MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

On the Solution of the Nonsymmetric T-Riccati Equation

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Outline

Problem: Find $X \in \mathbb{R}^{n\times n}$ such that

$$0 = \mathcal{R}_T(X) := \; DX + X^TA - X^TBX + C, \qquad A, B, C, D \in \mathbb{R}^{n\times n}.$$

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T-Riccati equations arise in

- solving large-scale Dynamic Stochastic General Equilibrium (DSGE) models [Binder/Pesaran 1997, Sims 2001, Schmitt-Grohé/Uribe 2004];
- designing H_∞ controllers for descriptor systems (with additional constraints not considered here);
- special cases appear in Hamiltonian dynamics, queuing theory, etc.
- ...

The T-Riccati Equation

We consider the "nonnegative matrix" setting, i.e., we look for nonnegative solutions to the T-Riccati equation, similar to settings often considered for nonsymmetric classical algebraic Riccati equations, cf., e.g., [Guo 2001, Bini/Iannazzo/Meini (SIAM) 2012, B./Kürschner/Saak 2016].

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Assumptions

- B is nonnegative, $B \geq 0$.
- C is nonpositive, $C \leq 0$.
- $I \otimes D + (A^T \otimes I)\Pi$ is a nonsingular M-matrix, where $\Pi \in \mathbb{R}^{n^2 \times n^2}$ is the permutation matrix given by $\Pi := \sum_{i,j=1}^{n} e_i e_j^T \otimes e_j e_i^T$.

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Corollary

The T-Sylvester operator

$$\mathcal{S}_T : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}, \quad \mathcal{S}_T(X) := DX + X^T A,$$

has a nonnegative inverse, i.e., $\mathcal{S}_T^{-1}(X) \geq 0$ for $X \geq 0$.

As a consequence, the T-Sylvester equation $\mathcal{S}_T(X) + C = 0$ has a unique solution which is nonnegative.

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Want: minimal nonnegative solution to T-Riccati equation

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Consider fixed-point iteration

$$\begin{aligned} X_0 &= 0, \\ \text{solve} \;\; DX_{k+1} + X_{k+1}^T A &= X_k^T B X_k - C \;\; \text{for } X_{k+1}, \quad k \geq 0. \end{aligned} \tag{1}$$

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Theorem ([B./Palitta 2019])

In addition to given assumptions, suppose $\exists\, Y \geq 0$: $\mathcal{R}_T(Y) \geq 0$. *Then,*

(i) *the iterates computed by the fixed-point iteration (1) form an increasing sequence, bounded from above by* Y*:*

$$Y \geq X_{k+1} \geq X_k \quad \text{for all } k \geq 0;$$

(ii) $\{X_k\}_{k \geq 0}$ *converges (from below) to the minimal nonnegative solution* $X_{\min}$ *of the T-Riccati equation.*

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- Total complexity depends on choice of T-Sylvester solver:
  - Small-scale, dense case: solver based on generalized Schur decomposition of (A, D^T) [De Terán/Dopico 2011].
  - Large-scale, sparse/low-rank case: (extended) block-Krylov subspace-type solver [Dopico/González/Kressner/Simoncini 2016].

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- Newton's method:

$$\mathcal{R}'_T[X](X_{k+1} - X_k) = -\mathcal{R}_T(X_k),$$

where $\mathcal{R}'_T[X]$ denotes the Fréchet derivative of $\mathcal{R}_T$ at X:

$$\begin{aligned}\mathcal{R}'_T[X](Y) &= DY + Y^TA - Y^TBX - X^TBY \\ &= (D - X^TB)Y + Y^T(A - BX).\end{aligned}$$

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- The resulting method is analogous to the Newton-Kleinman method for standard algebraic Riccati equations.

