MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Realization-independent Reduced-order Modeling of (Second-order) Dynamical Systems

Peter Benner

Joint work with Pawan K. Goyal and Igor Pontes Duff

5th International Workshop
Reduced Basis, POD and PGD Model Reduction Techniques
Paris, 20–22 November 2019

Supported by:

BiG max MAX PLANCK RESEARCH NETWORK

DFG

DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Table of Contents

1. Introduction
2. Model Reduction of Second-Order Systems
3. Data-driven Identification
4. Numerical Results
5. General Second-order Systems
6. Outlook

Second-order dynamical systems

A **second-order (SO) dynamical systems** is a control system of the form:

$$\Sigma_{\mathrm{SO}} := \begin{cases} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases}$$

where

- $\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{p\times m}$ are constant matrices,
- (generalized) states $\mathbf{x}(t) \in \mathbb{R}^n$,
- inputs (controls) $\mathbf{u}(t) \in \mathbb{R}^m$,
- outputs (measurements, quantities of interest) $\mathbf{y}(t) \in \mathbb{R}^q$.

We assume homogeneous initial conditions $\mathbf{x}(0) = \dot{\mathbf{x}}(0) = 0$.

Introduction

–Large-scale systems–

rigid behaviour or experimental/simulation data

Physical system

Continuum mechanics, structure dynamics

Differential Equations (ODEs) or **Frequency Response Data**

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned}$$

$$\{i\omega_k, \mathbf{G}(i\omega_k)\} \; k = 1, 2, \ldots$$

Partial Differential Equations (PDEs)

$$\overset{\rightrightarrows}{\nabla} \cdot \sigma = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\varepsilon = \frac{1}{2}\left(\nabla s + \nabla^T s\right)$$

$$\sigma = \lambda \, \mathrm{tr}\,(\varepsilon)\, \mathbf{I} + 2\mu\varepsilon$$

Introduction

–Large-scale systems–

Physical system

rigid behaviour or experimental/simulation data

Continuum mechanics, structure dynamics

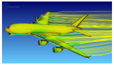
Differential Equations (ODEs) or **Frequency Response Data**

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$\{i\omega_k, \mathbf{G}(i\omega_k)\} \quad k = 1, 2, \ldots$$

Semidiscretisation

finite elements, finite differences, etc.



Partial Differential Equations (PDEs)

$$\vec{\nabla} \cdot \sigma = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\varepsilon = \frac{1}{2}\left(\nabla s + \nabla^T s\right)$$

$$\sigma = \lambda \operatorname{tr}(\varepsilon)\, \mathbf{I} + 2\mu\varepsilon$$

Introduction

–Large-scale systems–

rigid behaviour or experimental/simulation data

Physical system

Continuum mechanics, structure dynamics

Differential Equations (ODEs)
or **Frequency Response Data**

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

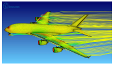
$$\{i\omega_k, \mathbf{G}(i\omega_k)\} \;\; k = 1, 2, \ldots$$

simulation,
control,
analysis,
optimisation, etc.

Large-scale

Semidiscretisation

finite elements,
finite differences, etc.



Partial Differential Equations (PDEs)

$$\vec{\nabla} \cdot \sigma = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\varepsilon = \frac{1}{2}\left(\nabla s + \nabla^T s\right)$$

$$\sigma = \lambda \operatorname{tr}(\varepsilon)\, \mathbf{I} + 2\mu\varepsilon$$

Introduction

–Large-scale systems–

Physical system

rigid behaviour or experimental/simulation data

Continuum mechanics, structure dynamics

Partial Differential Equations (PDEs)

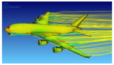
$$\vec{\nabla} \cdot \sigma = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\varepsilon = \frac{1}{2}\left(\nabla s + \nabla^T s\right)$$

$$\sigma = \lambda \,\mathrm{tr}\,(\varepsilon)\,\mathbf{I} + 2\mu\varepsilon$$

Semidiscretisation

finite elements, finite differences, etc.



Differential Equations (ODEs) or **Frequency Response Data**

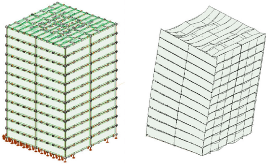
$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$\{i\omega_k, \mathbf{G}(i\omega_k)\}\ k = 1, 2, \ldots$$

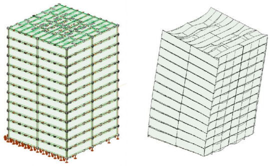
~~simulation, control, analysis, optimisation, etc.~~

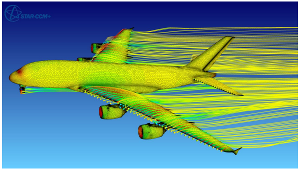
Large-scale

- Need of reduced-order models
- objective: alleviate numerical burden

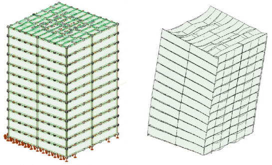


- Building model, SLICOT library,
  [Chahlaoui/Van Dooren '05]

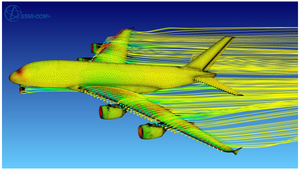




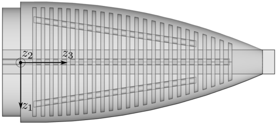
- Building model, SLICOT library, [Chahlaoui/Van Dooren '05]
- Flexible airplane [Poussot-Vassal et al. '15]



- Building model, SLICOT library, [Chahlaoui/Van Dooren '05]

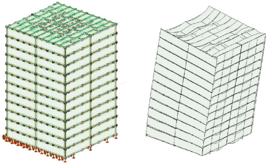


- Flexible airplane [Poussot-Vassal et al. '15]

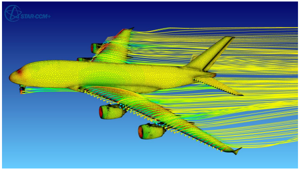


- Artificial Fishtail robot [Siebelts et al. '18]

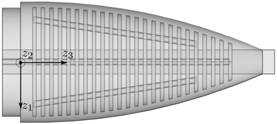
Introduction
–Motivating examples–



- Building model, SLICOT library, [Chahlaoui/Van Dooren '05]



- Flexible airplane [Poussot-Vassal et al. '15]



- Artificial Fishtail robot [Siebelts et al. '18]

- Bridge design

Second-order systems

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \dot{\mathbf{x}}(0) = 0,$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

Frequency representation or transfer function

Transform the system in the frequency domain (Laplace transform)

$$\mathbf{x}(t) \mapsto \mathbf{X}(s), \quad \dot{\mathbf{x}}(t) \mapsto s\mathbf{X}(s), \quad \ddot{\mathbf{x}}(t) \mapsto s^2\mathbf{X}(s)$$

Second-order systems

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \dot{\mathbf{x}}(0) = 0,$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

Frequency representation or transfer function

Transform the system in the frequency domain (Laplace transform)

$$\mathbf{x}(t) \mapsto \mathbf{X}(s), \quad \dot{\mathbf{x}}(t) \mapsto s\mathbf{X}(s), \quad \ddot{\mathbf{x}}(t) \mapsto s^2\mathbf{X}(s)$$

As a result, we have

$$\begin{aligned} s^2\mathbf{M}\mathbf{X}(s) + s\mathbf{D}\mathbf{X}(s) + \mathbf{K}\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s), \\ \mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s), \end{aligned}$$

Second-order systems

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \dot{\mathbf{x}}(0) = 0,$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

Frequency representation or transfer function

Transform the system in the frequency domain (Laplace transform)

$$\mathbf{x}(t) \mapsto \mathbf{X}(s), \quad \dot{\mathbf{x}}(t) \mapsto s\mathbf{X}(s), \quad \ddot{\mathbf{x}}(t) \mapsto s^2\mathbf{X}(s)$$

As a result, we have

$$s^2\mathbf{M}\mathbf{X}(s) + s\mathbf{D}\mathbf{X}(s) + \mathbf{K}\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s),$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s),$$

yielding the I/O map

$$\boxed{\mathbf{Y}(s) = \Big(\underbrace{\mathbf{C}\left(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K}\right)^{-1}\mathbf{B}}_{\mathbf{H}_{\mathsf{SO}}(s)}\Big)\mathbf{U}(s).}$$

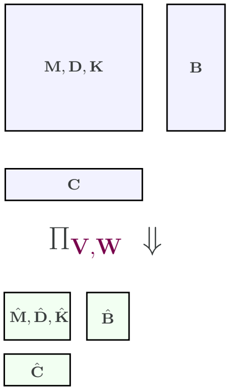
$\mathbf{H}_{\mathsf{SO}}(s)$ is the **transfer function** of the SO system.

