MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Computational Methods for Feedback Control of Flow Problems

Björn Baran Peter Benner Jens Saak Heiko Weichelt

Workshop on Numerical Analysis and Scientific Computing — on the occasion of Prof. Eberhard Bänsch's 60th birthday

Erlangen, 22–23 November 2019

Supported by:

Joint Work

Bremen, 1998 – 2000

Mainly working on

Bremen, 1998 – 2000

Mainly working on

- Schäufele

Joint Work

Bremen, 1998 – 2000

Mainly working on

- Schäufele
- St. Emilion

Joint Work

Bremen, 1998 – 2000

Mainly working on

- Schäufele
- St. Emilion
- rivalry of SVW vs. FCB.

vs.

Joint Work

Bremen, 1998 – 2000

Mainly working on

- Schäufele
- St. Emilion
- rivalry of SVW vs. FCB.

Chemnitz/Magdeburg – Erlangen, 2005 – 2013

Eventually, some idea of joint work evolved around call for proposals within DFG Priority Program 1253 Optimization with Partial Differential Equations . . .

Joint Work

Bremen, 1998 – 2000

Mainly working on

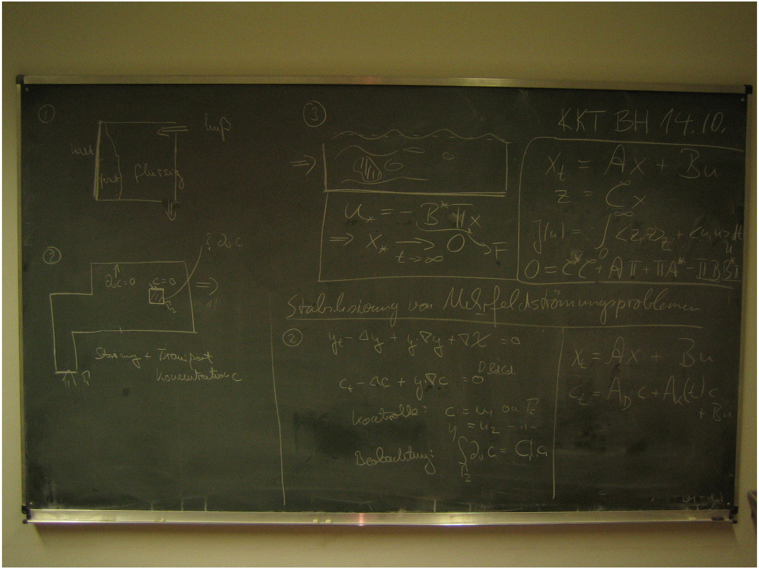
- Schäufele
- St. Emilion
- rivalry of SVW vs. FCB.

Chemnitz/Magdeburg – Erlangen, 2005 – 2013

Eventually, some idea of joint work evolved around call for proposals within DFG Priority Program 1253 Optimization with Partial Differential Equations . . .

$\rightsquigarrow$ **stabilization and feedback control of multi-field flow problems!**

...Chemnitz, October 12, 2005 (a few days before the deadline)



Our joint proposal for SPP1253
...hard at work

- **Physical transport** is one of the most fundamental dynamical processes in nature.
- **Prediction** and **manipulation** of transport processes are important research topics, e.g., to
  - avoid stall — for stable and safe flight;
  - save energy (or increase attainable speed) by minimizing drag coefficient;
  - use fluid flow for optimal transport (e.g., in blood veins).
- **Open-loop** controllers are widely used in various engineering fields.
  $\rightarrow$ **Not robust** regarding perturbation
- Dynamical systems are often influenced via so called **distributed control**.
  $\rightarrow$ **Unfeasible** in many real-world areas

$\Rightarrow$ **Boundary feedback stabilization (closed-loop)** can be used to increase robustness and feasibility.

Let (x_*, u_*) solve $\min_{u \in \mathcal{U}_{ad}} J(x, u)$ s.t. $\dot{x}(t) = f(x(t), u(t))$.

Let (x_*, u_*) solve $\min_{u \in \mathcal{U}_{ad}} J(x,u)$ s.t. $\dot{x}(t) = f(x(t), u(t))$.

Fundamental observation

Optimized trajectory $x_*(t; u_*)$ and precomputed optimal control $u_*(t)$ will not be attainable in practice due to

- modeling errors and/or unmodeled dynamics,
- model uncertainties,
- external perturbations,
- measurement errors.

Let (x_*, u_*) solve $\min_{u \in \mathcal{U}_{ad}} J(x, u)$ s.t. $\dot{x}(t) = f(x(t), u(t))$.

Fundamental observation

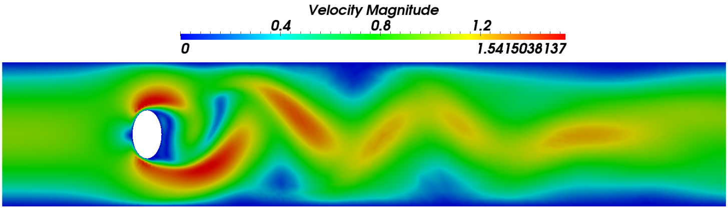
Optimized trajectory $x_*(t; u_*)$ and precomputed optimal control $u_*(t)$ will not be attainable in practice due to

- modeling errors and/or unmodeled dynamics,
- model uncertainties,
- external perturbations,
- measurement errors.

Consequence: for compensation of errors and correction of deviation from desired path, need feedback control

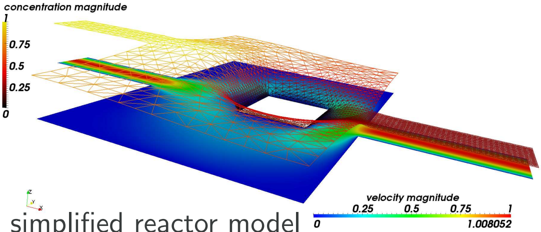
$$u(t) = u_*(t) + U(t, x(t) - x_*(t)).$$

- Consider 2D flow problems described by **incompressible Navier–Stokes equations**.
- Riccati feedback approach requires the solution of an **algebraic Riccati equation**.
- Conservation of mass introduces a **divergence-freeness** condition $\rightsquigarrow$ problems with mathematical basis of control design schemes.



Kármán vortex street

- Consider 2D flow problems described by **incompressible Navier–Stokes equations**.
- Riccati feedback approach requires the solution of an **algebraic Riccati equation**.
- Conservation of mass introduces a **divergence-freeness** condition $\rightsquigarrow$ problems with mathematical basis of control design schemes.
- **Coupling** flow problems with a **scalar reaction-advection-diffusion equation**.



simplified reactor model

1. Functional analytic control approach by Raymond ([Raymond '05–'07]) works in subspace of divergence-free functions.

1. Functional analytic control approach by Raymond ([Raymond '05–'07]) works in subspace of divergence-free functions.
   Establish a numerical realization for Leray projection.

1. Functional analytic control approach by Raymond ([Raymond '05–'07]) works in subspace of divergence-free functions.
   Establish a numerical realization for Leray projection.
2. NAVIER: FE package using $\mathcal{P}_2$-$\mathcal{P}_1$ Taylor–Hood elements.

1. Functional analytic control approach by Raymond ([Raymond '05–'07]) works in subspace of divergence-free functions.
   Establish a numerical realization for Leray projection.
2. NAVIER: FE package using $\mathcal{P}_2$-$\mathcal{P}_1$ Taylor–Hood elements.
   Incorporate unsteady boundary conditions and boundary control operator.

1. Functional analytic control approach by Raymond ([Raymond '05–'07]) works in subspace of divergence-free functions.
   Establish a numerical realization for Leray projection.
2. NAVIER: FE package using $\mathcal{P}_2$-$\mathcal{P}_1$ Taylor–Hood elements.
   Incorporate unsteady boundary conditions and boundary control operator.
3. LQR theory for generalized state-space systems.

Goal of our SPP1253 Proposal

Available Tools and Necessary Tasks at Project Start

1. Functional analytic control approach by Raymond ([Raymond '05–'07]) works in subspace of divergence-free functions.
   Establish a numerical realization for Leray projection.
2. NAVIER: FE package using $\mathcal{P}_2$-$\mathcal{P}_1$ Taylor–Hood elements.
   Incorporate unsteady boundary conditions and boundary control operator.
3. LQR theory for generalized state-space systems.
   Incorporate a DAE structure without using expensive DAE methods.