Newton-Kleinman iteration for T-Riccati equations:

$$X_0 = 0; \quad \text{for } k \geq 0 \text{ solve T-Sylvester equation}$$

$$(D - X_k^T B)X_{k+1} + X_{k+1}^T(A - BX_k) = -X_k^T BX_k - C.$$

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Theorem ([B./Palitta 2019])

Assume $\exists\, \overline{Y}$: $\mathcal{R}_T(\overline{Y}) > 0$, *then*

(i) $I \otimes (D - X_{\min}^T B) + ((A - BX_{\min})^T \otimes I)\Pi$ *is a nonsingular M-matrix.*

(ii) *The sequence* $\{X_k\}_{k\geq 0}$ *computed by the Newton-Kleinman iteration is well-defined since* $I \otimes (D - X_k^T B) + ((A - BX_k)^T \otimes I)\Pi$ *is a nonsingular M-matrix for all* $k \geq 0$.

(iii) $X_k \leq X_{k+1} \leq X_{\min}$ *for any* $k \geq 0$.

(iv) $\lim_{k\to\infty} X_k = X_{\min}$.

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Conjecture: $\mathcal{R}_T(\overline{Y}) > 0$ can be replaced by $\mathcal{R}_T(\overline{Y}) \geq 0$.

Newton-Kleinman iteration for T-Riccati equations:

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Implementation details:

- Need to solve a T-Sylvester equation in each step — we use generalized Schur approach from [De Terán/Dopico 2011].
- Local convergence is quadratic in all experiments, as expected.
- Initial convergence can be accelerated by exact line search as suggested in [B./Byers 1998] for continuous-time symmetric algebraic Riccati equations:
  - use Newton direction $S_k := X_{k+1} - X_k$ as a descent direction;
  - minimize $\|\mathcal{R}_T(X_k + \lambda_k S_k)\|_F$ — optimizer $\lambda_k^{\mathrm{opt}} \in [0, 2]$ can be computed analytically;
  - set $X_{k+1} := X_k + \lambda_k^{\mathrm{opt}} S_k$.
- Stop when $\|\mathcal{R}_T(X_{k+1})\|_F \leq \varepsilon \|C\|_F$ for user-specified tolerance $\varepsilon > 0$.

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- Here, we consider large-scale T-Riccati equations

$$0 \;=\; DX + X^TA - X^TBX + C.$$

with A, D sparse and B, C of low rank, i.e.,
  - $B = B_1B_2^T, \quad B_1, B_2 \in \mathbb{R}^{n\times p}$ with $p \ll n$,
  - $C = C_1C_2^T, \quad C_1, C_2 \in \mathbb{R}^{n\times q}$ with $q \ll n$,

so that $B, -C$ are nonnegative.

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so that $B, -C$ are nonnegative.

- The solution of large-scale T-Sylvester equation can then be approximated in low-rank format by (extended) block-Krylov subspace-type solver [Dopico/González/Kressner/Simoncini 2016].

Inexact Newton-Kleinman(-Krylov) method:

- T-Sylvester equation is solved using (extended) block-Krylov method only up to a residual

$$L_{k+1} := C_1^T C_2 + X_k^T B_1 B_2^T X_k - (D - X_k^T B_1 B_2^T)\widetilde{X}_{k+1} - \widetilde{X}_{k+1}^T (A - B_1 B_2^T X_k).$$

Note: L_{k+1} can be computed in low-rank format!

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- Accuracy parameter $0 < \eta_k < 1$ for T-Sylvester equation is chosen to achieve at least superlinear convergence:

  $$\|L_{k+1}\|_F \leq \eta_k \|\mathcal{R}_T(X_k)\|_F.$$

  Typical choice, also used here: $\eta_k = 1/(1 + k^3)$.

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- To ensure convergence, enforce sufficient decrease condition (Armijo rule) by line search

$$\|\mathcal{R}_T(X_k + \lambda_k S_k)\|_F \leq (1 - \lambda_k \alpha)\|\mathcal{R}_T(X_k)\|_F, \quad \alpha > 0.$$

Theorem ([B./Palitta 2019])

Under the assumptions for convergence of the exact Newton-Kleinman iteration, suppose that furthermore, for all $k \geq 0$, $\exists\, \widetilde{X}_{k+1}$ *satisfying*

$$(D - X_k^T B_1 B_2^T)\widetilde{X}_{k+1} + \widetilde{X}_{k+1}^T (A - B_1 B_2^T X_k) = -X_k^T B_1 B_2^T X_k - C_1^T C_2 + L_{k+1}$$

where $\|L_{k+1}\|_F \leq \eta_k \|\mathcal{R}_T(X_k)\|_F$. *Then:*

(i) *If the step size parameters* λ_k *are bounded away from zero, i.e.,* $\lambda_k \geq \lambda_{\min} > 0$ *for all* k, *then* $\|\mathcal{R}_T(X_k)\|_F \to 0$.