Model Reduction of Second-Order Systems

–Projection-based framework–

Given a large-scale **SO** of order $n \gg 1$

$$\mathbf{H}_{\mathsf{SO}}(s) = \mathbf{C}\left(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K}\right)^{-1}\mathbf{B},$$



Given a large-scale **SO** of order $n \gg 1$

$$\mathbf{H}_{\mathsf{SO}}(s) = \mathbf{C}\left(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K}\right)^{-1}\mathbf{B},$$

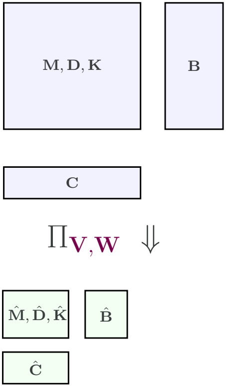
find projection matrices

$$\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n\times r},$$

(with $r \ll n$), such that

$$\hat{\mathbf{H}}_{\mathsf{SO}}(s) = \hat{\mathbf{C}}\left(s^2\hat{\mathbf{M}} + s\hat{\mathbf{D}} + \hat{\mathbf{K}}\right)^{-1}\hat{\mathbf{B}}, \text{ where}$$

$$\boxed{\begin{aligned}&\hat{\mathbf{M}} = \mathbf{W}^T\mathbf{M}\mathbf{V},\ \ \hat{\mathbf{D}} = \mathbf{W}^T\mathbf{D}\mathbf{V},\ \ \hat{\mathbf{K}} = \mathbf{W}^T\mathbf{K}\mathbf{V},\\ &\hat{\mathbf{B}} = \mathbf{W}^T\mathbf{B},\ \text{ and }\ \hat{\mathbf{C}} = \mathbf{C}\mathbf{V}.\end{aligned}}$$



Given a large-scale **SO** of order $n \gg 1$

$$\mathbf{H}_{\mathsf{SO}}(s) = \mathbf{C}\left(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K}\right)^{-1}\mathbf{B},$$

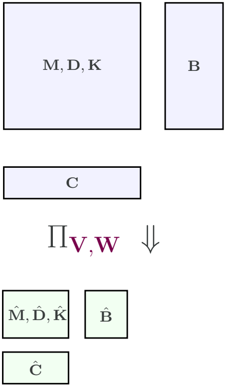
find projection matrices

$$\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n\times r},$$

(with $r \ll n$), such that

$$\hat{\mathbf{H}}_{\mathsf{SO}}(s) = \hat{\mathbf{C}}\left(s^2\hat{\mathbf{M}} + s\hat{\mathbf{D}} + \hat{\mathbf{K}}\right)^{-1}\hat{\mathbf{B}}, \text{ where}$$

$$\boxed{\begin{aligned}&\hat{\mathbf{M}} = \mathbf{W}^T\mathbf{M}\mathbf{V},\ \ \hat{\mathbf{D}} = \mathbf{W}^T\mathbf{D}\mathbf{V},\ \ \hat{\mathbf{K}} = \mathbf{W}^T\mathbf{K}\mathbf{V},\\ &\hat{\mathbf{B}} = \mathbf{W}^T\mathbf{B},\ \text{ and }\ \hat{\mathbf{C}} = \mathbf{C}\mathbf{V}.\end{aligned}}$$



- $\|\mathbf{H}_{\mathsf{SO}}(\cdot) - \hat{\mathbf{H}}_{\mathsf{SO}}(\cdot)\| \ll 1$, for some norm $(\mathcal{H}_2, \mathcal{H}_\infty, \ldots)$.
- Note that

$$\|\mathbf{y} - \hat{\mathbf{y}}\|_{L_\infty} \leq \|\mathbf{H}_{\mathsf{SO}}(\cdot) - \hat{\mathbf{H}}_{\mathsf{SO}}(\cdot)\|_{\mathcal{H}_2}\|\mathbf{u}\|_{L_2},$$
$$\|\mathbf{y} - \hat{\mathbf{y}}\|_{L_2} \leq \|\mathbf{H}_{\mathsf{SO}}(\cdot) - \hat{\mathbf{H}}_{\mathsf{SO}}(\cdot)\|_{\mathcal{H}_\infty}\|\mathbf{u}\|_{L_2}.$$

Interpolation-based methods

- Interpolation-based methods for structure-preserving model reduction.
  [Beattie/Gugercin '05, Beattie/Gugercin '09, Benner/G./Pontes '19]

Interpolation-based methods

- Interpolation-based methods for structure-preserving model reduction.
  [Beattie/Gugercin '05, Beattie/Gugercin '09, Benner/G./Pontes '19]

Balancing truncation methods

- Position and velocity Gramians. [Chahlaoui et al. '06, Reis/Stykel '08]

Interpolation-based methods

- Interpolation-based methods for structure-preserving model reduction. [Beattie/Gugercin '05, Beattie/Gugercin '09, Benner/G./Pontes '19]

Balancing truncation methods

- Position and velocity Gramians. [Chahlaoui et al. '06, Reis/Stykel '08]

Modal approximation

- Dominant pole algorithm for second order systems. [Rommes/Martins '06]

Interpolation-based methods

- Interpolation-based methods for structure-preserving model reduction. [Beattie/Gugercin '05, Beattie/Gugercin '09, Benner/G./Pontes '19]

Balancing truncation methods

- Position and velocity Gramians. [Chahlaoui et al. '06, Reis/Stykel '08]

Modal approximation

- Dominant pole algorithm for second order systems. [Rommes/Martins '06]

Realization-independent methods

× Assume the realization $(\mathbf{M}, \mathbf{D}, \mathbf{K}, \mathbf{B}, \mathbf{C})$ is not available.
  Hence, cannot apply these methods.

Interpolation-based methods

- Interpolation-based methods for structure-preserving model reduction. [Beattie/Gugercin '05, Beattie/Gugercin '09, Benner/G./Pontes '19]

Balancing truncation methods

- Position and velocity Gramians. [Chahlaoui et al. '06, Reis/Stykel '08]

Modal approximation

- Dominant pole algorithm for second order systems. [Rommes/Martins '06]

Realization-independent methods

× Assume the realization $(\mathbf{M}, \mathbf{D}, \mathbf{K}, \mathbf{B}, \mathbf{C})$ is not available.
Hence, cannot apply these methods.

✓ Access to only frequency response, i.e., $\mathbf{H}(\imath\omega_j)$, $\omega_j \in \mathbb{R}$.

Interpolation-based methods

- Interpolation-bas … uction.

