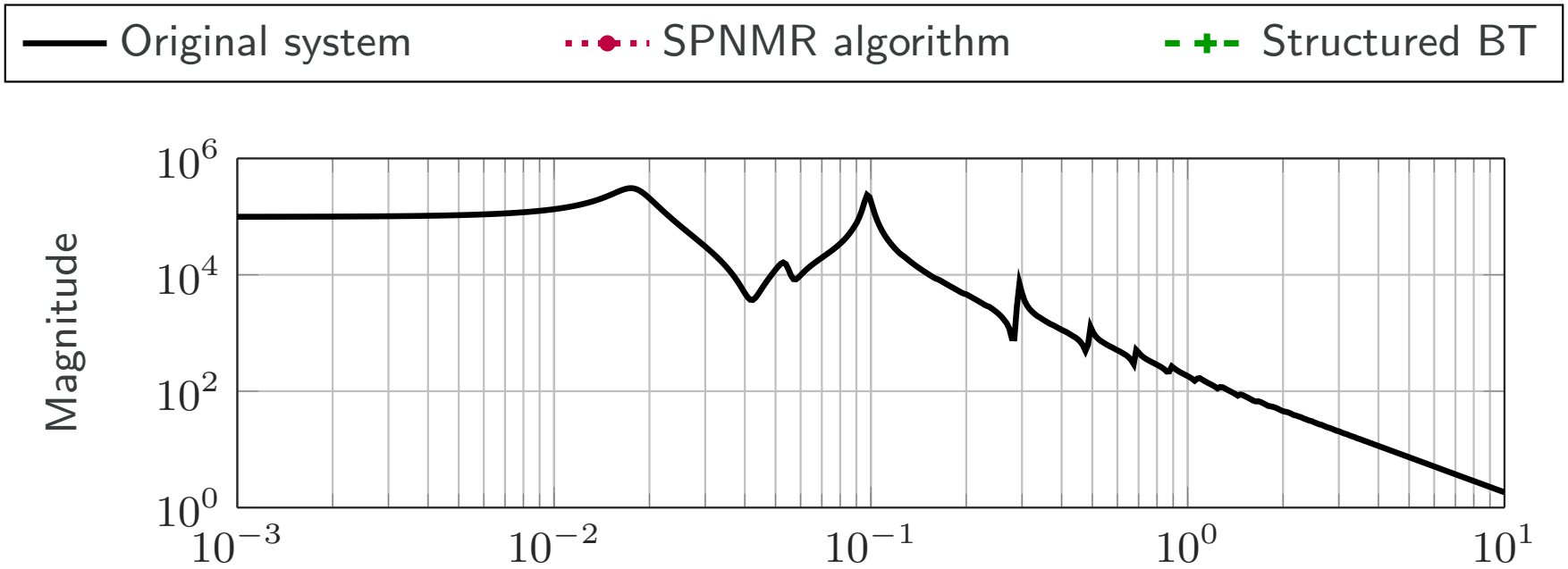

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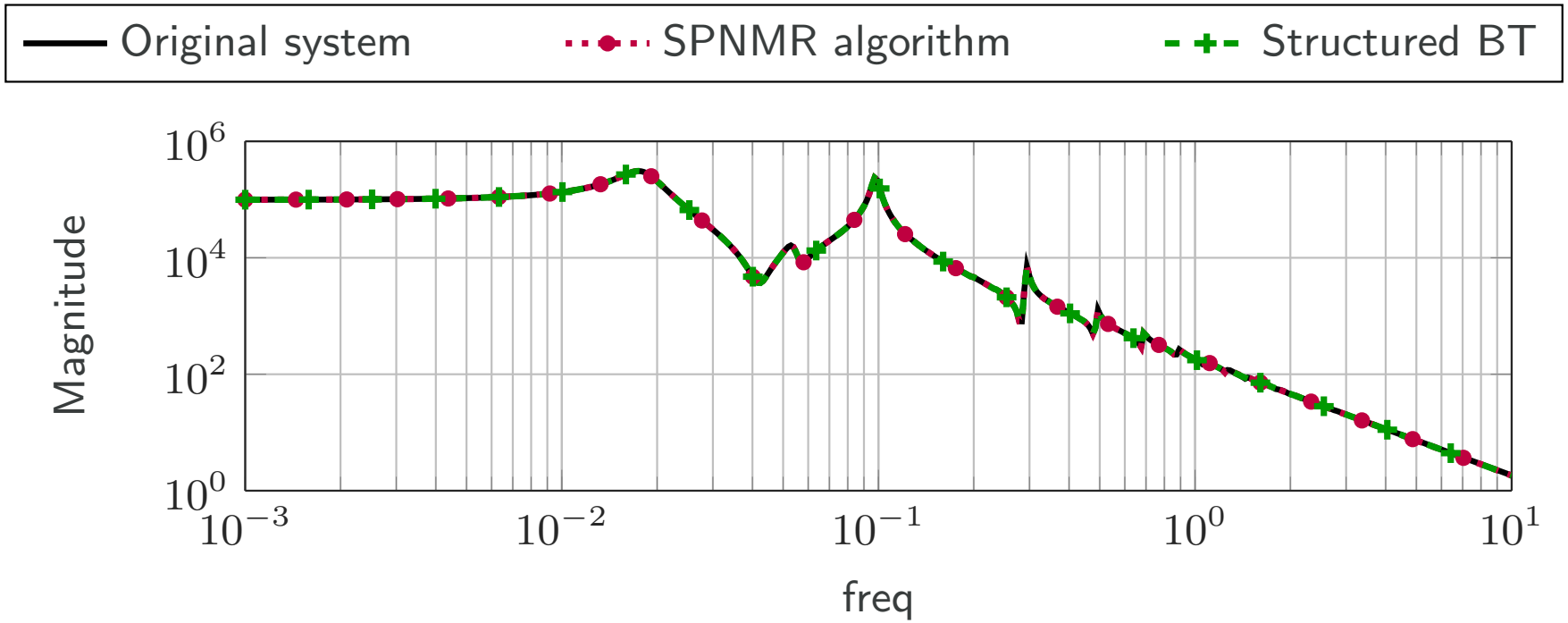