Goal of our SPP1253 Proposal

Available Tools and Necessary Tasks at Project Start

1. Functional analytic control approach by Raymond ([RAYMOND '05–'07]) works in subspace of divergence-free functions.
   Establish a numerical realization for Leray projection.
2. NAVIER: FE package using $\mathcal{P}_2$-$\mathcal{P}_1$ Taylor–Hood elements.
   Incorporate unsteady boundary conditions and boundary control operator.
3. LQR theory for generalized state-space systems.
   Incorporate a DAE structure without using expensive DAE methods.
4. Kleinman–Newton-ADI framework for solving generalized algebraic Riccati equations.

1. Functional analytic control approach by Raymond ([RAYMOND '05–'07]) works in subspace of divergence-free functions.
   Establish a numerical realization for Leray projection.
2. NAVIER: FE package using $\mathcal{P}_2$-$\mathcal{P}_1$ Taylor–Hood elements.
   Incorporate unsteady boundary conditions and boundary control operator.
3. LQR theory for generalized state-space systems.
   Incorporate a DAE structure without using expensive DAE methods.
4. Kleinman–Newton-ADI framework for solving generalized algebraic Riccati equations.
   Incorporate the divergence-free condition without explicit projection.

Goal of our SPP1253 Proposal

Available Tools and Necessary Tasks at Project Start

1. Functional analytic control approach by Raymond ([Raymond '05–'07]) works in subspace of divergence-free functions.
   Establish a numerical realization for Leray projection.
2. NAVIER: FE package using $\mathcal{P}_2$-$\mathcal{P}_1$ Taylor–Hood elements.
   Incorporate unsteady boundary conditions and boundary control operator.
3. LQR theory for generalized state-space systems.
   Incorporate a DAE structure without using expensive DAE methods.
4. Kleinman–Newton-ADI framework for solving generalized algebraic Riccati equations.
   Incorporate the divergence-free condition without explicit projection.
5. Preconditioned iterative methods to solve stationary Navier–Stokes systems.

Goal of our SPP1253 Proposal

Available Tools and Necessary Tasks at Project Start

1. Functional analytic control approach by Raymond ([RAYMOND '05–'07]) works in subspace of divergence-free functions.
   Establish a numerical realization for Leray projection.
2. NAVIER: FE package using $\mathcal{P}_2$-$\mathcal{P}_1$ Taylor–Hood elements.
   Incorporate unsteady boundary conditions and boundary control operator.
3. LQR theory for generalized state-space systems.
   Incorporate a DAE structure without using expensive DAE methods.
4. Kleinman–Newton-ADI framework for solving generalized algebraic Riccati equations.
   Incorporate the divergence-free condition without explicit projection.
5. Preconditioned iterative methods to solve stationary Navier–Stokes systems.
   Develop techniques to deal with complex-shifted multi-field flow

Navier–Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta \vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{f}$$
$$\operatorname{div} \vec{v} = 0$$

$A,\ M \in \mathbb{R}^{n\times n},\ \widehat{G} \in \mathbb{R}^{n\times n_{\mathbf{p}}}$

$B \in \mathbb{R}^{n\times n_r},\ C \in \mathbb{R}^{n_a\times n}$

$\mathbf{u}(t) \in \mathbb{R}^{n_r},\ \mathbf{y}(t) \in \mathbb{R}^{n_a}$

$\operatorname{rank}\left(\widehat{G}\right) = n_{\mathbf{p}}$

Linearize + Discretize → Index-2 DAE

$$M\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = A\mathbf{v}(t) + \widehat{G}\mathbf{p}(t) + B\mathbf{u}(t)$$
$$0 = \widehat{G}^T\mathbf{v}(t)$$
$$\mathbf{y}(t) = C\mathbf{v}(t)$$

$M = M^T \succ 0$

$\mathbf{v}(t) \in \mathbb{R}^n,\ \mathbf{p}(t) \in \mathbb{R}^{n_{\mathbf{p}}}$

$n = n_{\mathbf{v}},\ N = n + n_{\mathbf{p}}$

Show that projection in [Hei/Sor/Sun '08] is discretized version of Leray projector in [Ray '06].

$$M\Pi^T = \Pi M \quad \wedge \quad \Pi^T\mathbf{v} = \mathbf{v}_{\mathrm{div},0}$$

[**Bänsch**/B./Saak/**Stoll**/Weichelt **'13**,**'15**]

Navier–Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta \vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{f}$$

$$\operatorname{div} \vec{v} = 0$$

Concentration Equation

$$\frac{\partial c^{(\vec{v})}}{\partial t} - \frac{1}{\mathrm{Re}\,\mathrm{Sc}}\Delta c^{(\vec{v})} + (\vec{v}\cdot\nabla)c^{(\vec{v})} = 0$$

$A,\ M \in \mathbb{R}^{n\times n}$, $\widehat{G} \in \mathbb{R}^{n\times n_p}$

$B \in \mathbb{R}^{n\times n_r}$, $C \in \mathbb{R}^{n_a\times n}$

$\mathbf{u}(t) \in \mathbb{R}^{n_r}$, $\mathbf{y}(t) \in \mathbb{R}^{n_a}$

$\operatorname{rank}\left(\widehat{G}\right) = n_p$

Linearize + Discretize → Index-2 DAE

$$M\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = A\mathbf{x}(t) + \widehat{G}\mathbf{p}(t) + B\mathbf{u}(t)$$

$$0 = \widehat{G}^T\mathbf{x}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t)$$

$M = M^T \succ 0$

$$\mathbf{x}(t) = \begin{bmatrix}\mathbf{v}(t)\\ \mathbf{c}(t)\end{bmatrix} \in \mathbb{R}^n$$

$n = n_v + n_c,\ N = n + n_p$

Show that projection in [Hei/Sor/Sun '08] is discretized version of Leray projector in [Ray '06].

$$M\Pi^T = \Pi M \quad \wedge \quad \Pi^T\mathbf{v} = \mathbf{v}_{\mathrm{div},0}$$

[**Bänsch**/B./Saak/**Stoll**/Weichelt **'13,'15**]

Extension to coupled flow case, i.e.,

$$\widehat{\Pi} := \begin{bmatrix}\Pi & 0\\ 0 & I\end{bmatrix} \quad \wedge \quad \begin{bmatrix}\Pi^T & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{c}\end{bmatrix} = \begin{bmatrix}\mathbf{v}_{\mathrm{div},0}\\ \mathbf{c}\end{bmatrix}.$$

[Bänsch/B./Saak/Weichelt '14]

Minimize

$$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \lambda^2 ||\mathbf{y}||^2 + ||\mathbf{u}||^2 \, \mathrm{d}t$$

subject to

$$\widehat{\Theta}_r^T M \widehat{\Theta}_r \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\boldsymbol{x}}(t) = \widehat{\Theta}_r^T A \widehat{\Theta}_r \widetilde{\boldsymbol{x}}(t) + \widehat{\Theta}_r^T B\mathbf{u}(t)$$

$$\mathbf{y}(t) = C\widehat{\Theta}_r\widetilde{\boldsymbol{x}}(t)$$

with $\widehat{\Pi} = \widehat{\Theta}_l\widehat{\Theta}_r^T$ such that $\widehat{\Theta}_r^T\widehat{\Theta}_l = I \in \mathbb{R}^{(n-n_{\mathbf{p}})\times(n-n_{\mathbf{p}})}$ and $\widetilde{\boldsymbol{x}} = \widehat{\Theta}_l^T\mathbf{x}$.

Minimize

$$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \lambda^2||\mathbf{y}||^2 + ||\mathbf{u}||^2 \,\mathrm{d}t$$

subject to

$$\begin{aligned} \mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\boldsymbol{x}}(t) &= \mathcal{A}\widetilde{\boldsymbol{x}}(t) + \mathcal{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathcal{C}\widetilde{\boldsymbol{x}}(t) \end{aligned}$$

with $\mathcal{M} = \mathcal{M}^T \succ 0$.