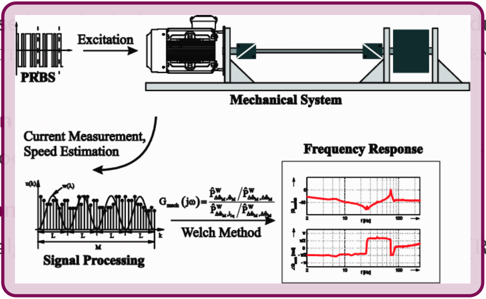
  [Beat … nner/G./Pontes '19]

Balancing truncatio…

- Position and velo … 06, Reis/Stykel '08]

Modal approximatio…

- Dominant pole a … Rommes/Martins '06]



Realization-independent methods

× Assume the realization $(\mathbf{M}, \mathbf{D}, \mathbf{K}, \mathbf{B}, \mathbf{C})$ is not available.

  Hence, cannot apply these methods.

✓ Access to only frequency response, i.e., $\mathbf{H}(\imath\omega_j)$, $\omega_j \in \mathbb{R}$.

Data-driven Identification

–Problem formulation –

Rational interpolation problem

Given shift points $\{\sigma_1, \ldots, \sigma_{2\ell}\} \subset \mathbb{C}$ and sample values $\{\gamma_1, \ldots, \gamma_{2\ell}\} \subset \mathbb{C}$, **construct** a rational function $\mathbf{H}(s) = \mathbf{C}\left(s\mathbf{E} - \mathbf{A}\right)^{-1}\mathbf{B}$, satisfying

$$\mathbf{H}(\sigma_j) = \gamma_j, \; j = 1, \ldots, 2\ell.$$

Data-driven Identification

–Problem formulation –

Rational interpolation problem

Given shift points $\{\sigma_1, \ldots, \sigma_{2\ell}\} \subset \mathbb{C}$ and sample values $\{\gamma_1, \ldots, \gamma_{2\ell}\} \subset \mathbb{C}$, **construct** a rational function $\mathbf{H}(s) = \mathbf{C}\left(s\mathbf{E} - \mathbf{A}\right)^{-1}\mathbf{B}$, satisfying

$$\mathbf{H}(\sigma_j) = \gamma_j, \; j = 1, \ldots, 2\ell.$$

- One solution approach is the **Loewner framework**! [Mayo/Antoulas '05]

Rational interpolation problem

Given shift points $\{\sigma_1, \ldots, \sigma_{2\ell}\} \subset \mathbb{C}$ and sample values $\{\gamma_1, \ldots, \gamma_{2\ell}\} \subset \mathbb{C}$, **construct** a rational function $\mathbf{H}(s) = \mathbf{C}\left(s\mathbf{E} - \mathbf{A}\right)^{-1}\mathbf{B}$, satisfying

$$\mathbf{H}(\sigma_j) = \gamma_j,\ j = 1, \ldots, 2\ell.$$

- One solution approach is the **Loewner framework**! [Mayo/Antoulas '05]

Question: can this be applied directly to **second-order systems**?

Rational interpolation problem

Given shift points $\{\sigma_1, \ldots, \sigma_{2\ell}\} \subset \mathbb{C}$ and sample values $\{\gamma_1, \ldots, \gamma_{2\ell}\} \subset \mathbb{C}$, **construct a rational function** $\mathbf{H}(s) = \mathbf{C}\left(s\mathbf{E} - \mathbf{A}\right)^{-1}\mathbf{B}$, satisfying

$$\mathbf{H}(\sigma_j) = \gamma_j,\ j = 1, \ldots, 2\ell.$$

- One solution approach is the **Loewner framework**! [Mayo/Antoulas '05]

Question: can this be applied directly to **second-order systems**?

Second-order identification problem

Given shift points $\{\sigma_1, \ldots, \sigma_{2\ell}\} \subset \mathbb{C}$ and sample values $\{\gamma_1, \ldots, \gamma_{2\ell}\} \subset \mathbb{C}$, construct a second-order system $\mathbf{H}_{\mathrm{SO}}(s) = \mathbf{C}\left(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K}\right)^{-1}\mathbf{B}$, satisfying

$$\mathbf{H}_{\mathrm{SO}}(\sigma_j) = \gamma_j,\ j = 1, \ldots, 2\ell.$$

- Let us recall the Loewner framework in the single-input single-output case.

- Let us recall the Loewner framework in the single-input single-output case.
- Data set consists of $\begin{cases} \text{shift points} & \sigma_k \in \mathbb{C}, \\ \text{sample values} & \gamma_k \in \mathbb{C}, \end{cases}$ for $k = 1, \ldots, 2\ell$.

- Let us recall the Loewner framework in the single-input single-output case.
- Data set consists of $\begin{cases} \text{shift points} & \sigma_k \in \mathbb{C}, \\ \text{sample values} & \gamma_k \in \mathbb{C}, \end{cases}$ for $k = 1, \ldots, 2\ell$.
- Partition the data into the left & right sets ($\ell + \ell = 2\ell$):

$$\{(\sigma_k, \gamma_k)\} = \{(\mu_i, \mathbf{v}_i) \cup (\lambda_i, \mathbf{w}_i)\}, \quad k = 1, \ldots, 2\ell, \quad i = 1, \ldots, \ell.$$

- Let us recall the Loewner framework in the single-input single-output case.
- Data set consists of $\begin{cases} \text{shift points} & \sigma_k \in \mathbb{C}, \\ \text{sample values} & \gamma_k \in \mathbb{C}, \end{cases}$ for $k = 1, \ldots, 2\ell$.
- Partition the data into the left & right sets ($\ell + \ell = 2\ell$):

$$\{(\sigma_k, \gamma_k)\} = \{(\mu_i, \mathbf{v}_i) \cup (\lambda_i, \mathbf{w}_i)\}, \quad k = 1, \ldots, 2\ell, \quad i = 1, \ldots, \ell.$$

Objective

Find a rational function $\mathbf{H}(s) = \mathbf{C}\,(s\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{B}$ such that

$$\mathbf{H}(\mu_j) = \mathbf{v}_j, \quad \mathbf{H}(\lambda_i) = \mathbf{w}_i.$$

- Let us recall the Loewner framework in the single-input single-output case.
- Data set consists of $\begin{cases} \text{shift points} & \sigma_k \in \mathbb{C}, \\ \text{sample values} & \gamma_k \in \mathbb{C}, \end{cases}$ for $k = 1, \ldots, 2\ell$.
- Partition the data into the left & right sets ($\ell + \ell = 2\ell$):

$$\{(\sigma_k, \gamma_k)\} = \{(\mu_i, \mathbf{v}_i) \cup (\lambda_i, \mathbf{w}_i)\}, \quad k = 1, \ldots, 2\ell, \quad i = 1, \ldots, \ell.$$

Objective

Find a rational function $\mathbf{H}(s) = \mathbf{C}\,(s\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{B}$ such that

$$\mathbf{H}(\mu_j) = \mathbf{v}_j, \quad \mathbf{H}(\lambda_i) = \mathbf{w}_i.$$

Organize the data as follows:

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_\ell), \qquad \Omega = \operatorname{diag}(\mu_1, \ldots, \mu_\ell),$$
$$\mathbf{V} = \left[\mathbf{H}(\lambda_1), \ldots, \mathbf{H}(\lambda_\ell)\right]^T, \text{ and } \mathbf{W} = \left[\mathbf{H}(\mu_1), \ldots, \mathbf{H}(\mu_\ell)\right]^T.$$

Loewner Approach (Matrix form)

- Let $\mathbb{L}$ and $\mathbb{L}_\sigma$ satisfy:

$$\boxed{\begin{aligned} -\mathbb{L}\Lambda + \mathbb{L}_\sigma &= \mathbf{V}\mathbf{1}^T, \\ -\mathbb{L}^T\Omega + {\mathbb{L}_\sigma}^T &= \mathbf{W}\mathbf{1}^T, \end{aligned}}$$

where $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_\ell)$, $\Omega = \mathrm{diag}\,(\mu_1, \ldots, \mu_\ell)$, $\mathbf{V} = \left[\mathbf{H}(\lambda_1), \ldots, \mathbf{H}(\lambda_\ell)\right]^T$, and $\mathbf{W} = \left[\mathbf{H}(\mu_1), \ldots, \mathbf{H}(\mu_\ell)\right]^T$.