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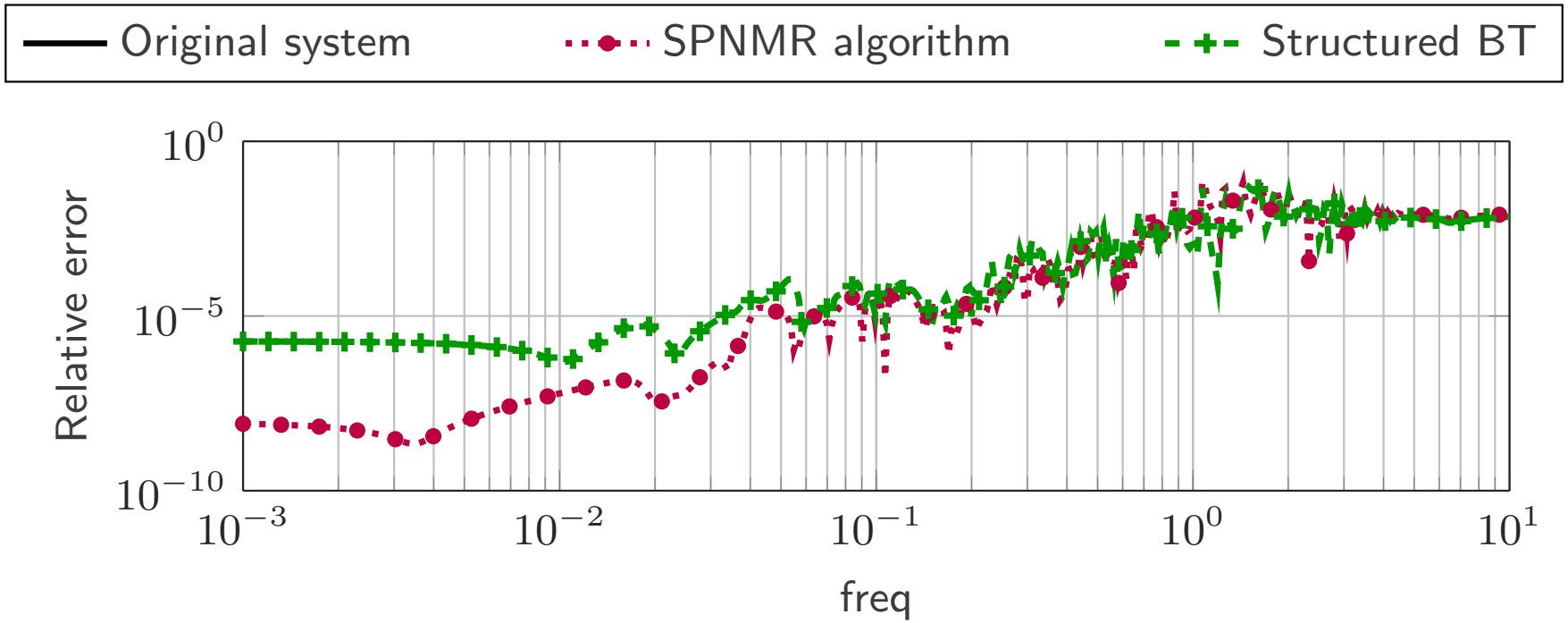

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$$\mathbf{H}(s,p) = \mathcal{C}(s,p)\mathcal{K}(s,p)^{-1}\mathcal{B}(s,p).$$

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Example 1: Minimal Realization of a Small-scale System

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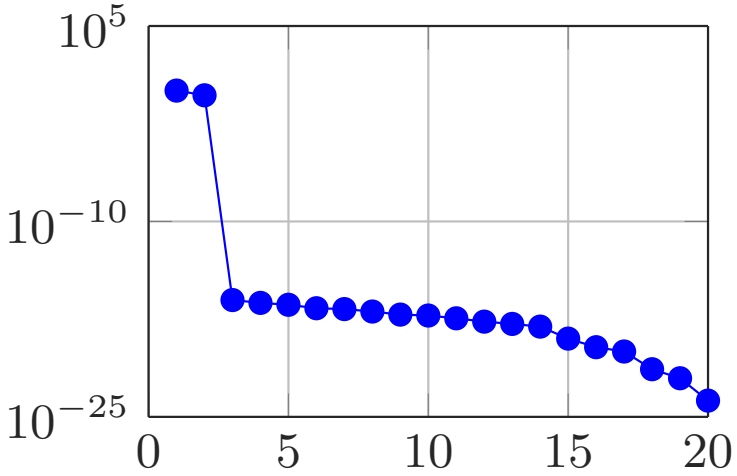
Decay of Singular values