Minimize

$$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \lambda^2||\mathbf{y}||^2 + ||\mathbf{u}||^2 \,\mathrm{d}t$$

subject to

$$\begin{aligned}\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\boldsymbol{x}}(t) &= \mathcal{A}\widetilde{\boldsymbol{x}}(t) + \mathcal{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathcal{C}\widetilde{\boldsymbol{x}}(t)\end{aligned}$$

with $\mathcal{M} = \mathcal{M}^T \succ 0$.

Riccati Based Feedback Approach

- Optimal control: $\mathbf{u}(t) = -\mathcal{K}\widetilde{\boldsymbol{x}}(t)$, with feedback: $\mathcal{K} = \mathcal{B}^T\mathcal{X}\mathcal{M}$,

  where $\mathcal{X}$ is the solution of the generalized continuous-time algebraic Riccati equation (GCARE)

$$\mathcal{R}(\mathcal{X}) = \lambda^2\mathcal{C}^T\mathcal{C} + \mathcal{A}^T\mathcal{X}\mathcal{M} + \mathcal{M}\mathcal{X}\mathcal{A} - \mathcal{M}\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}\mathcal{M} = 0.$$

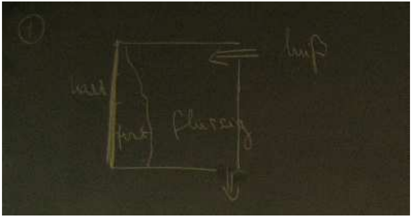
Main Contributions of our SPP1253 Project

- Analyzed **Riccati-based feedback** for **scalar** and **vector-valued transport** problems.
- Wide-spread usability tailored for standard **inf-sup stable finite element** discretizations.
- Established **specially tailored Kleinman–Newton-ADI** that **avoids explicit projections**.
- **Suitable preconditioners** for multi-field flow problems have been developed.
- Major run time improvements due to combination of **inexact Newton** and **line search**.
- Established **new convergence proofs** that were verified by **extensive numerical tests**.
- $\Rightarrow$ Showed **usability** of new approach by a **closed-loop forward simulation**.

- E. Bänsch and P. Benner, *Stabilization of incompressible flow problems by Riccati-based feedback*, in Constrained Optimization and Optimal Control for Partial Differential Equations, vol. 160 of International Series of Numerical Mathematics, Birkhäuser, 2012, pp. 5–20.
- P. Benner, J. Saak, M. Stoll, and H. K. Weichelt, *Efficient solution of large-scale saddle point systems arising in Riccati-based boundary feedback stabilization of incompressible Stokes flow*, **SIAM J. Sci. Comput.**, 35 (2013), pp. S150–S170.
- E. Bänsch, P. Benner, J. Saak, and H. K. Weichelt, *Optimal control-based feedback stabilization of multi-field flow problems*, in Trends in PDE Constrained Optimization, vol. 165 of Internat. Ser. Numer. Math., Birkhäuser, Basel, 2014, pp. 173–188.
- E. Bänsch, P. Benner, J. Saak, and H. K. Weichelt, *Riccati-based boundary feedback stabilization of incompressible Navier-Stokes flows*, **SIAM J. Sci. Comput.**, 37 (2015), pp. A832–A858.
- P. Benner, M. Heinkenschloss, J. Saak, and H. K. Weichelt, *Efficient solution of large-scale algebraic Riccati equations associated with index-2 DAEs via the inexact low-rank Newton-ADI method*, **Appl. Numer. Math.**, accepted 2019-11-21 (2019).

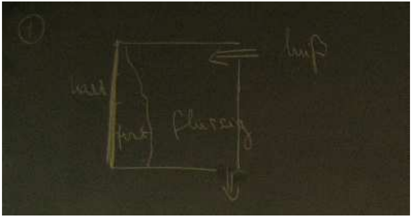
What we did not achieve. . .

- We never started working on Scenario 3 in the SPP1253 proposal . . .



What we did not achieve. . .

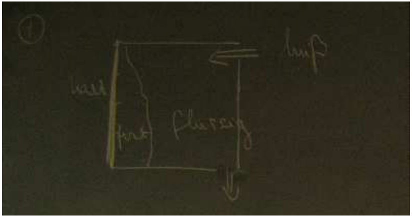
- We never started working on Scenario 3 in the SPP1253 proposal . . .



I.e., we promised to stabilize the interface at solid/liquid phase transitions with convection . . .

What we did not achieve...

- We never started working on Scenario 3 in the SPP1253 proposal ...

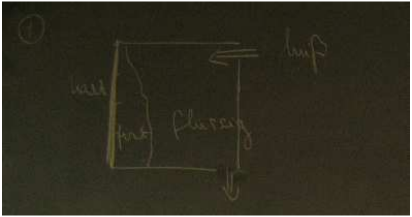


I.e., we promised to stabilize the interface at solid/liquid phase transitions with convection ...

- The first birthday present: **It's (partially) done!**

What we did not achieve...

- We never started working on Scenario 3 in the SPP1253 proposal ...



  I.e., we promised to stabilize the interface at solid/liquid phase transitions with convection ...

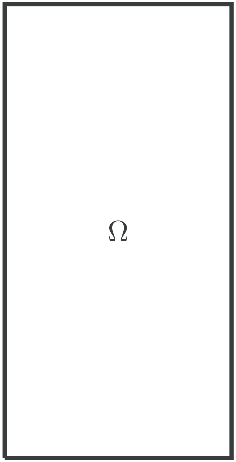
- The first birthday present: **It's (partially) done!**
- Rest of the talk: feedback control of the Stefan problem.

Linear Quadratic Regulator

Stefan Problem

Structural Properties

Numerical Examples



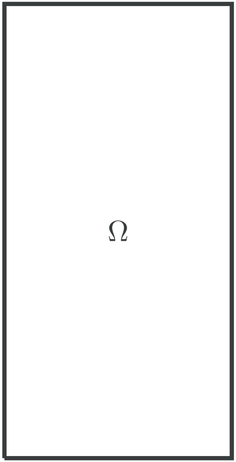
Minimize $$\frac{1}{2}\int_0^{t_f} \lambda \left\|\theta - \theta_d\right\|^2 + \left\|\mathbf{u}\right\|^2 \; dt$$

subject to

$$\begin{aligned} \frac{d}{dt}\theta - k\Delta\theta &= 0, & \Omega \\ g(\theta) &= \mathbf{u}, & \partial\Omega \end{aligned}$$

Motivation

Feedback Stabilization of the Heat Equation with Riccati-feedback



Minimize $$\frac{1}{2}\int_0^{t_f} \lambda \left\|y - y_d\right\|^2 + \left\|\mathbf{u}\right\|^2 \; dt$$

subject to

$$\begin{aligned} \frac{d}{dt}\theta - k\Delta\theta &= 0, && \Omega \\ g(\theta) &= \mathbf{u}, && \partial\Omega \end{aligned}$$

Discretization:

A, M large sparse, B, C skinny

$$\begin{aligned} M\dot{x} &= Ax + B\mathbf{u} \\ y &= Cx \end{aligned}$$

$$M\dot{x} = Ax + B\mathbf{u}$$
$$y = Cx$$

Riccati-based Feedback Approach **e.g.,**[Sontag '98]

- Stabilizing feedback: $\mathcal{K} = \mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M}$

  where $\mathbf{X}$ is the solution of the generalized differential Riccati equation (DRE)

$$-\mathcal{M}^\mathsf{T}\dot{\mathbf{X}}\mathcal{M} = \mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\mathbf{X}\mathcal{M} + \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{A} - \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{B}\mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M}$$

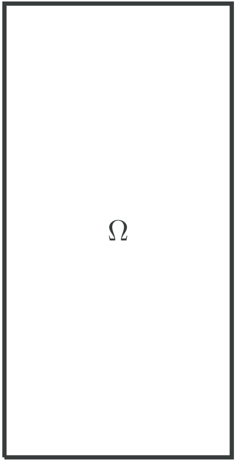
- Optimal control: $\mathbf{u} = -\mathcal{K}x$

$$-\mathcal{M}^{\mathsf{T}}\dot{\mathbf{X}}\mathcal{M} = \mathcal{C}^{\mathsf{T}}\mathcal{C} + \mathcal{A}^{\mathsf{T}}\mathbf{X}\mathcal{M} + \mathcal{M}^{\mathsf{T}}\mathbf{X}\mathcal{A} - \mathcal{M}^{\mathsf{T}}\mathbf{X}\mathcal{B}\mathcal{B}^{\mathsf{T}}\mathbf{X}\mathcal{M}$$

$$\mathbf{X} \approx ZZ^{\mathsf{T}} \qquad \text{or} \qquad \mathbf{X} \approx LDL^{\mathsf{T}}$$

- Backward Differentiation Formula (BDF)
- Rosenbrock
- Peer methods
- Splitting schemes
- Krylov subspace methods
- ...

