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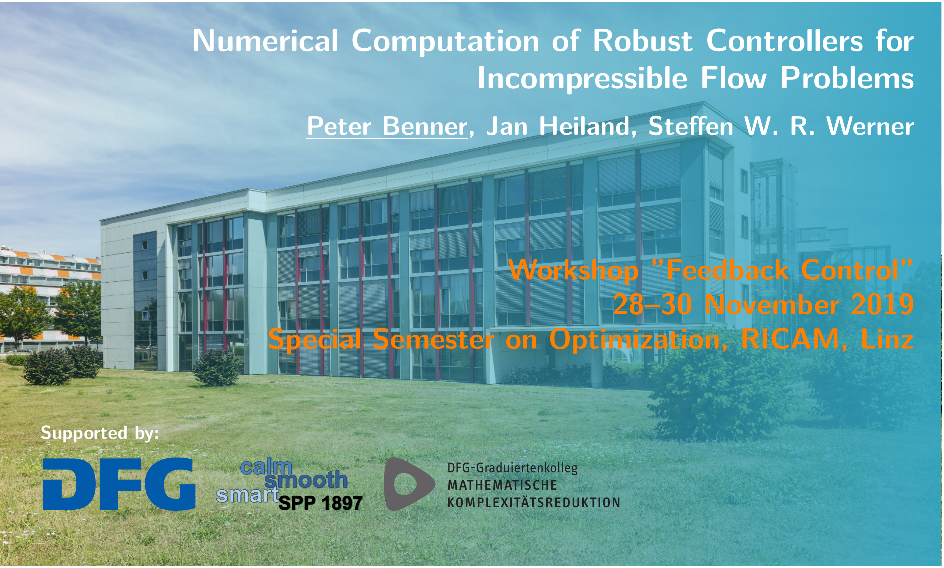
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY



Numerical Computation of Robust Controllers for Incompressible Flow Problems

Peter Benner, Jan Heiland, Steffen W. R. Werner

Workshop "Feedback Control"
28–30 November 2019
Special Semester on Optimization, RICAM, Linz

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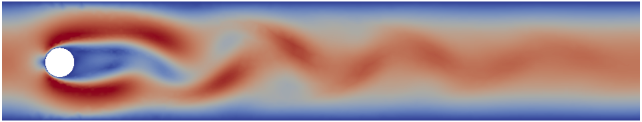
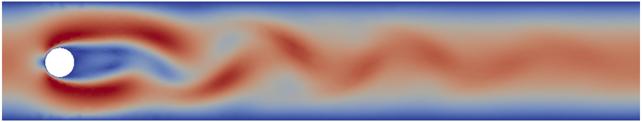
Outline



Problem: The steady state tends to unstable flow due to unavoidable system perturbations.

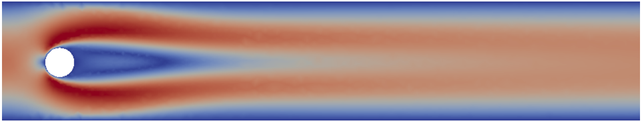


Feedback Control

Problem: The steady state tends to unstable flow due to unavoidable system perturbations.

Goal: Stabilizing feedback controller that can handle:

- limited measurements,
- short evaluation time,
- system uncertainties.



Idea: Linearization-based feedback control for stabilization of the steady state.
[Raymond '06, Breiten/Kunisch '14, B./Heiland '15]

$$\dot{v} + (v \cdot \nabla v) - \frac{1}{Re}\Delta v + \nabla p = Bu, \qquad \nabla \cdot v = 0$$

Linearization & Semi-Discretization $\Rightarrow$

$$E\dot{v} - Av - J^{\mathsf{T}} p = Bu, \qquad Jv = 0$$

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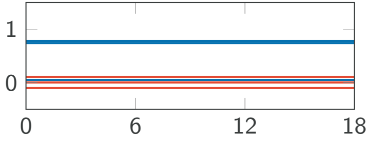
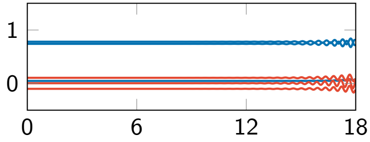
Fragility of Observer-Based Controllers [Doyle '78]

LQG controllers have no guaranteed robustness margins and will likely fail in the presence of system uncertainties.