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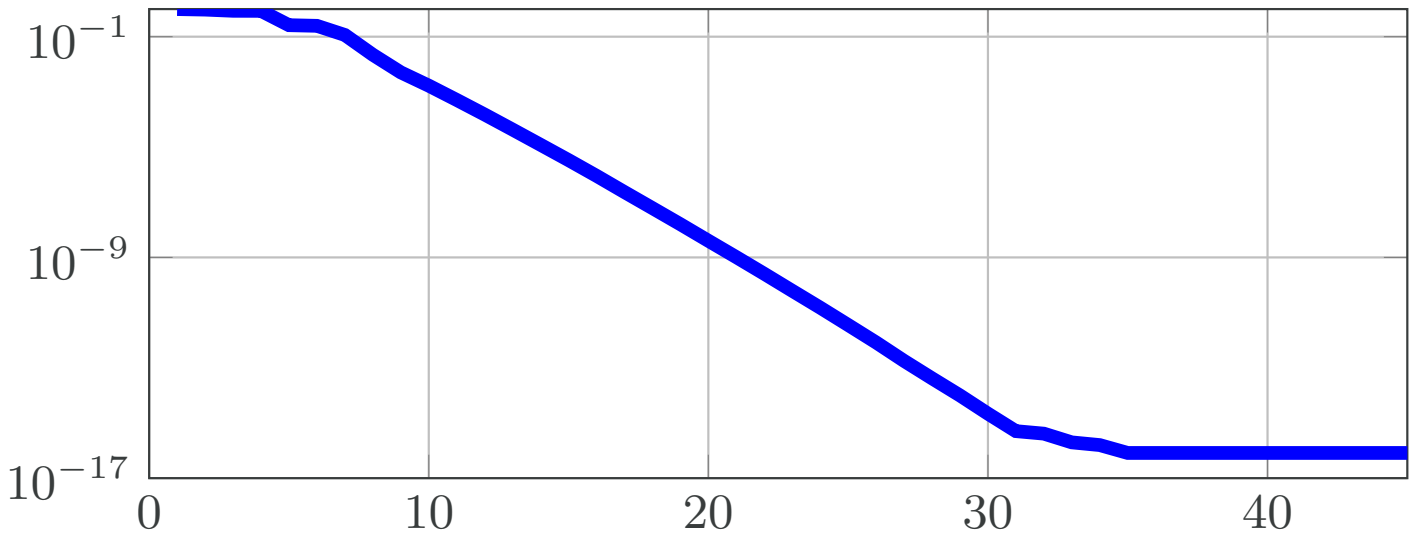
Singular values of the Loewner matrix



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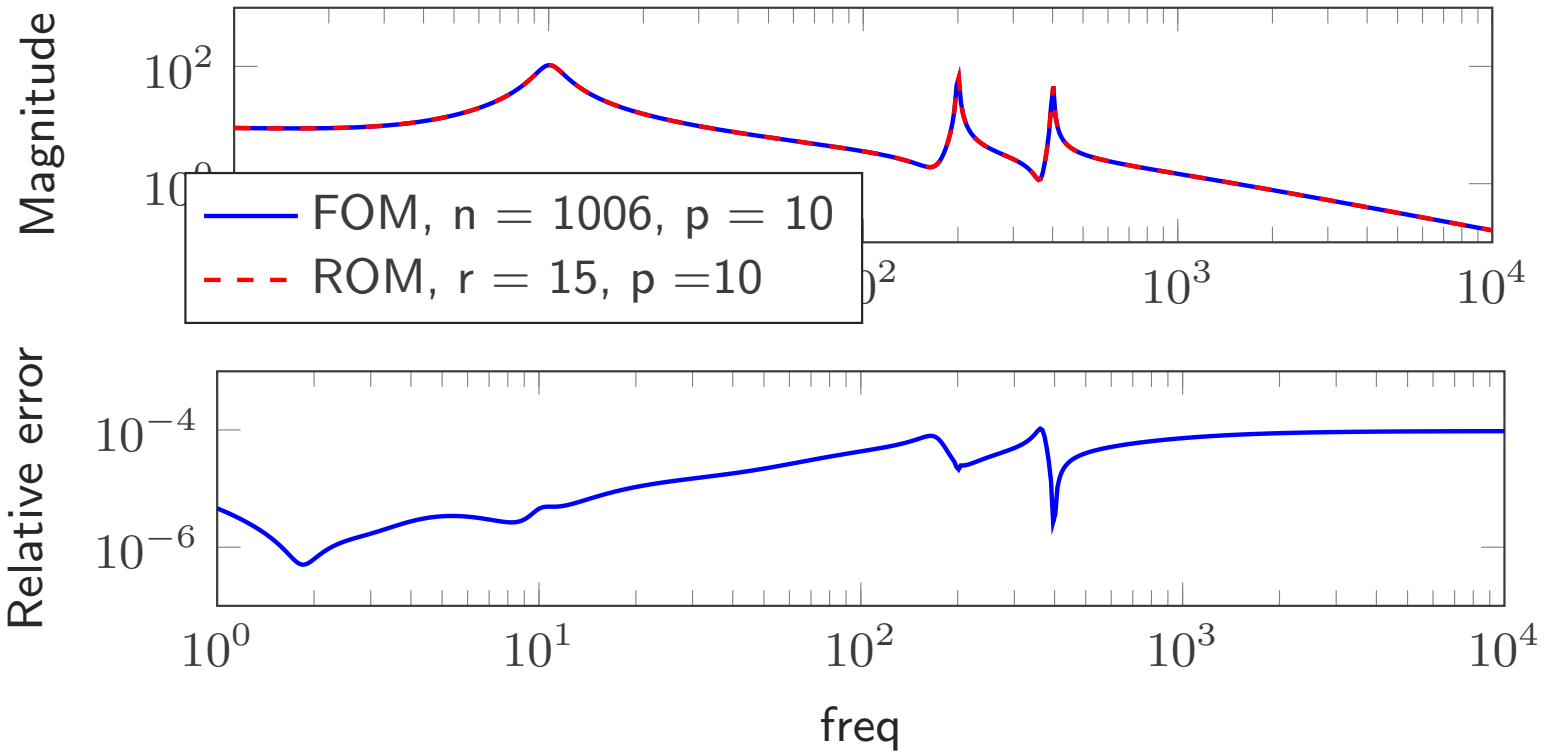