[Behr/Benner/Heiland '18, Kirsten/Simoncini '19, Koskela/Mena '18, Lang '17, Mena '07, Ostermann/Piazzola/Walach '18, Stillfjord '15, Stillfjord '18, ...]



Time dependent heat conductivity $k(t)$

$$
\begin{aligned}
\frac{d}{dt}\theta - k(t)\Delta\theta &= 0, \qquad && \Omega \\
g(\theta) &= \mathbf{u}, \qquad && \partial\Omega
\end{aligned}
$$

Discretization:

$$
\begin{aligned}
M\dot{x} &= A(t)x + B\mathbf{u} \\
y &= Cx \\
A(t) &= k(t)A_0 \qquad k\colon \mathbb{R} \to \mathbb{R}
\end{aligned}
$$

$$-\mathcal{M}^\mathsf{T}\dot{\mathbf{X}}\mathcal{M} = \mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}(t)^\mathsf{T}\mathbf{X}\mathcal{M} + \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{A}(t) - \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{B}\mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M}$$
$$\mathbf{X} \approx LDL^\mathsf{T}$$

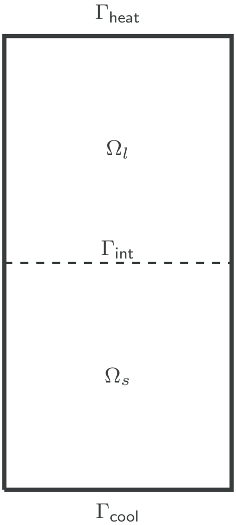
- BDF
- Rosenbrock
- Peer methods
- ...

[Lang '17, Mena '07, ...]

Linear Quadratic Regulator

$$M\dot{x} = Ax + B\mathbf{u}$$
$$y = Cx$$

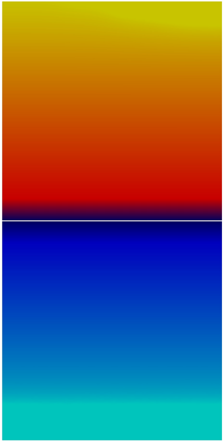
Stefan Problem



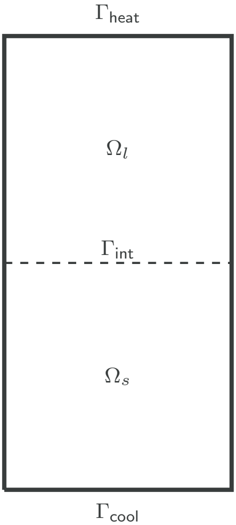
$$
\begin{aligned}
\frac{d}{dt}\theta - \alpha\Delta\theta &= 0, && \Omega_l \cup \Omega_s\\
[k_s(\nabla\theta)_s - k_l(\nabla\theta)_l] \cdot \boldsymbol{n}_{\text{int}} &= LV_{\text{int}}, && \Gamma_{\text{int}}\\
\theta &= \theta_{\text{cool}}, && \Gamma_{\text{cool}}\\
\partial_{\boldsymbol{n}}\theta &= \mathbf{u}, && \Gamma_{\text{heat}}
\end{aligned}
$$

Non-autonomous Problem

Stefan Problem

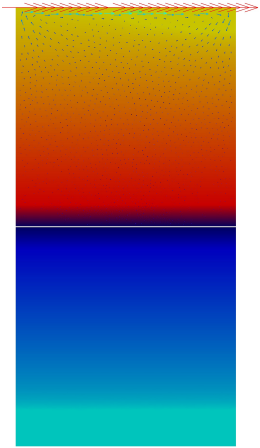


$$
\begin{aligned}
\frac{d}{dt}\theta - \alpha\Delta\theta &= 0, && \Omega_l \cup \Omega_s\\
\left[k_s(\nabla\theta)_s - k_l(\nabla\theta)_l\right]\cdot \boldsymbol{n}_{\mathsf{int}} &= LV_{\mathsf{int}}, && \Gamma_{\mathsf{int}}\\
\theta &= \theta_{\mathsf{cool}}, && \Gamma_{\mathsf{cool}}\\
\partial_{\boldsymbol{n}}\theta &= \mathbf{u}, && \Gamma_{\mathsf{heat}}
\end{aligned}
$$



$$
\begin{aligned}
\frac{d}{dt}\theta + v\cdot\nabla\theta - \alpha\Delta\theta &= 0, && \Omega_l \cup \Omega_s\\
[k_s(\nabla\theta)_s - k_l(\nabla\theta)_l]\cdot \boldsymbol{n}_{\text{int}} &= LV_{\text{int}}, && \Gamma_{\text{int}}\\
\theta &= \theta_{\text{cool}}, && \Gamma_{\text{cool}}\\
\partial_{\boldsymbol{n}}\theta &= \mathbf{u}, && \Gamma_{\text{heat}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}v - \eta\Delta v + \nabla p &= 0, && \Omega_l\\
\nabla\cdot v &= 0, && \Omega_l
\end{aligned}
$$



$$\frac{d}{dt}\theta + v \cdot \nabla\theta - \alpha\Delta\theta = 0, \qquad \Omega_l \cup \Omega_s$$

$$[k_s(\nabla\theta)_s - k_l(\nabla\theta)_l] \cdot \boldsymbol{n}_{\mathsf{int}} = LV_{\mathsf{int}}, \qquad \Gamma_{\mathsf{int}}$$

$$\theta = \theta_{\mathsf{cool}}, \qquad \Gamma_{\mathsf{cool}}$$

$$\partial_{\boldsymbol{n}}\theta = \mathbf{u}, \qquad \Gamma_{\mathsf{heat}}$$

$$\frac{d}{dt}v - \eta\Delta v + \nabla p = 0, \qquad \Omega_l$$

$$\nabla \cdot v = 0, \qquad \Omega_l$$

Nonlinear Problem $\rightsquigarrow$ Linearization

- Use open-loop control to compute a reference trajectory.
- Linearize around the reference trajectory.

Challenges

Nonlinear Problem $\rightsquigarrow$ Linearization

- Use open-loop control to compute a reference trajectory.
- Linearize around the reference trajectory.

Changing Domain $\rightsquigarrow$ Moving Mesh / Remeshing

- Non-autonomous system:

$$M(t)\dot{x} = A(t)x + B(t)\mathbf{u}$$
$$y = C(t)x$$

Nonlinear Problem $\rightsquigarrow$ Linearization

- Use open-loop control to compute a reference trajectory.
- Linearize around the reference trajectory.

Changing Domain $\rightsquigarrow$ Moving Mesh / Remeshing

- Non-autonomous system:

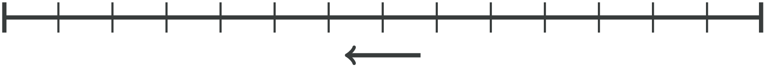
$$M(t)\dot{x} = A(t)x + B(t)\mathbf{u}$$
$$y = C(t)x$$

Memory Requirements

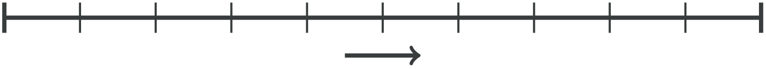
- Precompute matrices for every required time step.
- Avoid extra time steps in DRE solvers.