- The rational function $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{C}$ interpolates the data, where

$$\boxed{\mathbf{E} = -\mathbb{L}, \quad \mathbf{A} = -\mathbb{L}_\sigma, \quad \mathbf{B} = \mathbf{V}, \quad \text{and} \quad \mathbf{C} = \mathbf{W},}$$

and the pencil $(\mathbb{L}, \mathbb{L}_\sigma)$ is regular.

Loewner Approach (Matrix form)

- Let $\mathbb{L}$ and $\mathbb{L}_\sigma$ satisfy:

$$\boxed{\begin{aligned} -\mathbb{L}\Lambda + \mathbb{L}_\sigma &= \mathbf{V}\mathbf{1}^T, \\ -\mathbb{L}^T\Omega + {\mathbb{L}_\sigma}^T &= \mathbf{W}\mathbf{1}^T, \end{aligned}}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_\ell)$, $\Omega = \operatorname{diag}(\mu_1, \ldots, \mu_\ell)$, $\mathbf{V} = \left[\mathbf{H}(\lambda_1), \ldots, \mathbf{H}(\lambda_\ell)\right]^T$, and $\mathbf{W} = \left[\mathbf{H}(\mu_1), \ldots, \mathbf{H}(\mu_\ell)\right]^T$.

- The rational function $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{C}$ interpolates the data, where

$$\boxed{\mathbf{E} = -\mathbb{L}, \quad \mathbf{A} = -\mathbb{L}_\sigma, \quad \mathbf{B} = \mathbf{V}, \quad \text{and} \quad \mathbf{C} = \mathbf{W},}$$

and the pencil $(\mathbb{L}, \mathbb{L}_\sigma)$ is regular.

Remarks

- No need to solve Sylvester equations $\Rightarrow$ matrices $\mathbb{L}$ and $\mathbb{L}_\sigma$ have an analytic expression.
- $\operatorname{rank}(\mathbb{L}) =$ order of minimal realization $= r$.

Loewner Approach (Matrix form)

- Let $\mathbb{L}$ and $\mathbb{L}_\sigma$ satisfy:

$$\boxed{\begin{aligned} -\mathbb{L}\Lambda + \mathbb{L}_\sigma &= \mathbf{V}\mathbf{1}^T, \\ -\mathbb{L}^T\Omega + \mathbb{L}_\sigma{}^T &= \mathbf{W}\mathbf{1}^T, \end{aligned}}$$

  where $\Lambda = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_\ell\right)$, $\Omega = \mathrm{diag}\left(\mu_1, \ldots, \mu_\ell\right)$, $\mathbf{V} = \left[\mathbf{H}(\lambda_1), \ldots, \mathbf{H}(\lambda_\ell)\right]^T$, and $\mathbf{W} = \left[\mathbf{H}(\mu_1), \ldots, \mathbf{H}(\mu_\ell)\right]^T$.

- The rational function $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{C}$ interpolates the data, where

$$\boxed{\mathbf{E} = -\mathbb{L}, \quad \mathbf{A} = -\mathbb{L}_\sigma, \quad \mathbf{B} = \mathbf{V}, \quad \text{and} \quad \mathbf{C} = \mathbf{W},}$$

  and the pencil $(\mathbb{L}, \mathbb{L}_\sigma)$ is regular.

Remarks

- No need to solve Sylvester equations $\Rightarrow$ matrices $\mathbb{L}$ and $\mathbb{L}_\sigma$ have an analytic expression.
- $\mathrm{rank}\,(\mathbb{L})$ = order of minimal realization $= r$.
- Compression step can be performed to obtain (approximate) minimal realization.

- Data set consists of $\begin{cases} \text{shift points} & \sigma_k \in \mathbb{C}, \\ \text{sample values} & \gamma_k \in \mathbb{C}, \end{cases}$ for $k = 1, \ldots, 2\ell.$
- Partition the data into the left & right sets $(\ell + \ell = 2\ell)$:

$$\{(\sigma_k, \gamma_k)\} = \{(\mu_i, \mathbf{v}_i) \cup (\lambda_i, \mathbf{w}_i)\}, \quad k = 1, \ldots, 2\ell, \quad i = 1, \ldots, \ell.$$

Objective (second-order case)

Find a rational function $\mathbf{H}_{\mathsf{so}}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B}$ such that

$$\mathbf{H}_{\mathsf{so}}(\mu_j) = \mathbf{v}_j, \quad \mathbf{H}_{\mathsf{so}}(\lambda_i) = \mathbf{w}_i.$$

- Data set consists of $\begin{cases} \text{shift points} & \sigma_k \in \mathbb{C}, \\ \text{sample values} & \gamma_k \in \mathbb{C}, \end{cases}$ for $k = 1, \ldots, 2\ell$.
- Partition the data into the left & right sets ($\ell + \ell = 2\ell$):

$$\{(\sigma_k, \gamma_k)\} = \{(\mu_i, \mathbf{v}_i) \cup (\lambda_i, \mathbf{w}_i)\}, \quad k = 1, \ldots, 2\ell, \quad i = 1, \ldots, \ell.$$

Objective (second-order case)

Find a rational function $\mathbf{H}_{\text{so}}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B}$ such that

$$\mathbf{H}_{\text{so}}(\mu_j) = \mathbf{v}_j, \quad \mathbf{H}_{\text{so}}(\lambda_i) = \mathbf{w}_i.$$

Once again, we organize the data as follows:

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_\ell), \qquad \Omega = \operatorname{diag}(\mu_1, \ldots, \mu_\ell),$$

$$\mathbf{V} = \left[\mathbf{H}(\lambda_1), \ldots, \mathbf{H}(\lambda_\ell)\right]^T, \text{ and } \mathbf{W} = \left[\mathbf{H}(\mu_1), \ldots, \mathbf{H}(\mu_\ell)\right]^T.$$

Identification of second-order systems

[Unger et. al '16]

- Assume the matrices $\mathbb{L}_{\mathbf{M}}$, $\mathbb{L}_{\mathbf{D}}$, and $\mathbb{L}_{\mathbf{K}}$ satisfy:

$$\boxed{\begin{aligned} \mathbb{L}_{\mathbf{M}}\Lambda^2 + \mathbb{L}_{\mathbf{D}}\Lambda + \mathbb{L}_{\mathbf{K}} &= \mathbf{V}\mathbf{1}^T, \\ \mathbb{L}_{\mathbf{M}}{}^T\Omega^2 + \mathbb{L}_{\mathbf{D}}{}^T\Lambda + \mathbb{L}_{\mathbf{K}} &= \mathbf{W}\mathbf{1}^T, \end{aligned}}$$

  where $\Lambda = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$, $\Omega = \operatorname{diag}(\mu_1, \ldots, \mu_r)$, $\mathbf{V} = \left[\mathbf{H}(\mu_1), \ldots, \mathbf{H}(\mu_r)\right]^T$, and $\mathbf{W} = \left[\mathbf{H}(\sigma_1), \ldots, \mathbf{H}(\sigma_r)\right]^T$.

- The rational function $\mathbf{H}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{C}$ interpolates the data, where

$$\boxed{\mathbf{M} = \mathbb{L}_{\mathbf{M}}, \quad \mathbf{D} = \mathbb{L}_{\mathbf{D}}, \quad \mathbf{K} = \mathbb{L}_{\mathbf{K}}, \quad \mathbf{B} = \mathbf{V}, \quad \text{and} \quad \mathbf{C} = \mathbf{W}.}$$

  .