LQGBT-feedback



corrupted LQGBT-feedback



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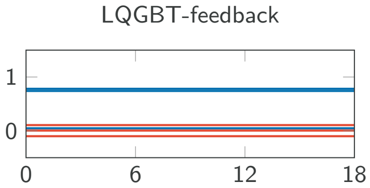

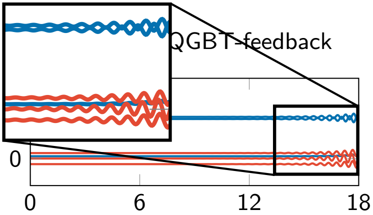

In general, an uncertainty $\boldsymbol{A_\Delta}$ in the linearization A ...

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{x}(t) &= [A + \boldsymbol{A_\Delta}]x(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

... is an additive uncertainty in the transfer function

$$G(s) = C(sI - A)^{-1}B \quad \Longrightarrow \quad \begin{aligned} G_\Delta(s) &= C(sI - A - \boldsymbol{A_\Delta})^{-1}B \\ &= G(s) + \widetilde{G}(s) \end{aligned}$$

where $\widetilde{G}(s) = C\boldsymbol{A_\Delta}(sI - A)^{-1}(sI - A - \boldsymbol{A_\Delta})^{-1}B$.

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Additive uncertainties can be compensated by robust $\mathcal{H}_\infty$ controller design.

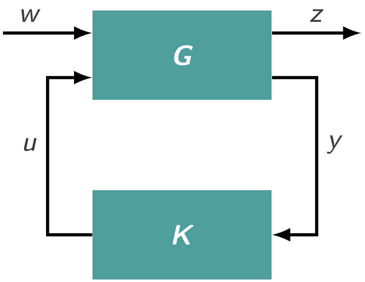
Linear Time-Invariant Systems

$$G : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w + D_{22} u, \end{cases}$$

where A, B_j, C_i, D_{ij} are matrices of suitable size, $j, i \in \{1, 2\}$.

- x ... internal state,
- w ... disturbance input,
- u ... control input,
- z ... performance output,
- y ... measured output,

and a controller $K : y \mapsto u$.



For such partitioned systems, the transfer function writes

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right],$$

such that

$$\begin{aligned} z(s) &= G_{11}(s)w(s) + G_{12}u(s), \\ y(s) &= G_{21}(s)w(s) + G_{22}u(s). \end{aligned}$$

Goal: Optimal $\mathcal{H}_\infty$ Control Problem

Find a controller K that minimizes the performance outputs of the closed-loop system

$$z = (G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21})w =: \mathcal{F}(G, K)w,$$

and internally stabilizes the plant, i.e.,

$$\min_{K \text{ stabilizing}} \|\mathcal{F}(G, K)\|_{\mathcal{H}_\infty}.$$

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Goal: Sub-Optimal $\mathcal{H}_\infty$ Control Problem

Find a controller K that minimizes the performance outputs of the closed-loop system

$$z = (G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21})w =: \mathcal{F}(G, K)w,$$

and internally stabilizes the plant, i.e., **for a given constant γ**

$$\|\mathcal{F}(G, K)\|_{\mathcal{H}_\infty} < \gamma\,.$$

Simplifying Assumptions:

1. $D_{11} = 0$
2. $D_{22} = 0$
3. (A, B_1) stabilizable and (A, C_1) detectable
4. (A, B_2) stabilizable and (A, C_2) detectable
5. $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$
6. $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$

Remarks:

- Assumptions 2., 5., 6. are for notational convenience.
- From 4., internal stabilizability follows.
- The points 1., 3. can be relaxed, but derivations become more complicated.