Open-loop control simulation:

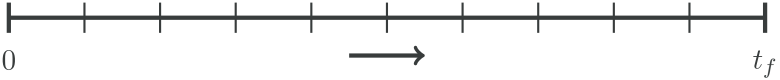
DRE solver:



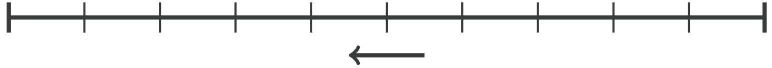
Feedback control simulation:



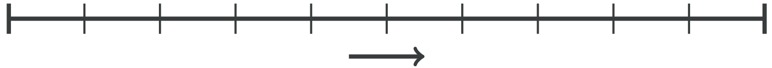
Open-loop control simulation:



DRE solver:



Feedback control simulation:



Autonomous DRE

$$-\mathcal{M}^\mathsf{T}\dot{\mathbf{X}}\mathcal{M} = \mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\mathbf{X}\mathcal{M} + \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{A} - \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{B}\mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M}$$

Non-autonomous DRE

$\mathcal{M} = \mathcal{M}(t), \mathcal{A} = \mathcal{A}(t), \mathcal{B} = \mathcal{B}(t), \mathcal{C} = \mathcal{C}(t)$

$$\begin{aligned}-\mathcal{M}^\mathsf{T}\dot{\mathbf{X}}\mathcal{M} =& \mathcal{C}^\mathsf{T}\mathcal{C} + (\dot{\mathcal{M}} + \mathcal{A})^\mathsf{T}\mathbf{X}\mathcal{M} + \mathcal{M}^\mathsf{T}\mathbf{X}(\dot{\mathcal{M}} + \mathcal{A}) \\ &- \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{B}\mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M}\end{aligned}$$

BDF Method

- low-rank multi-step algorithm to solve DREs
- algebraic Riccati equation in each time step
- main work: solution of shifted systems $(\mathcal{A} + p\mathcal{M})V = W$

BDF Method

- low-rank multi-step algorithm to solve DREs
- algebraic Riccati equation in each time step
- main work: solution of shifted systems $(\mathcal{A} + p\mathcal{M})V = W$

Non-autonomous DRE $\rightsquigarrow$ Non-autonomous BDF Method

- start-up with fixed number of very small extra time steps
- cannot reuse data from previous time step
- extra matrix-matrix multiplications in each time step

Challenges

BDF Method

- low-rank multi-step algorithm to solve DREs
- algebraic Riccati equation in each time step
- main work: solution of shifted systems $(\mathcal{A} + p\mathcal{M})V = W$

Non-autonomous DRE $\rightsquigarrow$ Non-autonomous BDF Method

- start-up with fixed number of very small extra time steps
- cannot reuse data from previous time step
- extra matrix-matrix multiplications in each time step

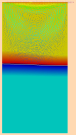
$$\mathcal{C}_k^{\mathsf{T}} = \left[\sqrt{\lambda}\mathcal{C}(t_k)^{\mathsf{T}}, \mathcal{M}(t_k)^{\mathsf{T}} L_{k-1}, \ldots, \mathcal{M}(t_k)^{\mathsf{T}} L_{k-\wp} \right]$$

$\wp$: order of the BDF method

Linear Quadratic Regulator

$$M\dot{x} = Ax + B\mathbf{u}$$
$$y = Cx$$

Stefan Problem



Structural Properties

$$\begin{bmatrix} \\ & \mathcal{M} & \\ \\ \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \\ \mathbf{x} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ & \mathcal{A} & \\ \\ \end{bmatrix} \begin{bmatrix} \\ \mathbf{x} \\ \\ \end{bmatrix} + \mathcal{B}\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} & \mathcal{C} & \end{bmatrix} \begin{bmatrix} \\ \mathbf{x} \\ \\ \end{bmatrix}$$

$$x \leftarrow \theta, v \qquad \Gamma \leftarrow V_{\text{int}} \qquad \Phi \leftarrow p$$

$$\begin{bmatrix} M_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} = \begin{bmatrix} A & J_1 & J_2 \\ G_1 & S & J_3 \\ G_2 & G_3 & 0 \end{bmatrix} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \mathcal{C}_x & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix}$$

$$\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Gamma \end{bmatrix} = \begin{bmatrix} A & J \\ G & S \end{bmatrix} \begin{bmatrix} x \\ \Gamma \end{bmatrix} + \mathcal{B}\mathbf{u}$$

e.g., [Gugercin, Stykel, Wyatt '13]

$$\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Gamma \end{bmatrix} = \begin{bmatrix} A & J \\ G & S \end{bmatrix} \begin{bmatrix} x \\ \Gamma \end{bmatrix} + \mathcal{B}\mathbf{u}$$

S regular : $\Gamma = -S^{-1}Gx$

e.g., [Gugercin, Stykel, Wyatt '13]

$$\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Gamma \end{bmatrix} = \begin{bmatrix} A & J \\ G & S \end{bmatrix} \begin{bmatrix} x \\ \Gamma \end{bmatrix} + \mathcal{B}\mathbf{u}$$

S regular : $\qquad \Gamma = -S^{-1}Gx$

$$M_x \frac{d}{dt} x = \tilde{A} x + \mathcal{B}\mathbf{u}$$

$$\tilde{A} = A - JS^{-1}G$$

e.g., [Gugercin, Stykel, Wyatt '13]

$$\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Phi \end{bmatrix} = \begin{bmatrix} A & J \\ G & 0 \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}$$

e.g., [Gugercin, Stykel, Wyatt '13]

$$\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Phi \end{bmatrix} = \begin{bmatrix} A & J \\ G & 0 \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}$$

J, G full rank :
$$\Pi_l = I - J(GM_x^{-1}J)^{-1}GM_x^{-1}$$
$$\Pi_r = I - M_x^{-1}J(GM_x^{-1}J)^{-1}G$$

e.g., [Gugercin, Stykel, Wyatt '13]

$$\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Phi \end{bmatrix} = \begin{bmatrix} A & J \\ G & 0 \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}$$

J, G full rank :

$$\Pi_l = I - J(GM_x^{-1}J)^{-1}GM_x^{-1}$$
$$\Pi_r = I - M_x^{-1}J(GM_x^{-1}J)^{-1}G$$

$$\hat{M}_x \frac{d}{dt} x = \hat{A} x + \hat{\mathcal{B}}\mathbf{u}$$

$$\hat{A} = \Pi_l A \Pi_r \qquad \hat{M}_x = \Pi_l M_x \Pi_r \qquad \hat{\mathcal{B}} = \Pi_l \mathcal{B}$$

e.g., [Gugercin, Stykel, Wyatt '13]

$$
\begin{bmatrix} M_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} = \begin{bmatrix} A & J_1 & J_2 \\ G_1 & S & J_3 \\ G_2 & G_3 & 0 \end{bmatrix} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}
$$

$$\begin{bmatrix} M_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} = \begin{bmatrix} A & J_1 & J_2 \\ G_1 & S & J_3 \\ G_2 & G_3 & 0 \end{bmatrix} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}$$

$$\Gamma = -S^{-1}(G_1 x + J_3 \Phi), \qquad \tilde{A} = A - J_1 S^{-1} G_1, \ldots$$

$$\begin{bmatrix} M_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} = \begin{bmatrix} A & J_1 & J_2 \\ G_1 & S & J_3 \\ G_2 & G_3 & 0 \end{bmatrix} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}$$

$$\Gamma = -S^{-1}(G_1 x + J_3 \Phi), \qquad \tilde{A} = A - J_1 S^{-1} G_1, \ldots$$

$$\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Phi \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{J} \\ \tilde{G} & 0 \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}$$

$$\begin{bmatrix} M_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} = \begin{bmatrix} A & J_1 & J_2 \\ G_1 & S & J_3 \\ G_2 & G_3 & 0 \end{bmatrix} \begin{bmatrix} x \\ \Gamma \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}$$

$$\Gamma = -S^{-1}(G_1 x + J_3 \Phi), \qquad \tilde{A} = A - J_1 S^{-1} G_1, \ldots$$

$$\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \Phi \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{J} \\ \tilde{G} & 0 \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix} + \mathcal{B}\mathbf{u}$$

$$\hat{\mathcal{M}} \frac{d}{dt} x = \hat{A} x + \hat{\mathcal{B}}\mathbf{u}$$

$$\hat{A} = \Pi_l \tilde{A} \Pi_r \qquad \hat{M} = \Pi_l M_x \Pi_r \qquad \hat{\mathcal{B}} = \Pi_l \mathcal{B}$$