Identification of second-order systems [Unger et. al '16]

- Assume the matrices $\mathbb{L}_\mathbf{M}$, $\mathbb{L}_\mathbf{D}$, and $\mathbb{L}_\mathbf{K}$ satisfy:

$$\boxed{\begin{aligned} \mathbb{L}_\mathbf{M}\Lambda^2 + \mathbb{L}_\mathbf{D}\Lambda + \mathbb{L}_\mathbf{K} &= \mathbf{V}\mathbf{1}^T, \\ \mathbb{L}_\mathbf{M}{}^T\Omega^2 + \mathbb{L}_\mathbf{D}{}^T\Lambda + \mathbb{L}_\mathbf{K} &= \mathbf{W}\mathbf{1}^T, \end{aligned}}$$

  where $\Lambda = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_r)$, $\Omega = \mathrm{diag}\,(\mu_1, \ldots, \mu_r)$, $\mathbf{V} = \left[\mathbf{H}(\mu_1), \ldots, \mathbf{H}(\mu_r)\right]^T$, and $\mathbf{W} = \left[\mathbf{H}(\sigma_1), \ldots, \mathbf{H}(\sigma_r)\right]^T$.

- The rational function $\mathbf{H}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{C}$ interpolates the data, where

$$\boxed{\mathbf{M} = \mathbb{L}_\mathbf{M}, \quad \mathbf{D} = \mathbb{L}_\mathbf{D}, \quad \mathbf{K} = \mathbb{L}_\mathbf{K}, \quad \mathbf{B} = \mathbf{V}, \quad \text{and} \quad \mathbf{C} = \mathbf{W}.}$$

  .

Problem:

- Number of equations $(2\ell^2)$ (much) smaller than the number of variables $(3\ell^2)$.
- Hence, infinitely many possible realizations.
- Which one should be picked?

Case 1: SO Rayleigh Damped Systems

Second-order Rayleigh damped systems

- For now, consider the case that the Rayleigh damping hypothesis holds, i.e.,

$$\mathbf{D} = \alpha \mathbf{M} + \beta \mathbf{K}, \quad \alpha, \beta \geq 0.$$

Identification problem in matrix form

Assume the matrices $\mathbb{L}_{\mathbf{M}}$, $\mathbb{L}_{\mathbf{D}}$, and $\mathbb{L}_{\mathbf{K}}$ satisfy:

$$\boxed{\begin{aligned} \mathbb{L}_{\mathbf{M}} \Lambda^2 + \mathbb{L}_{\mathbf{D}} \Lambda + \mathbb{L}_{\mathbf{K}} &= \mathbf{V} \mathbf{1}^T, \\ \mathbb{L}_{\mathbf{M}}^T \Omega^2 + \mathbb{L}_{\mathbf{D}}^T \Omega + \mathbb{L}_{\mathbf{K}}^T &= \mathbf{W} \mathbf{1}^T. \end{aligned}}$$

Remarks

- Very similar equations as in the classical linear case.
- $\mathbb{L}_{\mathbf{M}}$ and $\mathbb{L}_{\mathbf{K}}$ $\Rightarrow$ determined using (modified) Loewner and shifted Loewner matrices.
- $\operatorname{rank}(\mathbb{L}_{\mathbf{M}})$ = order of minimal realization $= r$.

Case 1: SO Rayleigh Damped Systems

Second-order Rayleigh damped systems

- For now, consider the case that the Rayleigh damping hypothesis holds, i.e.,

$$\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}, \quad \alpha, \beta \geq 0.$$

Identification problem in matrix form

Assume the matrices $\mathbb{L}_{\mathbf{M}}$, $\mathbb{L}_{\mathbf{D}}$, and $\mathbb{L}_{\mathbf{K}}$ satisfy:

$$\mathbb{L}_{\mathbf{M}}\Lambda^2 + (\alpha\mathbb{L}_{\mathbf{M}} + \beta\mathbb{L}_{\mathbf{K}})\,\Lambda + \mathbb{L}_{\mathbf{K}} = \mathbf{V}\mathbf{1}^T,$$
$$\mathbb{L}_{\mathbf{M}}{}^T\Omega^2 + (\alpha\mathbb{L}_{\mathbf{M}} + \beta\mathbb{L}_{\mathbf{K}})^T\,\Omega + \mathbb{L}_{\mathbf{K}}{}^T = \mathbf{W}\mathbf{1}^T,$$

Remarks

- Very similar equations as in the classical linear case.
- $\mathbb{L}_{\mathbf{M}}$ and $\mathbb{L}_{\mathbf{K}}$ $\Rightarrow$ determined using (modified) Loewner and shifted Loewner matrices.
- $\operatorname{rank}(\mathbb{L}_{\mathbf{M}})$ = order of minimal realization $= r$.

Case 1: SO Rayleigh Damped Systems

Second-order Rayleigh damped systems

- For now, consider the case that the Rayleigh damping hypothesis holds, i.e.,

$$\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}, \quad \alpha, \beta \geq 0.$$

Identification problem in matrix form

Assume the matrices $\mathbb{L}_\mathbf{M}$, $\mathbb{L}_\mathbf{D}$, and $\mathbb{L}_\mathbf{K}$ satisfy:

$$\boxed{\begin{aligned} \mathbb{L}_\mathbf{M} f(\Lambda) + \mathbb{L}_\mathbf{K} g(\Lambda) &= \mathbf{V}\mathbf{1}^T, \\ \mathbb{L}_\mathbf{M}{}^T f(\Omega) + \mathbb{L}_\mathbf{K}{}^T g(\Omega) &= \mathbf{W}\mathbf{1}^T, \end{aligned}}$$

where $f(\Sigma) := \Sigma^2 + \alpha\Sigma$ and $g(\Sigma) := (\mathbf{I} + \beta\Sigma)$.

Remarks

- Very similar equations as in the classical linear case.
- $\mathbb{L}_\mathbf{M}$ and $\mathbb{L}_\mathbf{K}$ $\Rightarrow$ determined using (modified) Loewner and shifted Loewner matrices.
- $\operatorname{rank}(\mathbb{L}_\mathbf{M})$ = order of minimal realization $= r$.

Demo Example

- Consider a second-order system of order $n = 2$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{B} = \mathbf{C}^T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

with $\mathbf{D} = \alpha \mathbf{M} + \beta \mathbf{K}$, where $\alpha = 0.01$ and $\beta = 0.02$.

Transfer function: $\mathbf{H}_{\mathsf{SO}}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B}$.

Demo Example

- Consider a second-order system of order $n = 2$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{B} = \mathbf{C}^T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

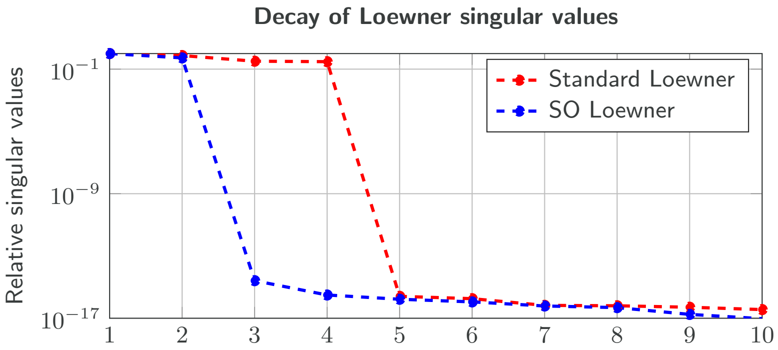
  with $\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}$, where $\alpha = 0.01$ and $\beta = 0.02$.

  Transfer function: $\mathbf{H}_{\mathsf{SO}}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B}$.

- We collect 20 samples $(\sigma_j, \mathbf{H}_{\mathsf{SO}}(\sigma_j))$, for $\sigma_j \in i[10^{-1}, 10^{1}]$ logarithmically spaced.