(ii) *If, in addition to (i), the matrices* L_{k+1} *are nonnegative for all* $k \geq 0$, *then the sequence* $\{X_k\}_{k\geq 0}$ *generated by the inexact Newton-Kleinman method with* $X_0 = 0$ *is well-defined and* $X_k \leq X_{k+1} \leq X_{\min}$. *Moreover,*

$$\lim_{k\to\infty} X_k = X_{\min}.$$

Outline

- We use the same coefficient matrices as in "Numerical test 7.1" from [Dopico et al. 2016]: $D, A \in \mathbb{R}^{n\times n}$ represent the finite difference discretization on the unit square of the 2-dimensional differential operators

$$\begin{aligned} \mathcal{L}_D(u) &= -u_{xx} - u_{yy} + y(1-x)u_x + 10^4 u, \\ \mathcal{L}_A(u) &= -u_{xx} - u_{yy} = -\Delta u \end{aligned}$$

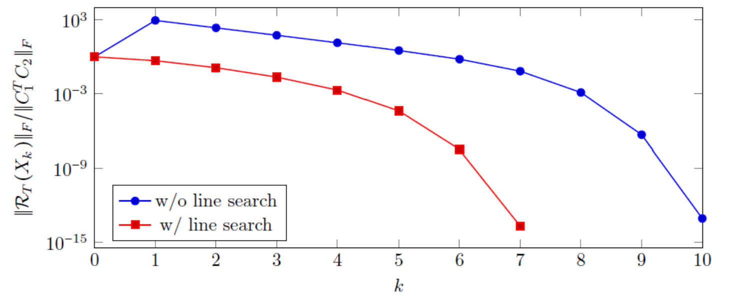
  with homogeneous Dirichlet boundary conditions.
- For small-scale tests, $B, C \in \mathbb{R}^{n\times n}$ are full random matrices.
- For large-scale tests, we consider low-rank matrices
  - $B = B_1 B_2^T$, $B_1, B_2 \in \mathbb{R}^{n\times p}$,
  - $C = C_1 C_2^T$, $C_1, C_2 \in \mathbb{R}^{n\times q}$,

  such that B_i, C_i have unit norm and random entries for $i = 1, 2$.

Results for small-scale / "exact" Newton-Kleinman

	n	Its	Rel. Res	Time (secs)
w/o line search	324	8	9.0e-15	11.28
w/ line search		5	1.1e-14	7.54
w/o line search	784	10	7.5e-14	99.94
w/ line search		7	2.4e-14	73.73

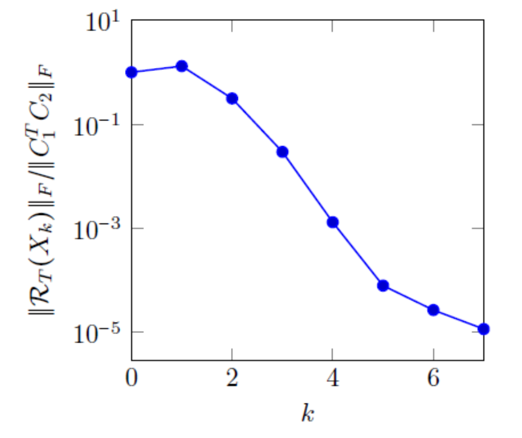
Results for small-scale ($n = 784$) / "exact" Newton-Kleinman