Explicit **dense: $\mathrm{O}(n^2)$ memory**

$$\tilde{A} = A - J_1 S^{-1} G_1$$

$$\hat{A} = \Pi_l \tilde{A} \Pi_r \qquad \hat{M} = \Pi_l M_x \Pi_r$$

$$(\hat{A} + p\hat{M})V = W \text{ and } \Pi_r V = V$$

Explicit dense: $\mathcal{O}(n^2)$ memory

$$\tilde{A} = A - J_1 S^{-1} G_1$$

$$\hat{A} = \Pi_l \tilde{A} \Pi_r \qquad \hat{M} = \Pi_l M_x \Pi_r$$

$$(\hat{A} + p\hat{M})V = W \text{ and } \Pi_r V = V$$

$$\Longleftrightarrow$$

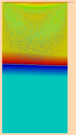
Implicit low-rank sparse: $\mathcal{O}(n)$ memory

$$\begin{bmatrix} A + pM_x & J_1 & J_2 \\ G_1 & S & J_3 \\ G_2 & G_3 & 0 \end{bmatrix} \begin{bmatrix} V \\ \Gamma \\ \Phi \end{bmatrix} = \begin{bmatrix} W \\ 0 \\ 0 \end{bmatrix}$$

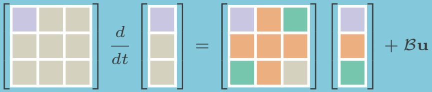
Linear Quadratic Regulator

$$M\dot{x} = Ax + B\mathbf{u}$$
$$y = Cx$$

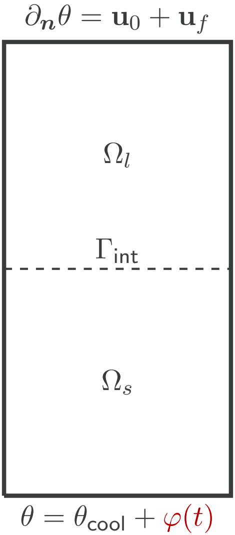
Stefan Problem



Structural Properties



Numerical Examples

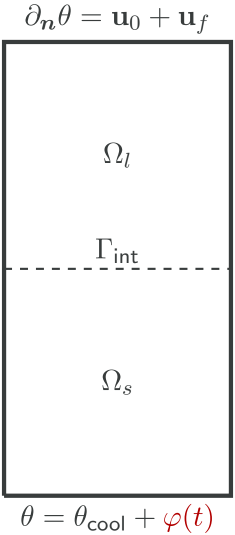


Reference: Use trajectory and control $\mathbf{u}_0$ from open-loop control.

Aim: Stabilize the interface position Γ_{int} with $\mathbf{u}_f = -\mathcal{K}x$.

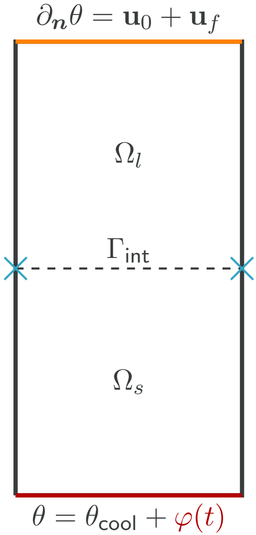
Perturbation: Add $\varphi(t)$ at the cooling boundary.

Software: FEniCS, M-M.E.S.S.



$$\frac{1}{2}\int_0^{t_f} \lambda \left\| y - y_d \right\|^2 + \left\| \mathbf{u} \right\|^2 \; dt$$