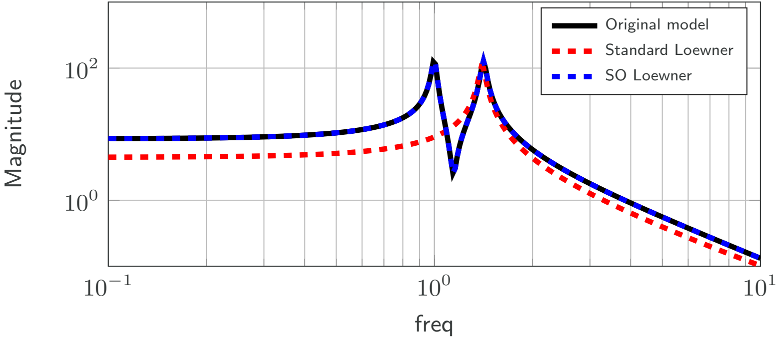
Demo Example

- Consider a second-order system of order $n = 2$.
- We collect 20 samples $(\sigma_j, \mathbf{H}_{\mathtt{SO}}(\sigma_j))$, for $\sigma_j \in i[10^{-1}, 10^{1}]$ logarithmically spaced.
- Build standard and second-order Loewner matrices $(\operatorname{rank}(\mathbb{L}) = 4$ and $\operatorname{rank}(\mathbb{L}^{\mathtt{SO}}) = 2)$.



Demo Example

- Consider a second-order system of order $n = 2$.
- We collect 20 samples $(\sigma_j, \mathbf{H}_{\mathtt{SO}}(\sigma_j))$, for $\sigma_j \in i[10^{-1}, 10^{1}]$ logarithmically spaced.
- Build standard and second-order Loewner matrices ($\mathrm{rank}\,(\mathbb{L}) = 4$ and $\mathrm{rank}\,(\mathbb{L}^{\mathtt{SO}}) = 2$).
- Construct rational and second-order model (both of order $r = r_{\mathtt{SO}} = 2$).



- Challenge: In many applications, the parameters α and β are not known.
- Hence, we propose a heuristic approach to estimate good values of the parameters.

Heuristic Approach (ML inspired)

1. Suppose that $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ are discretized in suitable intervals.

- Challenge: In many applications, the parameters α and β are not known.
- Hence, we propose a heuristic approach to estimate good values of the parameters.

Heuristic Approach (ML inspired)

1. Suppose that $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ are discretized in suitable intervals.
2. Split the measured data into training $\mathcal{D}_{\text{training}}$ and test set $\mathcal{D}_{\text{test}}$ (ratio 80:20).

- Challenge: In many applications, the parameters α and β are not known.
- Hence, we propose a heuristic approach to estimate good values of the parameters.

Heuristic Approach (ML inspired)

1. Suppose that $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ are discretized in suitable intervals.
2. Split the measured data into training $\mathcal{D}_{\text{training}}$ and test set $\mathcal{D}_{\text{test}}$ (ratio 80:20).
3. For $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, compute the SO Loewner model $\hat{\mathbf{H}}_{\text{SO}}(s, \alpha, \beta)$ using the training set $\mathcal{D}_{\text{training}}$.

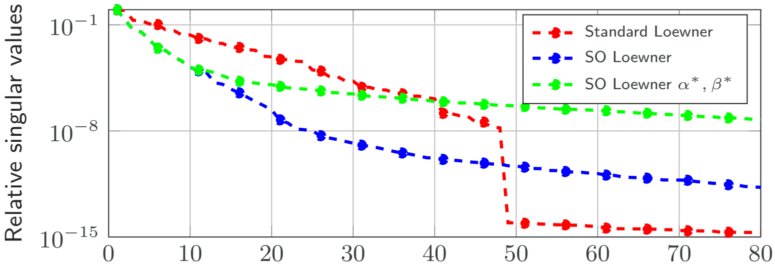
- Challenge: In many applications, the parameters α and β are not known.
- Hence, we propose a heuristic approach to estimate good values of the parameters.

Heuristic Approach (ML inspired)

1. Suppose that $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ are discretized in suitable intervals.
2. Split the measured data into training $\mathcal{D}_{\text{training}}$ and test set $\mathcal{D}_{\text{test}}$ (ratio 80:20).
3. For $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, compute the SO Loewner model $\hat{\mathbf{H}}_{\text{SO}}(s, \alpha, \beta)$ using the training set $\mathcal{D}_{\text{training}}$.
4. Find α^* and β^* solving

$$\min_{\alpha \in \mathcal{A},\, \beta \in \mathcal{B}} \mathcal{J}(\alpha, \beta) = \sum_{(\sigma_k, \gamma_k) \in \mathcal{D}_{\text{test}}} \left\| \hat{\mathbf{H}}_{\text{SO}}(\sigma_k, \alpha, \beta) - \gamma_k \right\|^2$$

- Modal model of order $n = 48$. Rayleigh damped with $\alpha \approx 0.4947$ and $\beta \approx 0.0011$.
- Measured data: 200 samples $\mathbf{H}(i\omega)$, with $\omega \in [10^0, 10^2]$. Reduced order $r = 16$.
- Optimal parameters: $\alpha^* = 0.495$ and $\beta^* = 0.001$.



- Modal model of order $n = 48$. Rayleigh damped with $\alpha \approx 0.4947$ and $\beta \approx 0.0011$.
- Measured data: 200 samples $\mathbf{H}(i\omega)$, with $\omega \in [10^0, 10^2]$. Reduced order $r = 16$.
- Optimal parameters: $\alpha^* = 0.495$ and $\beta^* = 0.001$.

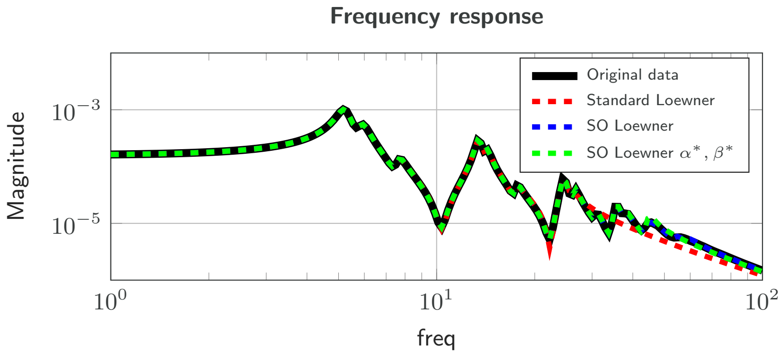


Figure: Absolute value of the original and reduced transfer functions.

- Modal model of order $n = 48$. Rayleigh damped with $\alpha \approx 0.4947$ and $\beta \approx 0.0011$.
- Measured data: 200 samples $\mathbf{H}(i\omega)$, with $\omega \in [10^0, 10^2]$. Reduced order $r = 16$.
- Optimal parameters: $\alpha^* = 0.495$ and $\beta^* = 0.001$.

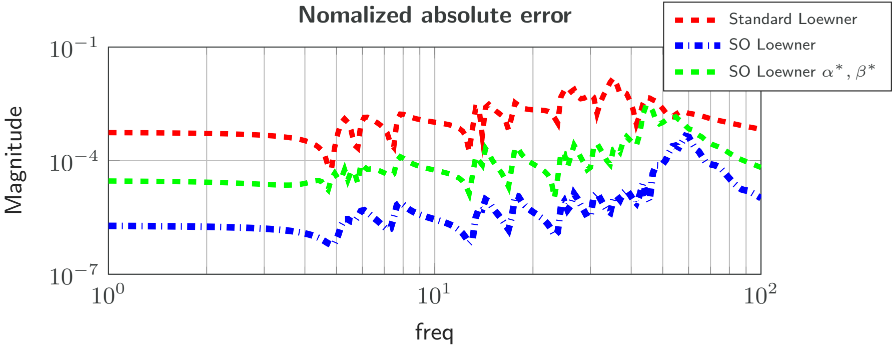
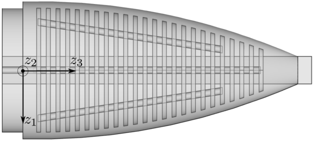
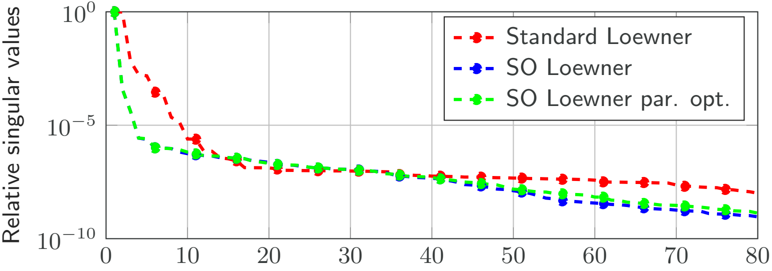


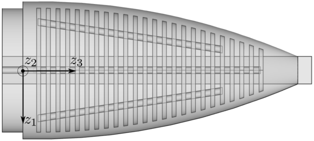
Figure: Absolute error between the original and reduced transfer functions.