$\mathcal{H}_\infty$ Riccati Equations [Doyle/Glover/Khargonekar/Francis '89, Van Keulen '93]

Given the assumptions 1.–6., there exists an admissible controller $K(s)$ $\Longleftrightarrow$:

1. There exists a stabilizing solution $X_\infty = X_\infty^T \geq 0$ to the regulator Riccati equation
$$A^T X_\infty + X_\infty A + C_1^T C_1 + X_\infty(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty = 0.$$
2. There exists a stabilizing solution $Y_\infty = Y_\infty^T \geq 0$ to the filter Riccati equation
$$A Y_\infty + Y_\infty A^T + B_1 B_1^T + Y_\infty(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y_\infty = 0.$$
3. It holds $\gamma^2 > \lambda_{\max}(Y_\infty X_\infty)$.

The central (or minimum entropy) controller $\widehat{K}(s) = \widehat{C}(sI_n - \widehat{A})^{-1}\widehat{B}$ is given by
$$\widehat{A} = A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty - Z_\infty Y_\infty C_2^T C_2, \quad \widehat{B} = Z_\infty Y_\infty C_2^T, \quad \widehat{C} = -B_2^T X_\infty,$$
with $Z_\infty = (I_n - \gamma^{-2} X_\infty Y_\infty)^{-1}$.

Large-Scale DAE Matrix Equations

How to solve the arising large-scale sparse DAE Riccati equations?

- Use **low-rank Riccati iteration** for approximation of the solution.
  [Lanzon/Feng/Anderson '07, B./Heiland/Werner '19]
- Apply **implicit projections** to handle DAE structure.
  [Heinkenschloss/Sorensen/Sun '08, Bänsch/B./Saak/Weichelt '15]

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High Dimensional Robust Controller

How much uncertainty can be compensated by a low-dimensional robust controller?

- Robustness margin gives condition on **coprime factor perturbations**.
- Use model order reduction based on X_∞ and Y_∞. $\Rightarrow$ **$\mathcal{H}_\infty$ Balanced Truncation**

Given a transfer function $G(s)$ of a linear system,

$$G(s) = M^{-1}(s)N(s)$$

is a **(left) coprime factorization** if there exist $U(s)$ and $V(s)$ such that

$$N(s)U(s) + M(s)V(s) = I_p,$$

where M, N, U, V are transfer functions with all poles in the open left complex half-plane, i.e., corresponding to stable linear systems.

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Coprime Factor Perturbations

The transfer functions $G(s)$ and $G_\Delta(s)$ differ by a coprime factor disturbance if

$$G(s) = M(s)^{-1}N(s) \quad \text{and} \quad G_\Delta(s) = M_\Delta(s)^{-1}N_\Delta(s),$$

with $\Delta = [N - N_\Delta, M - M_\Delta]$ stable.

- Uncertainties in the linearization result in coprime factor perturbations.

Assumption: Normalized LQG problem, i.e., $B_1 = B_2 = B$, $C_1 = C_2 = C$.

Theorem [McFarlane/Glover '90]

Let the controller K stabilize the plant G with

$$\left\| \begin{bmatrix} K \\ I_p \end{bmatrix} (I_p - GK)^{-1} M^{-1} \right\|_{\mathcal{H}_\infty} < \gamma^{-1},$$

then K stabilizes G_Δ if the perturbation Δ is small, namely $\|\Delta\|_{\mathcal{H}_\infty} < \gamma$.

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Let L be a gain such that

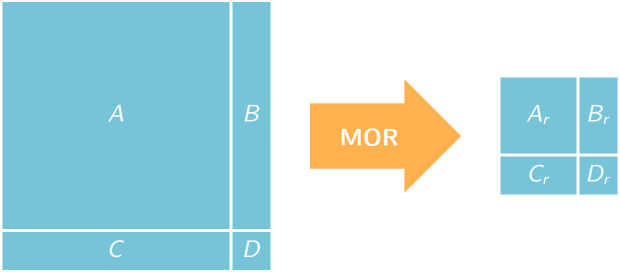
$$A - LC \quad \text{and} \quad A + A_\Delta - LC \quad \text{are both stable,}$$

then realizations of the coprime factorizations $G(s) = M(s)^{-1}N(s)$ and $G_\Delta(s) = M_\Delta(s)^{-1}N_\Delta(s)$ are given by

$$\begin{bmatrix} N & M \end{bmatrix} = \left[\begin{array}{c|cc} A - LC & B & -L \\ \hline C & 0 & I_p \end{array}\right], \quad \begin{bmatrix} N_\Delta & M_\Delta \end{bmatrix} = \left[\begin{array}{c|cc} A + A_\Delta - LC & B & -L \\ \hline C & 0 & I_p \end{array}\right].$$

Motivation

In practice, the number of differential equations becomes very large.
$\Rightarrow$ Large demand of computational resources, i.e., time and memory.