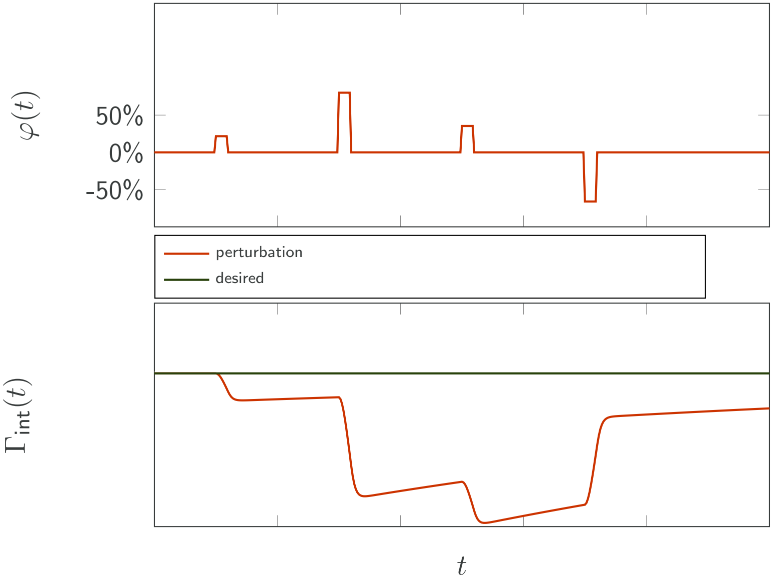
$$\lambda = 10^4$$



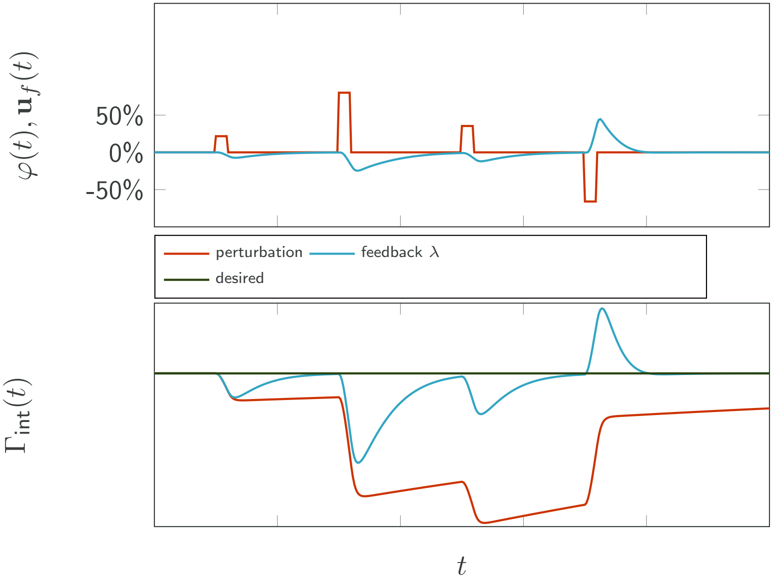
input

outputs:
temperature
measurments

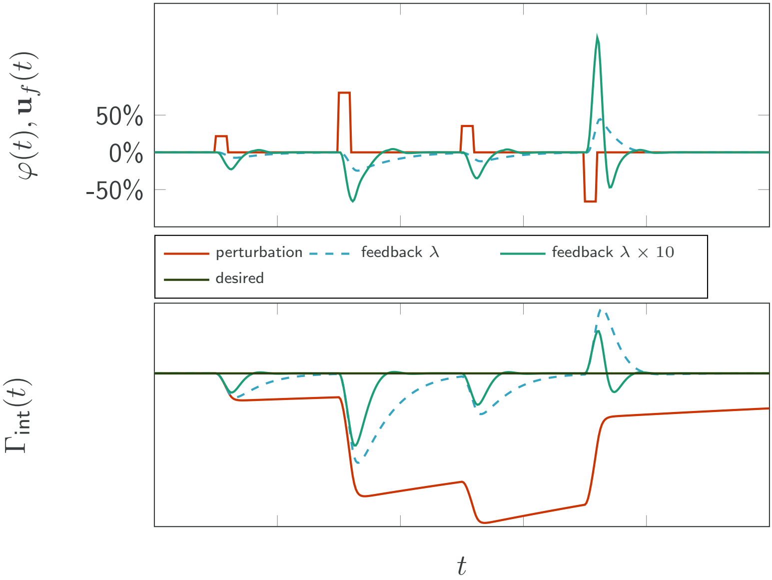
perturbation



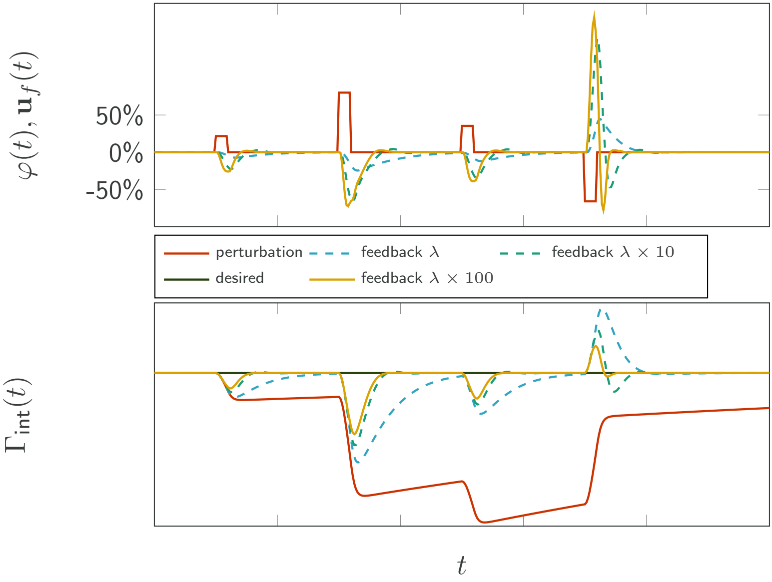
$\varphi(t)$
50%
0%
-50%
perturbation
desired
$\Gamma_{\text{int}}(t)$
t



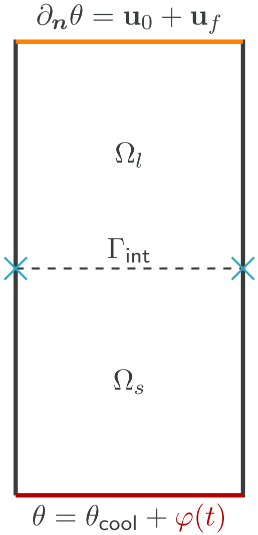
$\varphi(t), \mathbf{u}_f(t)$
50%
0%
-50%
perturbation
feedback λ
desired
$\Gamma_{\text{int}}(t)$
t



$\varphi(t), \mathbf{u}_f(t)$
50%
0%
-50%
perturbation
feedback λ
feedback $\lambda \times 10$
desired
$\Gamma_{\text{int}}(t)$
t



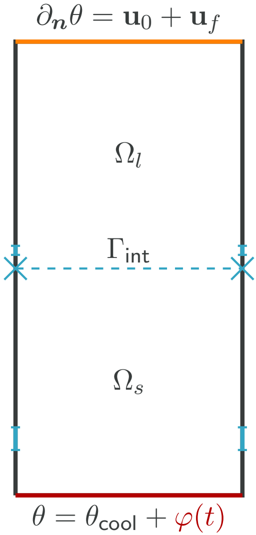
$\varphi(t), \mathbf{u}_f(t)$
50%
0%
-50%
perturbation
feedback λ
feedback $\lambda \times 10$
desired
feedback $\lambda \times 100$
$\Gamma_{\text{int}}(t)$
t

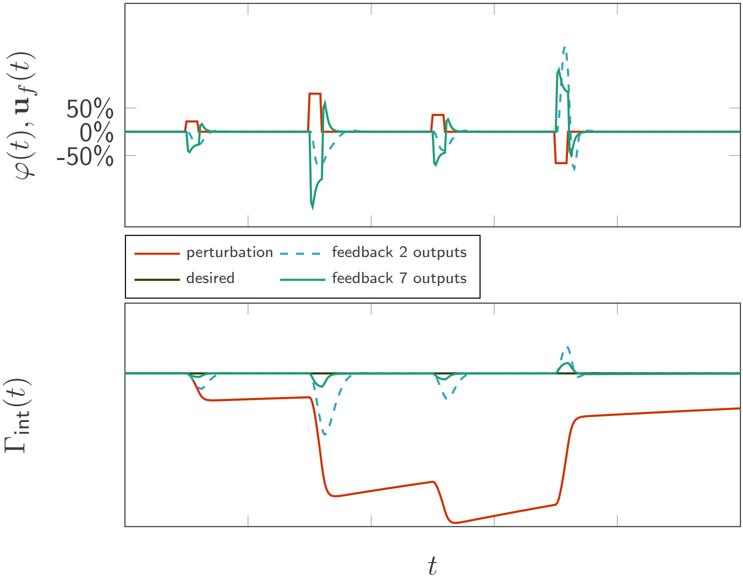


input

outputs:
temperature
measurments
+
interface
velocity

perturbation

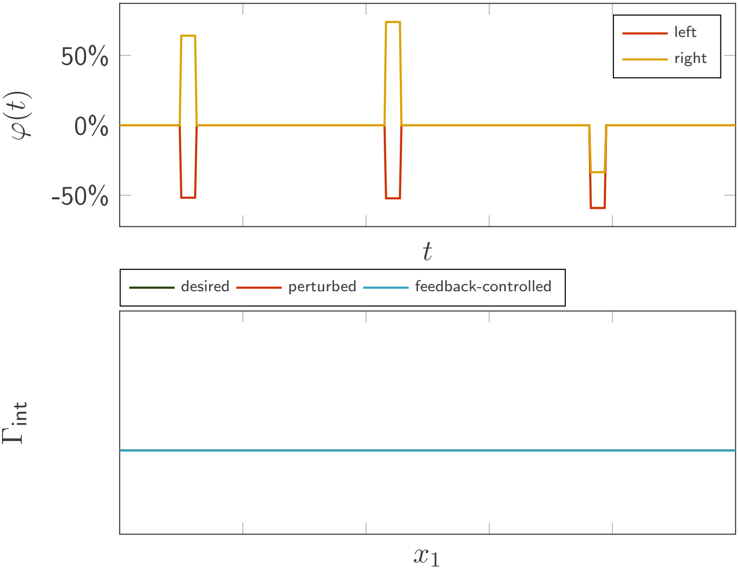


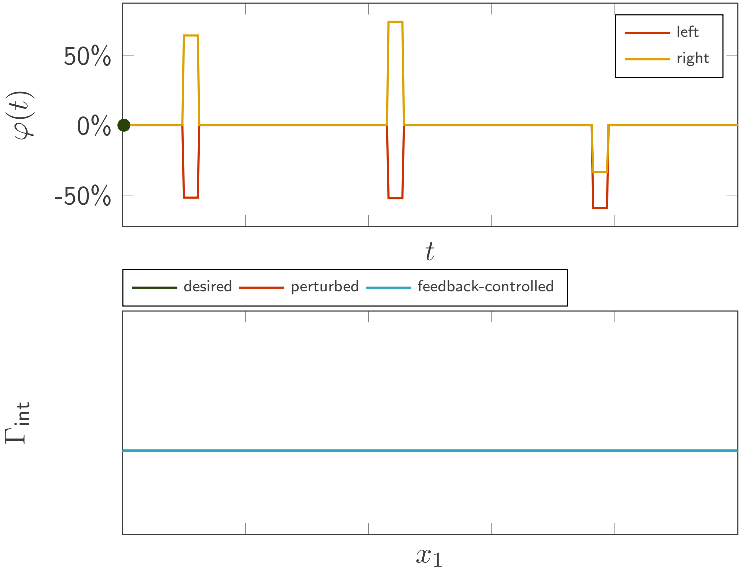


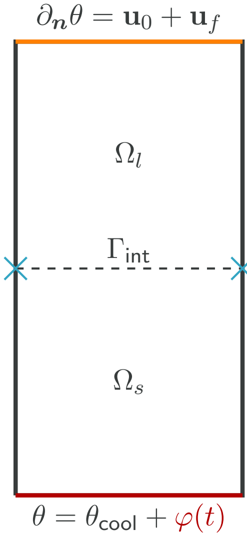
$\varphi(t), \mathbf{u}_f(t)$
50%
0%
-50%
perturbation
desired
feedback 2 outputs
feedback 7 outputs
$\Gamma_{\text{int}}(t)$
t