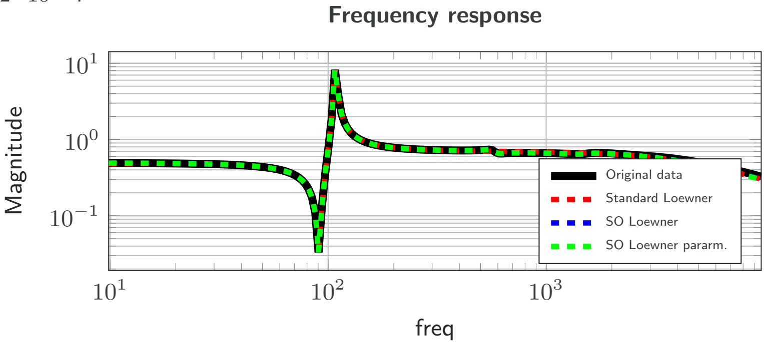
- FEM second-order model of order $n = 779,232$.
- $\alpha = 1.0 \cdot 10^{-4}$, $\beta = 2 \cdot 10^{-4}$.
- Mesured data: 200 samples $\mathbf{H}(i\omega)$, with $\omega \in [10^1, 10^4]$.
- Identify a model of order $r = 8$.
- Optimal parameters: $\alpha^* = 1.19 \cdot 10^{-4}$ and $\beta^* = 2 \cdot 10^{-4}$.



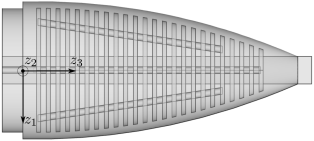


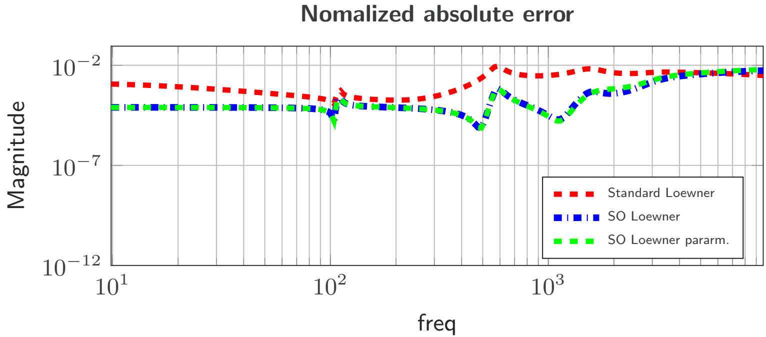
- FEM second-order model of order $n = 779,232$.
- $\alpha = 1.0 \cdot 10^{-4}$, $\beta = 2 \cdot 10^{-4}$.
- Mesured data: 200 samples $\mathbf{H}(i\omega)$, with $\omega \in [10^1, 10^4]$.
- Identify a model of order $r = 8$.
- Optimal parameters: $\alpha^* = 1.19 \cdot 10^{-4}$ and $\beta^* = 2 \cdot 10^{-4}$.





- FEM second-order model of order $n = 779,232$.
- $\alpha = 1.0 \cdot 10^{-4}$, $\beta = 2 \cdot 10^{-4}$.
- Mesured data: 200 samples $\mathbf{H}(i\omega)$, with $\omega \in [10^1, 10^4]$.
- Identify a model of order $r = 8$.
- Optimal parameters: $\alpha^* = 1.19 \cdot 10^{-4}$ and $\beta^* = 2 \cdot 10^{-4}$.





- So far, we have discussed the identification of Rayleigh-damped second-order systems, i.e., $\mathbf{D} := \alpha\mathbf{M} + \beta\mathbf{K}$.

General second-order systems

Assume the matrices $\mathbb{L}_{\mathbf{M}}$, $\mathbb{L}_{\mathbf{D}}$, and $\mathbb{L}_{\mathbf{K}}$ satisfy:

$$\boxed{\begin{aligned} \mathbb{L}_{\mathbf{M}}\Lambda^2 + \mathbb{L}_{\mathbf{D}}\Lambda + \mathbb{L}_{\mathbf{K}} &= \mathbf{V}\mathbf{1}^T, \\ {\mathbb{L}_{\mathbf{M}}}^T\Omega^2 + {\mathbb{L}_{\mathbf{D}}}^T\Lambda + \mathbb{L}_{\mathbf{K}} &= \mathbf{W}\mathbf{1}^T, \end{aligned}}$$

where $\Lambda = \operatorname{diag}(\sigma_1, \ldots, \sigma_l)$, $\Omega = \operatorname{diag}(\mu_1, \ldots, \mu_l)$, $\mathbf{V} = \left[\mathbf{H}(\mu_1), \ldots, \mathbf{H}(\mu_l)\right]^T$, and $\mathbf{V} = \left[\mathbf{H}(\sigma_1), \ldots, \mathbf{H}(\sigma_l)\right]^T$. Let the system be constructed as follows:

$$\boxed{\mathbf{M} = \mathbb{L}_{\mathbf{M}}, \quad \mathbf{D} = \mathbb{L}_{\mathbf{D}}, \quad \mathbf{K} = \mathbb{L}_{\mathbf{K}}, \quad \mathbf{B} = \mathbf{V}, \quad \mathbf{C} = \mathbf{W}^T.}$$

Then, the following interpolation conditions are satisfied, i.e.,

$$\mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B} = \mathbf{H}(s), \quad s \in \{\sigma_i, \mu_i\}.$$

The question remains how to choose a good Loewner-based model?

What we know ...

Some results

- Minimal order of second-order systems $= 2 \times$ minimal order of first-order system
  [Laub/Arnold '84]
- $\operatorname{rank}\left(\left[\mathbb{L}_{\mathbf{M}}, \mathbb{L}_{\mathbf{D}}, \mathbb{L}_{\mathbf{K}}\right]\right) =$ minimal order of second-order system
  [B./Goyal/Pontes '19]
- The minimal order of first-order system can be found using the classical Loewner approach (say, $2r$)

  $\Longleftarrow$ the minimal order of a second-order realization would then be r.
- This implies, $\operatorname{rank}\left(\left[\mathbb{L}_{\mathbf{M}}, \mathbb{L}_{\mathbf{D}}, \mathbb{L}_{\mathbf{K}}\right]\right) = r$.