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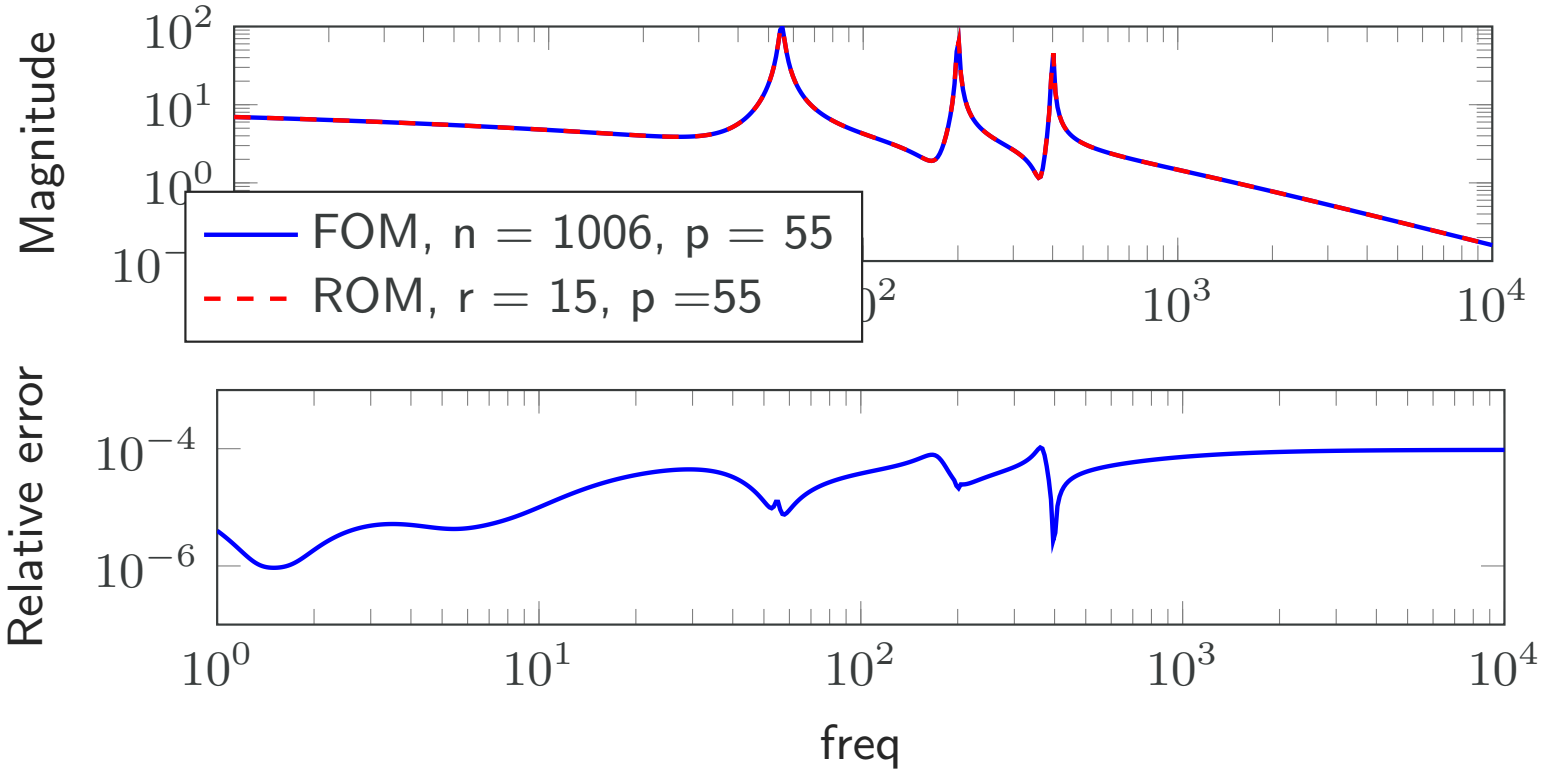