Numerical Examples

Test: Curvature



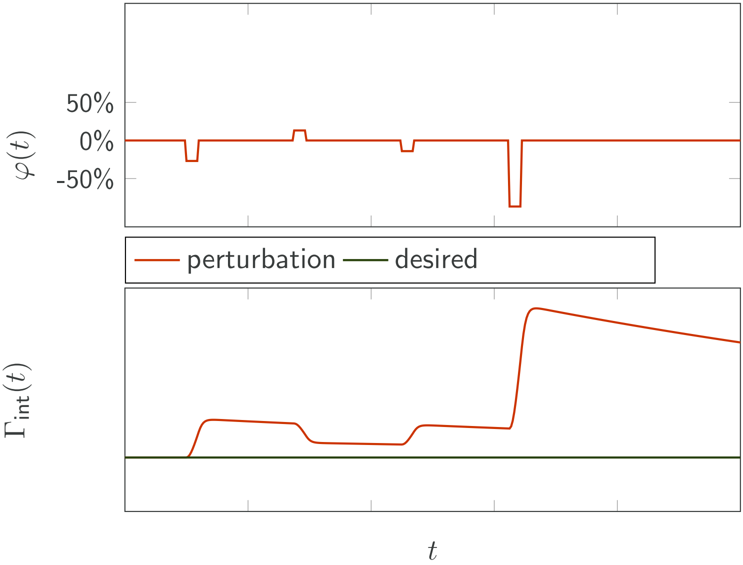

left
right
50%
0%
-50%
$\varphi(t)$
t
desired
perturbed
feedback-controlled
Γ_{int}
x_1



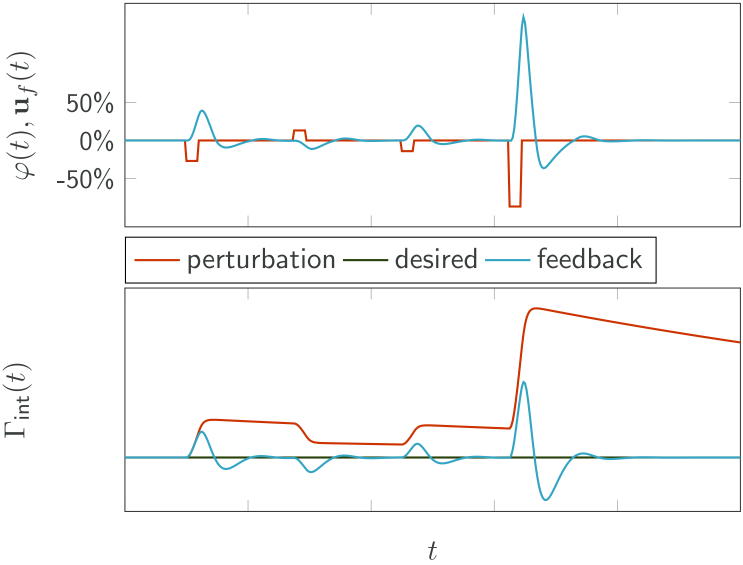
input

outputs:
temperature
measurments

perturbation



50%
0%
-50%
$\varphi(t)$
perturbation
desired
$\Gamma_{\text{int}}(t)$
t



$\varphi(t), \mathbf{u}_f(t)$
50%
0%
-50%
perturbation
desired
feedback
$\Gamma_{\text{int}}(t)$
t

Conclusions

Presented

- Riccati-feedback stabilization of the Stefan problem comes with challenges:
  - Non-linear problem,
  - Time-dependent matrices,
  - Non-autonomous DRE,
  - Differential algebraic structure.
- The quality of the feedback depends on the discretization and cost functional.

Conclusions

Presented

- Riccati-feedback stabilization of the Stefan problem comes with challenges:
  - Non-linear problem,
  - Time-dependent matrices,
  - Non-autonomous DRE,
  - Differential algebraic structure.
- The quality of the feedback depends on the discretization and cost functional.

Outlook

- Further test feedback performance.
- Implement more sophisticated time integrator.
- Couple with Navier-Stokes equations in the liquid phase.

Happy Birthday!

Happy Birthday!

. . . et Santé !

References I

Baran, B. (2016).
Optimal control of a Stefan problem with gradient-based methods in FEniCS.
Master's thesis, Otto-von-Guericke-Universität, Magdeburg, Germany.

Baran, B., Benner, P., Heiland, J., and Saak, J. (2018).
Optimal control of a Stefan problem fully coupled with incompressible Navier–Stokes equations and mesh movement.
Analele Stiintifice ale Universitatii Ovidius Constanta: Seria Matematica, XXVI(2):11–40.

Behr, M., Benner, P., and Heiland, J. (2018).
Solution formulas for differential Sylvester and Lyapunov equations.
Calcolo, to appear. (arXiv 1811.08327).

Benner, P., Saak, J., and Uddin, M. M. (2016).
Structure preserving model order reduction of large sparse second-order index-1 systems and application to a mechatronics model.
Math. Comput. Model. Dyn. Syst., 22(6):509–523.

Gugercin, S., Stykel, T., and Wyatt, S. (2013).
Model reduction of descriptor systems by interpolatory projection methods.
SIAM J. Sci. Comput., 35(5):B1010–B1033.

Kirsten, G. and Simoncini, V. (2019).
Order reduction methods for solving large-scale differential matrix riccati equations.
e-print 1905.12119, arXiv.
math.NA.

Koskela, A. and Mena, H. (2017).
Analysis of krylov subspace approximation to large scale differential riccati equations.
e-print 1705.07507, arXiv.
math.NA.

References II

Lang, N. (2017).
Numerical Methods for Large-Scale Linear Time-Varying Control Systems and related Differential Matrix Equations.
Dissertation, Germany.
Logos-Verlag, Berlin, ISBN 978-3-8325-4700-4.

Mena, H. (2007).
Numerical Solution of Differential Riccati Equations Arising in Optimal Control of Partial Differential Equations.
PhD thesis, Escuela Politécnica Nacional, Quito, Ecuador.

Mena, H., Ostermann, A., Pfurtscheller, L.-M., and Piazzola, C. (2018).
Numerical low-rank approximation of matrix differential equations.
Journal of Computational and Applied Mathematics, 340:602 – 614.

Ostermann, A., Piazzola, C., and Walach, H. (2018).
Convergence of a low-rank Lie–Trotter splitting for stiff matrix differential equations.
e-print 1803.10473, arXiv.
math.NA.

Saak, J., Köhler, M., and Benner, P. (2016).
M-M.E.S.S.-1.0.1 – the matrix equations sparse solvers library.
DOI:10.5281/zenodo.50575.
see also:www.mpi-magdeburg.mpg.de/projects/mess.

Sontag, E. D. (1998).
Mathematical Control Theory.
Springer-Verlag, New York, NY, 2nd edition.

Stillfjord, T. (2015).
Splitting schemes for nonlinear parabolic problems.
PhD thesis, Lund University.

References III

Stillfjord, T. (2018).
Singular value decay of operator-valued differential Lyapunov and Riccati equations.
SIAM J. Control Optim., 56:3598–3618.

Stefan Blocks

$$\begin{bmatrix} & & \\ & \mathcal{M}(t) & \\ & & \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \\ \mathbf{x}(t) \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & \mathcal{A}(t) & \\ & & \end{bmatrix} \begin{bmatrix} \\ \mathbf{x}(t) \\ \\ \end{bmatrix} + \mathcal{B}(t)\mathbf{u}(t),$$

$$\mathbf{y} = \begin{bmatrix} & \mathcal{C}(t) & \end{bmatrix} \begin{bmatrix} \\ \mathbf{x}(t) \\ \\ \end{bmatrix}.$$

Stefan Blocks

$$\begin{bmatrix} \mathcal{M}_v & 0 & 0 & 0 \\ 0 & \mathcal{M}_\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ \theta \\ V_{\text{mesh}} \\ p \end{bmatrix} = \begin{bmatrix} A_{vv} & 0 & A_{vV} & A_{vp} \\ A_{\theta v} & A_{\theta\theta} & A_{\theta V} & 0 \\ 0 & A_{V\theta} & A_{VV} & 0 \\ A_{pv} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \theta \\ V_{\text{mesh}} \\ p \end{bmatrix} + \mathcal{B}(t)\mathbf{u}(t),$$

$$\mathbf{y} = \begin{bmatrix} 0 & \mathcal{C}_\theta & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \theta \\ V_{\text{mesh}} \\ p \end{bmatrix}.$$