Results for large-scale / "inexact" Newton-Kleinman

n	p	q	Its (inner)	Mem.	Rank(X)	Rel. Res.	Time (s)
10,000	1	1	13 (7.46)	192	24	6.7e-7	15.65
	1	5	8 (8.5)	672	93	6.3e-7	52.15
	5	10	6 (6.3)	1560	213	4.5e-7	110.12
22,500	1	1	14 (9.86)	256	30	5.9e-7	69.19
	1	5	no convergence				
	5	10	no convergence				
32,400	1	1	10 (10.6)	384	26	6.8e-7	127.61
	1	5	no convergence				
	5	10	no convergence				

Results for large-scale / "inexact" Newton-Kleinman



Similar to Example 6.1 in [Guo 2001], define

$$
\begin{aligned}
R &= \texttt{rand}(2n, 2n) \in \mathbb{R}^{2n\times 2n},\\
W &= \mathrm{diag}(R\mathbf{1}) - R, \text{ where } \mathbf{1} = (1,\dots,1)^T \in \mathbb{R}^{2n}.
\end{aligned}
$$

$A, D \in \mathbb{R}^{n\times n}$ are chosen according to the partition

$$
W = \begin{bmatrix} D & M \\ N & A \end{bmatrix}, \quad \text{while } B = -N/\|N\|_F.
$$

Then define $X_{\texttt{exact}}$ with uniformly distributed random entries and unit norm, and compute

$$
C := DX_{\texttt{exact}} + X_{\texttt{exact}}^T A - X_{\texttt{exact}}^T B X_{\texttt{exact}}.
$$

Results for small-scale / "exact" Newton-Kleinman

	n	Its	Rel. Res.	Rel. Err.	Time (s)
w/o line search	500	3	1.0e-14	1.1e-10	10.80
w/ line search		3	1.1e-14	7.6e-11	10.84
w/o line search	1,000	4	1.5e-14	2.2e-9	78.59
w/ line search		4	1.5e-14	4.2e-9	78.99

Here, line search brings no advantage.

- Compute two sparse nonnegative matrices $F, G \in \mathbb{R}^{n\times n}$ with random entries.
- Define

$$\begin{aligned} D &:= F + (\rho(F) + 1)I, \\ A &:= G + (\rho(G) + 20)I. \end{aligned}$$

- Construct low-rank matrices
  - $B = B_1 B_2^T,\ B_1, B_2 \in \mathbb{R}^{n\times p}$,
  - $C = C_1 C_2^T,\ C_1, C_2 \in \mathbb{R}^{n\times q}$

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Results for large-scale / "inexact" Newton-Kleinman

n	p	q	Its (inner)	Mem.	Rank(X)	Rel. Res.	Time (s)
10,000	1	1	4 (1.5)	32	10	6.1e-7	0.16
	1	5	5 (2.2)	192	36	5.5e-9	1.11
	5	10	5 (2)	360	76	4.9e-9	3.27
50,000	1	1	4 (1.5)	32	6	9.7e-8	0.79
	1	5	5 (2)	192	36	5.5e-9	5.44
	5	10	5 (2)	360	76	3.8e-9	14.88
100,000	1	1	4 (1.5)	32	6	5.8e-8	1.48
	1	5	5 (2.2)	192	36	5.5e-9	11.33
	5	10	5 (2)	360	76	3.9e-9	24.49

Numerical results confirm linear complexity of inexact Newton-Kleinman-Krylov solver!

Summary & Outlook

- We have established sufficient conditions for the existence and uniqueness of a minimal nonnegative solution of T-Riccati equations.
- The minimum nonnegative solution can be computed by a Newton(-Kleinman) method.
- Line search can accelerate the convergence.
- In the large scale setting, low-rank approximate solutions can be computed by inexact Newton-Kleinman, where line search guarantees convergence as long as T-Sylvester equations can be solved in the Newton steps.

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- In the large scale setting, low-rank approximate solutions can be computed by inexact Newton-Kleinman, where line search guarantees convergence as long as T-Sylvester equations can be solved in the Newton steps.
- Convergence analysis of (extended) Krylov-type solvers for T-Sylvester equations?
- Alternative solvers for T-Sylvester equations, e.g. of ADI-type, possible?
- Projection techniques working directly on the T-Riccati equations?
- Other (than nonnegative matrix) settings for T-Riccati equations?

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