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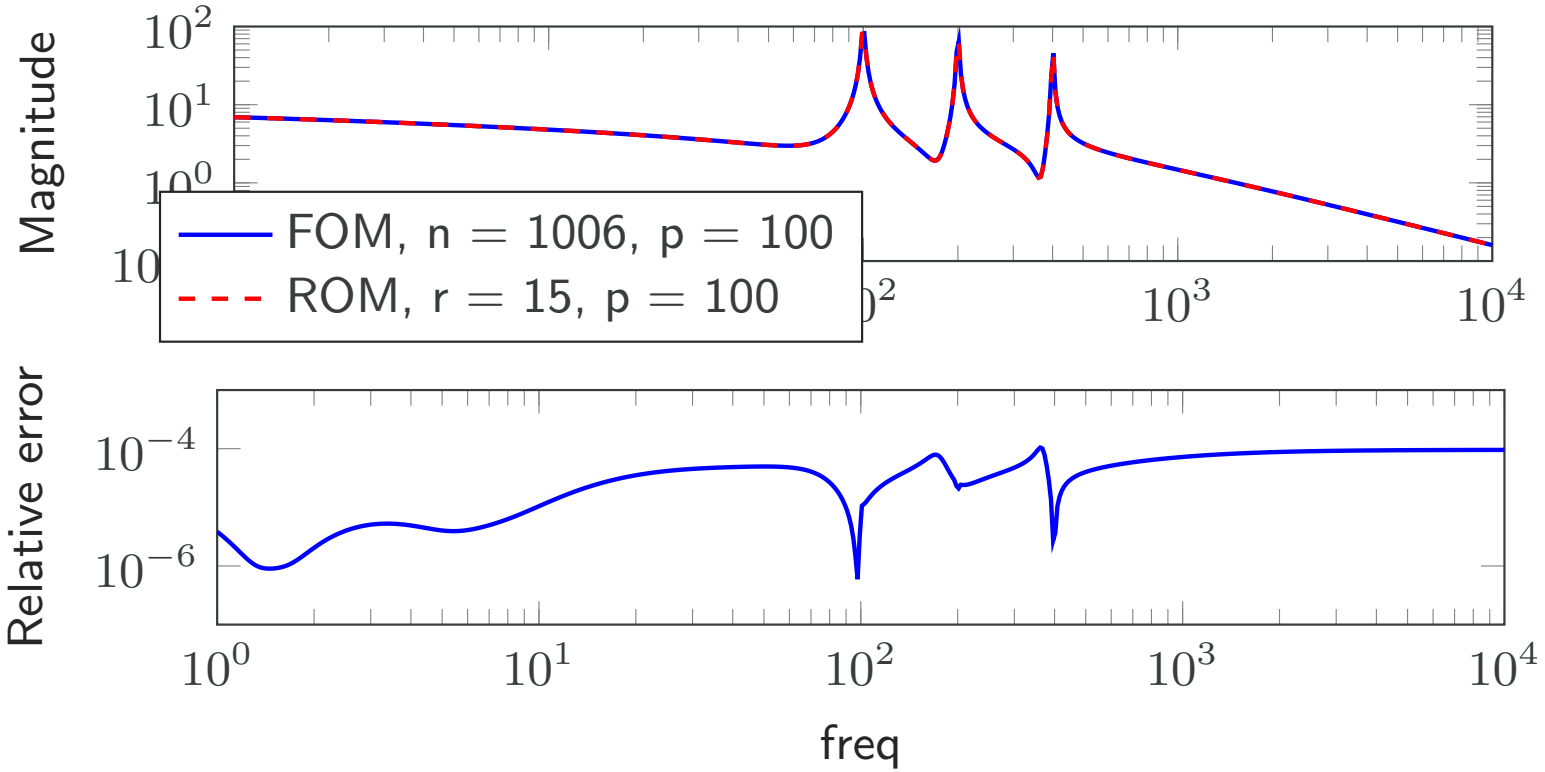

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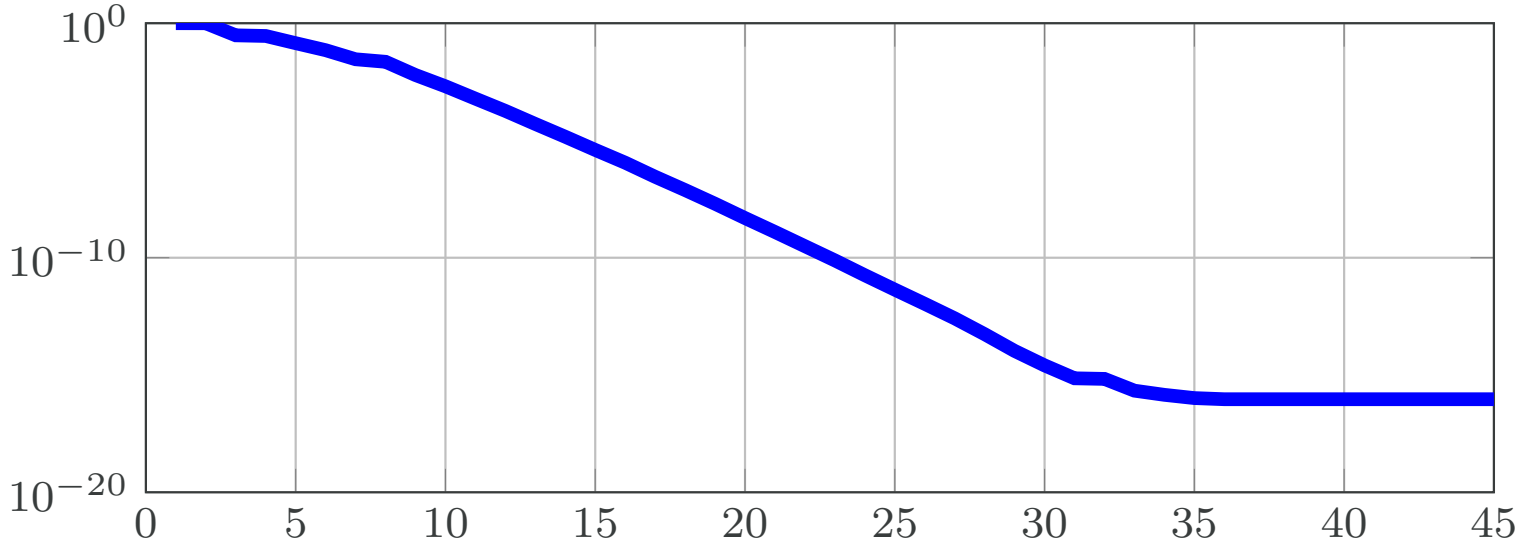