Goal

Approximation of the input-output behavior, i.e., $\|y - y_r\| \leq tol \cdot \|u\|$ for all admissible input signals u.

Balancing Related Model Reduction Methods

1. Solve primal and dual matrix equations defining the characteristic matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$.
2. Balance the system with respect to $\boldsymbol{P}$ and $\boldsymbol{Q}$.
3. Truncate states corresponding to small characteristic values of $\boldsymbol{PQ}$.

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- Linear-Quadratic Gaussian Balanced Truncation (LQGBT):
  [Jonckheere/Silverman '83]

$$A\boldsymbol{P}_{\boldsymbol{LQG}} + \boldsymbol{P}_{\boldsymbol{LQG}}A^{\mathsf{T}} - \boldsymbol{P}_{\boldsymbol{LQG}}C^{\mathsf{T}}C\boldsymbol{P}_{\boldsymbol{LQG}} + BB^{\mathsf{T}} = 0,$$
$$A^{\mathsf{T}}\boldsymbol{Q}_{\boldsymbol{LQG}} + \boldsymbol{Q}_{\boldsymbol{LQG}}A - \boldsymbol{Q}_{\boldsymbol{LQG}}BB^{\mathsf{T}}\boldsymbol{Q}_{\boldsymbol{LQG}} + C^{\mathsf{T}}C = 0.$$

- $\mathcal{H}_\infty$ Balanced Truncation (HINFBT): [Mustafa/Glover '91]

$$A\boldsymbol{P}_{\mathcal{H}_\infty} + \boldsymbol{P}_{\mathcal{H}_\infty}A^{\mathsf{T}} + \boldsymbol{P}_{\mathcal{H}_\infty}(\gamma^{-2}C^{\mathsf{T}}C - C^{\mathsf{T}}C)\boldsymbol{P}_{\mathcal{H}_\infty} + BB^{\mathsf{T}} = 0,$$
$$A^{\mathsf{T}}\boldsymbol{Q}_{\mathcal{H}_\infty} + \boldsymbol{Q}_{\mathcal{H}_\infty}A + \boldsymbol{Q}_{\mathcal{H}_\infty}(\gamma^{-2}BB^{\mathsf{T}} - BB^{\mathsf{T}})\boldsymbol{Q}_{\mathcal{H}_\infty} + C^{\mathsf{T}}C = 0.$$

Assumptions/Notations:

- normalized LQG problem, i.e., $B_1 = B_2 = B$, $C_1 = C_2 = C$,
- normalized left coprime factorizations $G = M^{-1}N$, $G_r = M_r^{-1}N_r$, $G_\Delta = M_\Delta^{-1}N_\Delta$ (for construction see [B./Heiland/Werner '19]),
- $\beta = \sqrt{1-\gamma^{-2}}$.

The approximation error of the $\mathcal{H}_\infty$ balanced truncation is given by

$$\left\| \begin{bmatrix} \beta(N - N_r) & M - M_r \end{bmatrix} \right\|_{\mathcal{H}_\infty} \leq \beta\epsilon = 2 \sum_{k=r+1}^{n} \frac{\beta\sigma_k^{\mathcal{H}_\infty}}{\sqrt{1 + \beta^2 \left(\sigma_k^{\mathcal{H}_\infty}\right)^2}}.$$

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Theorem [Mustafa/Glover '91, B./Heiland/Werner '19]

The reduced-order $\mathcal{H}_\infty$ controller is guaranteed to stabilize the disturbed system if

$$\epsilon(\beta + \gamma) < 1 \quad \text{and} \quad \left\| \begin{bmatrix} N - N_\Delta & M - M_\Delta \end{bmatrix} \right\|_{\mathcal{H}_\infty} < \gamma^{-1}.$$