Novel Problem Formulation

Identification of second-order systems

$$\mathbb{L}_{\mathbf{M}}\Lambda^2 + \mathbb{L}_{\mathbf{D}}\Lambda + \mathbb{L}_{\mathbf{K}} = \mathbf{V}\mathbf{1}^T,$$
$$\mathbb{L}_{\mathbf{M}}{}^T\Omega^2 + \mathbb{L}_{\mathbf{D}}{}^T\Omega + \mathbb{L}_{\mathbf{K}} = \mathbf{W}\mathbf{1}^T,$$

which can be written as a linear system:

$$\mathcal{A}\cdot\mathbf{x} = \mathbf{b}, \quad \text{where} \quad x = \mathrm{vect}\left(\left[\mathbb{L}_{\mathbf{M}}, \mathbb{L}_{\mathbf{D}}, \mathbb{L}_{\mathbf{K}}\right]\right).$$

- $\operatorname{rank}\left(\left[\mathbb{L}_{\mathbf{M}}, \mathbb{L}_{\mathbf{D}}, \mathbb{L}_{\mathbf{K}}\right]\right) = r.$ $\leftarrow$ apply Loewner approach
- Hence, $\left[\mathbb{L}_{\mathbf{M}}, \mathbb{L}_{\mathbf{D}}, \mathbb{L}_{\mathbf{K}}\right] = \mathbf{P}\mathbf{Q}^T$, where $\mathbf{P} \in \mathbb{R}^{\ell\times r}$ and $\mathbf{Q} \in \mathbb{R}^{3\ell\times r}$.

Novel Problem Formulation [B./Goyal/Pontes]

Solve

$$\mathcal{A}\cdot\mathrm{vec}\left(\mathbf{P}\mathbf{Q}^T\right) = b$$

for $\mathbf{P} \in \mathbb{R}^{\ell\times r}$ and $\mathbf{Q} \in \mathbb{R}^{3\ell\times r}$.

Novel Problem Formulation

Identification of second-order systems

$$\mathcal{A} \cdot \mathbf{x} = \mathbf{b}, \quad \text{where} \quad x = \text{vect}\left(\left[\mathbb{L}_{\mathbf{M}}, \mathbb{L}_{\mathbf{D}}, \mathbb{L}_{\mathbf{K}}\right]\right).$$

- $\operatorname{rank}\left(\left[\mathbb{L}_{\mathbf{M}}, \mathbb{L}_{\mathbf{D}}, \mathbb{L}_{\mathbf{K}}\right]\right) = r.$ $\leftarrow$ apply Loewner approach
- Hence, $\left[\mathbb{L}_{\mathbf{M}}, \mathbb{L}_{\mathbf{D}}, \mathbb{L}_{\mathbf{K}}\right] = \mathbf{P}\mathbf{Q}^T$, where $\mathbf{P} \in \mathbb{R}^{\ell \times r}$ and $\mathbf{Q} \in \mathbb{R}^{3\ell \times r}$.

Novel Problem Formulation [B./Goyal/Pontes]

Solve

$$\mathcal{A} \cdot \text{vec}\left(\mathbf{P}\mathbf{Q}^T\right) = b$$

for $\mathbf{P} \in \mathbb{R}^{\ell \times r}$ and $\mathbf{Q} \in \mathbb{R}^{3\ell \times r}$.

Remarks:

- Such a problem arises in several applications, e.g., matrix completion, de-noising, phase retrieval.
- Non-convex with several local minima.

A Demo Example

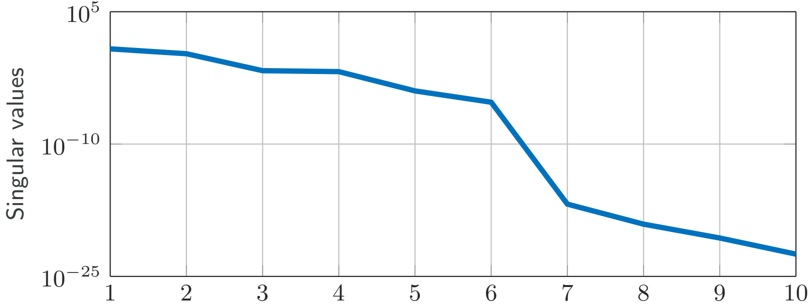
- Consider a second-order system of order $n = 3$ (randomly generated).

A Demo Example

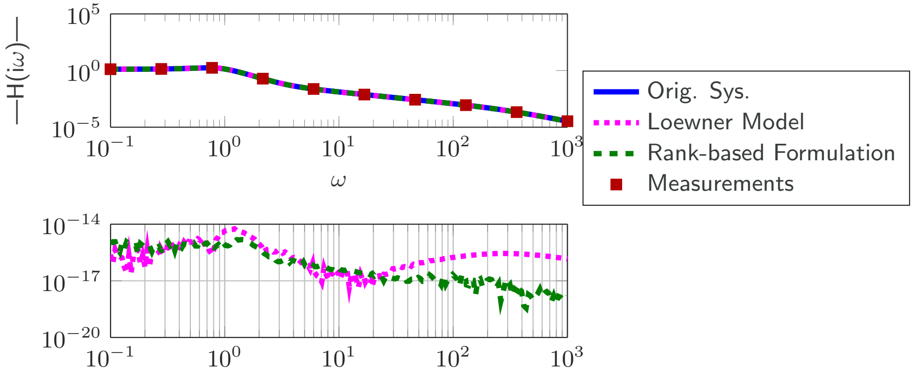
- Consider a second-order system of order $n = 3$ (randomly generated).
- We collect 10 samples $(\sigma_j, \mathbf{H}_{\mathtt{SO}}(\sigma_j))$, for $\sigma_j \in i[10^{-1}, 10^3]$ logarithmically spaced.

A Demo Example

- Consider a second-order system of order $n = 3$ (randomly generated).
- We collect 10 samples $(\sigma_j, \mathbf{H}_{\mathtt{SO}}(\sigma_j))$, for $\sigma_j \in i[10^{-1}, 10^3]$ logarithmically spaced.
- Applying the Loewner approach to the resulting data, one observes the rank 6.



- Consider a second-order system of order $n = 3$ (randomly generated).
- We collect 10 samples $(\sigma_j, \mathbf{H}_{\text{SO}}(\sigma_j))$, for $\sigma_j \in i[10^{-1}, 10^3]$ logarithmically spaced.
- Applying the Loewner approach to the resulting data, one observes the rank 6.
- Now, solve the optimization problem by fixing the rank $r = 3$.



Outlook

Contribution of this talk

- Loewner framework for second-order systems for Rayleigh damped second-order systems.
- Identification of general second-order systems.
- Demonstrated the applicability by means of numerical examples.

P. Benner, P. Goyal, and I. Pontes Duff.
Identification of dominant subspaces for linear structured parametric systems and model reduction.
arXiv:1910.13945, October 2019.

I. Pontes Duff, P. Goyal, and P. Benner.
Data-driven identification of Rayleigh-damped second-order systems.
arXiv:1910.00838, [math.OC], October 2019.

Outlook

Contribution of this talk

- Loewner framework for second-order systems for Rayleigh damped second-order systems.
- Identification of general second-order systems.
- Demonstrated the applicability by means of numerical examples.

P. Benner, P. Goyal, and I. Pontes Duff.
Identification of dominant subspaces for linear structured parametric systems and model reduction.
arXiv:1910.13945, October 2019.

I. Pontes Duff, P. Goyal, and P. Benner.
Data-driven identification of Rayleigh-damped second-order systems.
arXiv:1910.00838, [math.OC], October 2019.

Open questions and future work

- Focus on developing robust optimization methods.
- Use of analogue of barycentric representation $\Rightarrow$ AAA SO algorithm?

Outlook

Contribution of this talk

- Loewner framework for second-order systems for Rayleigh damped second-order systems.
- Identification of general second-order systems.
- Demonstrated the applicability by means of numerical examples.

P. Benner, P. Goyal, and I. Pontes Duff.
Identification of dominant subspaces for linear structured parametric systems and model reduction.
arXiv:1910.13945, October 2019.

I. Pontes Duff, P. Goyal, and P. Benner.
Data-driven identification of Rayleigh-damped second-order systems.
arXiv:1910.00838, [math.OC], October 2019.

Open questions and future work

- Focus on developing robust optimization methods.
- Use of analogue of barycentric representation $\rightarrow$ AAA-SO algorithms?

Thank you for your attention!!