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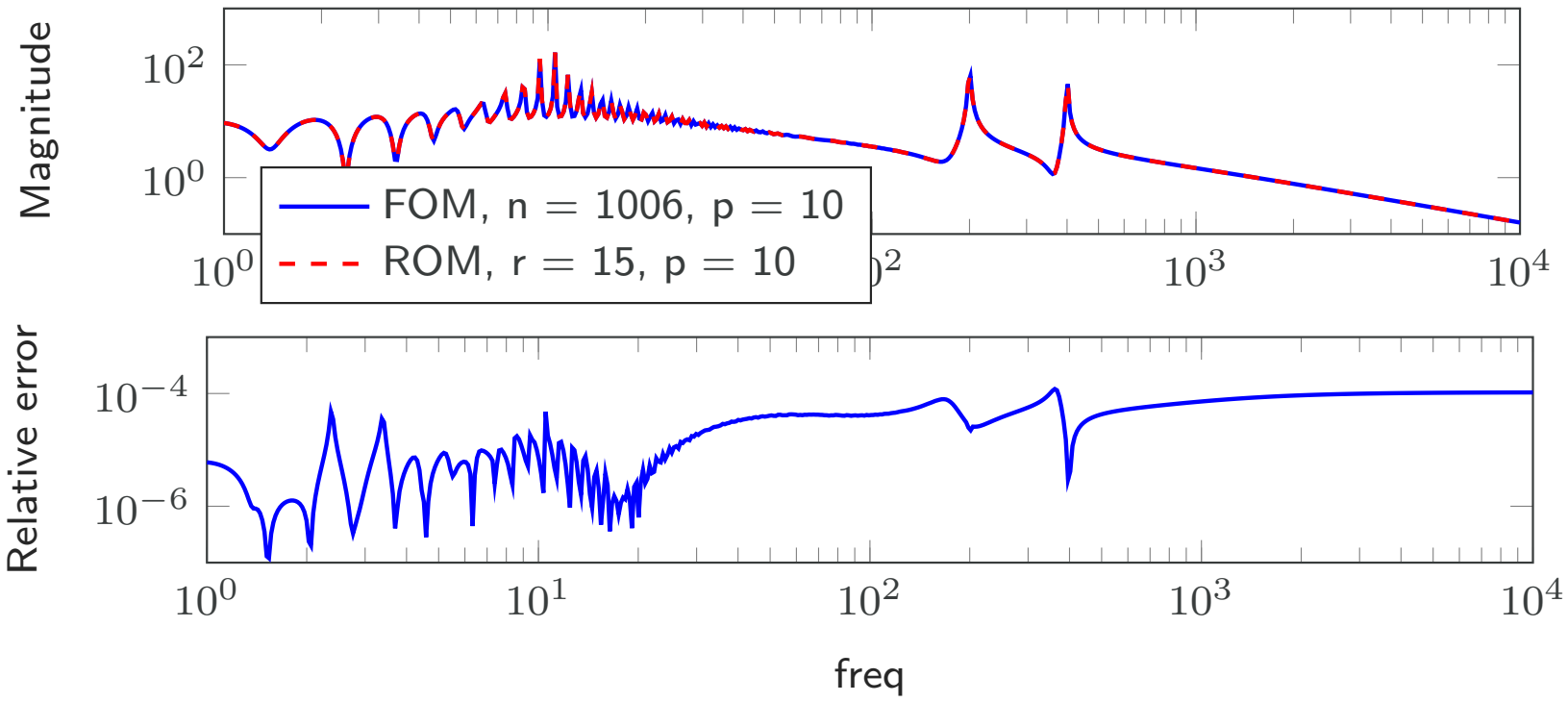

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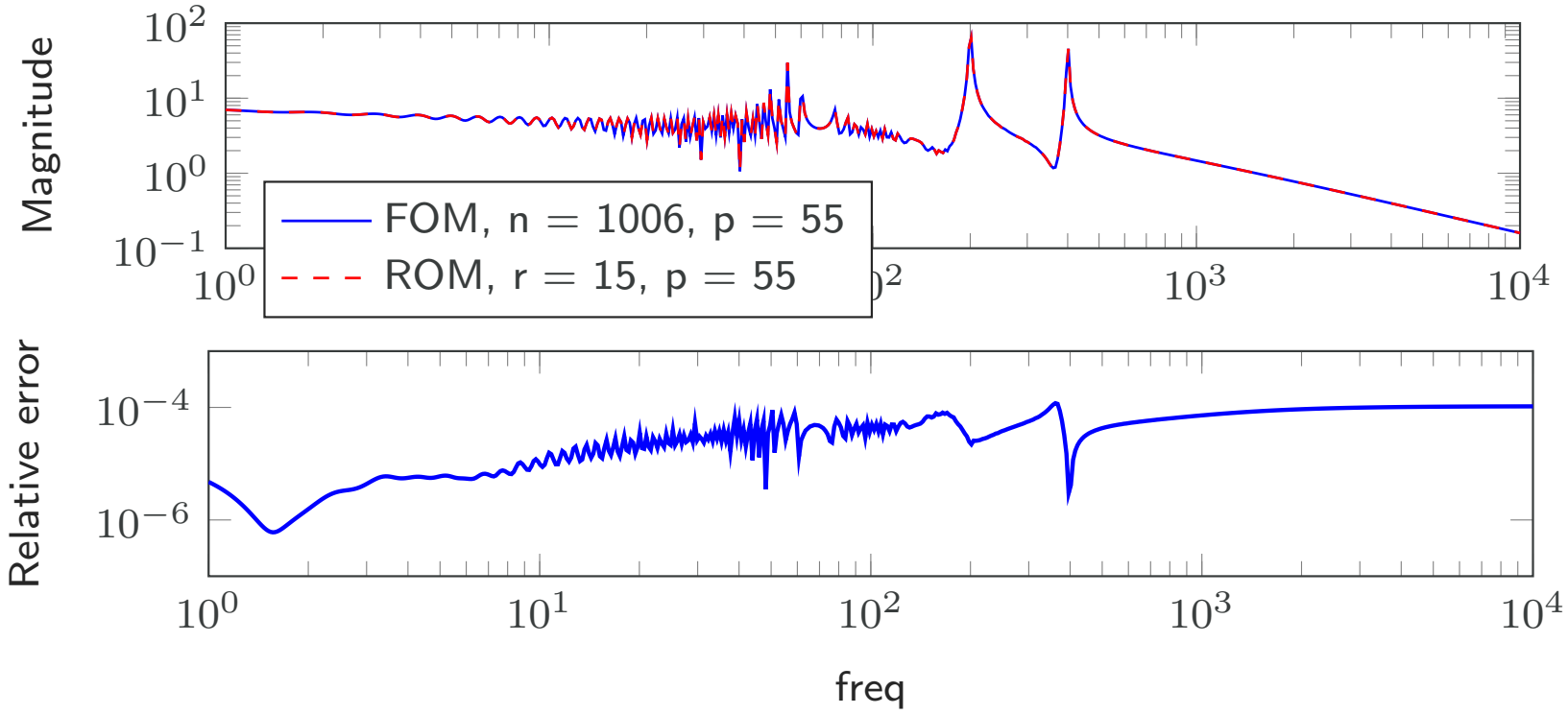