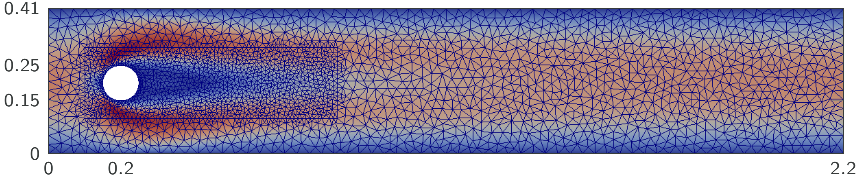


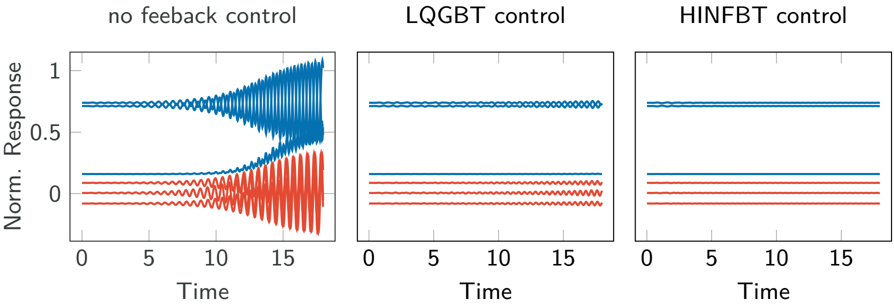
Fig.: 2D cylinder wake, discretized by Taylor-Hood (P_2/P_1) finite elements.

- Navier-Stokes equations
- Reynolds number 90
- 9,843 velocity nodes
- distributed observations:
  - 3 sensors in the wake
  - measuring both v components

Target

Stabilize the steady-state and compensate perturbations to suppress vortex shedding.

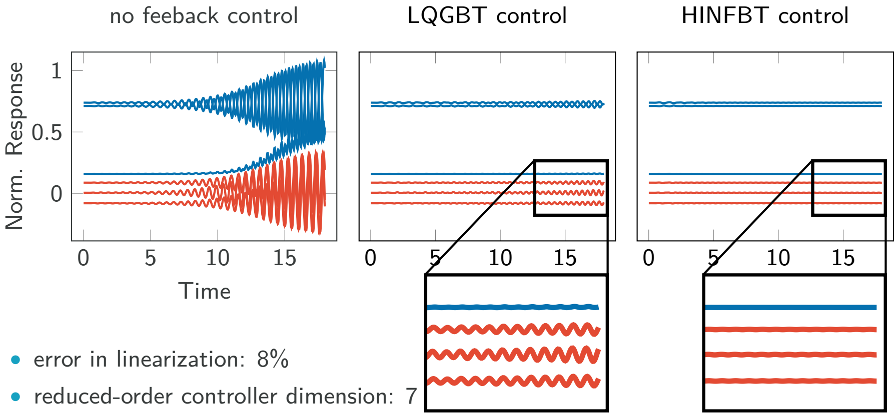
- boundary control:
  - 2 outlets at the cylinder periphery
  - control by injection and suction



- error in linearization: 8%
- reduced-order controller dimension: 7
- trigger instabilities by input disturbance on time interval $[0, 1]$:

$$u_\delta(t) = \begin{bmatrix} 0.01 \sin(2t\pi) \\ -0.01 \sin(2t\pi) \end{bmatrix}$$

Numerical Example
Results



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$\mathcal{H}_\infty$-BT controllers are

- output-based and of low dimension,
- robust against system uncertainties.

The application to flow stabilization becomes feasible by

- the implicit realization of the DAE structure,
- the low-rank Riccati iteration.

Conclusions

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The application to flow stabilization becomes feasible by

- the implicit realization of the DAE structure,
- the low-rank Riccati iteration.

Code Availability:

- The low-rank Riccati iteration will be available in the M-M.E.S.S. library version 2.0.
- HINFBT and LQGBT implementations can be found in the MORLAB toolbox.

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