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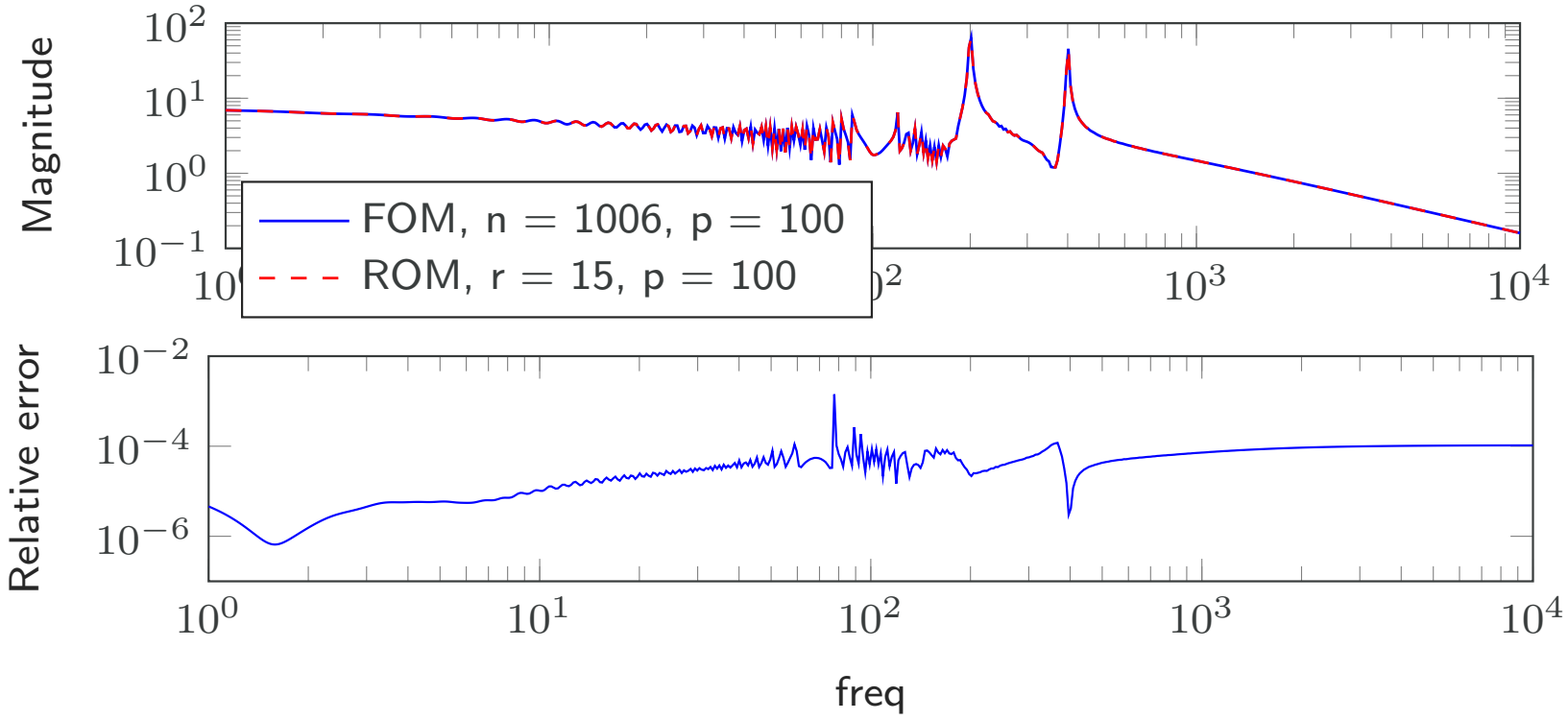

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Fitz-Hugh Nagumo model: Governing coupled equation

$$\begin{aligned} \epsilon v_t &= \epsilon^2 v_{xx} + v(v-0.1)(1-v) - w + u, \\ w_t &= hv - \gamma w + u \end{aligned} \qquad \text{on} \quad [0,T] \times [0,L]$$

with initial and boundary conditions

$$v(x,0) = 0, \quad w(x,0) = 0, \quad x \in (0,L), \qquad v_x(0,t) = i_0(t), \quad v_x(L,t) = 0, \quad t \geq 0.$$

- To employ the interpolation-based algorithm, we choose random 100 interpolation points in a logarithmic way between $\left[10^{-2}, 10^{2}\right]$ and set $\sigma_i = \mu_i$, $i \in \{1, \ldots, 100\}$.

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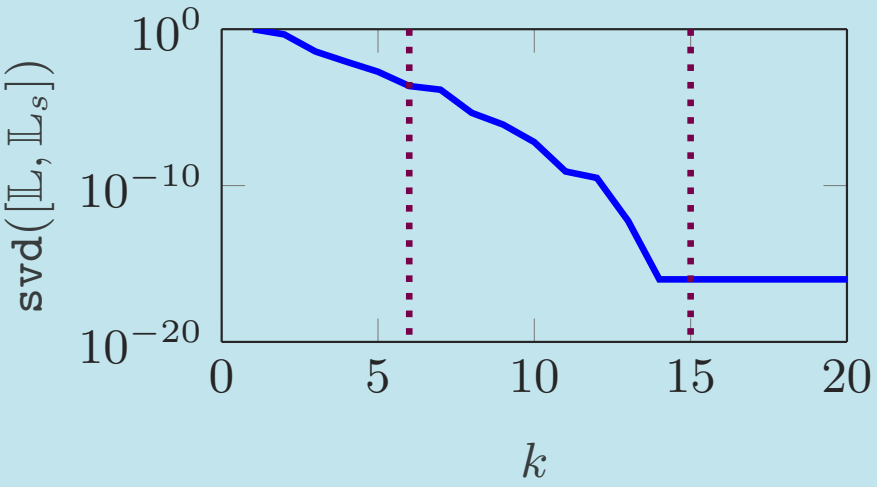

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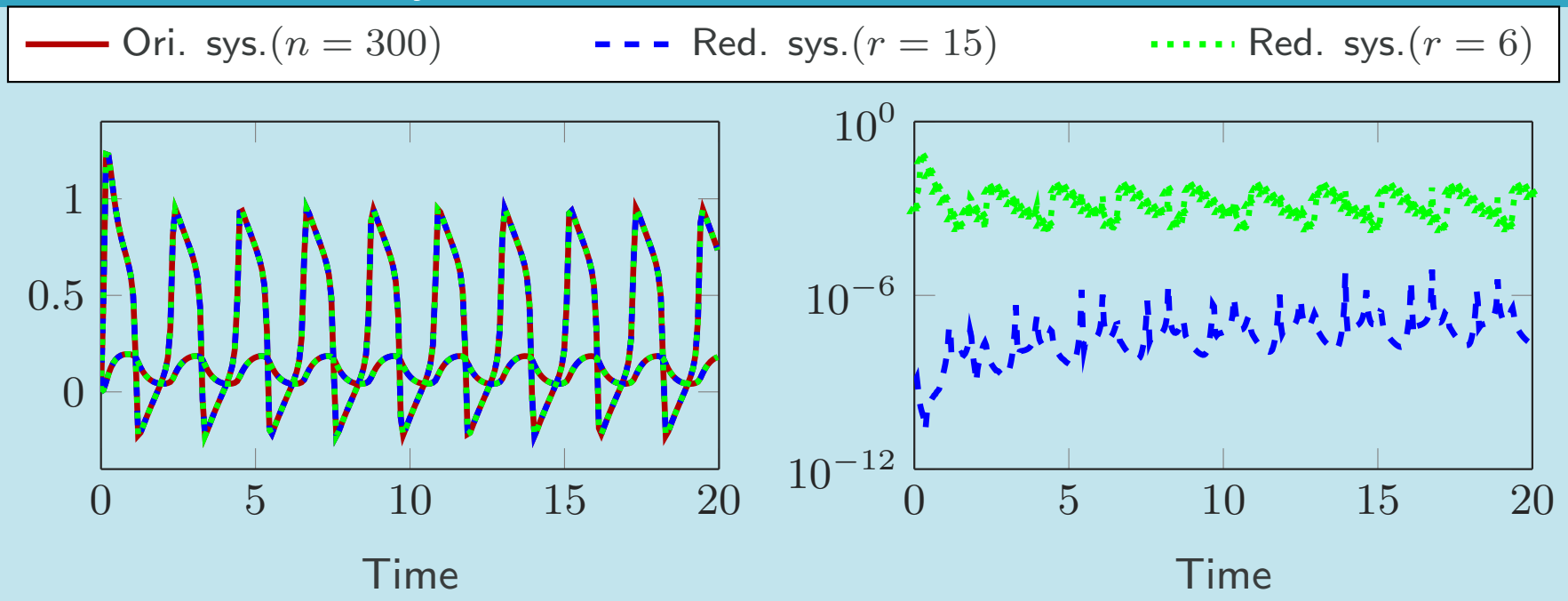
Construction of reduced systems



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Outlook and Conclusions

Contribution of this talk

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- Model reduction technique inspired by numerical rank of matrix $\mathbf{O}^T\mathbf{E}\mathbf{R}$.
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Open questions and future work

- Stability preservation and error bounds.
- Relation to pure Loewner-style approach [Schulze/Unger/Beattie/Gugercin '18]?
- How to find an intermediate full-order model for projection?
- Extension to nonlinear systems, first results for polynomial systems in [Benner/Goyal '19, arXiv:1904.11891].