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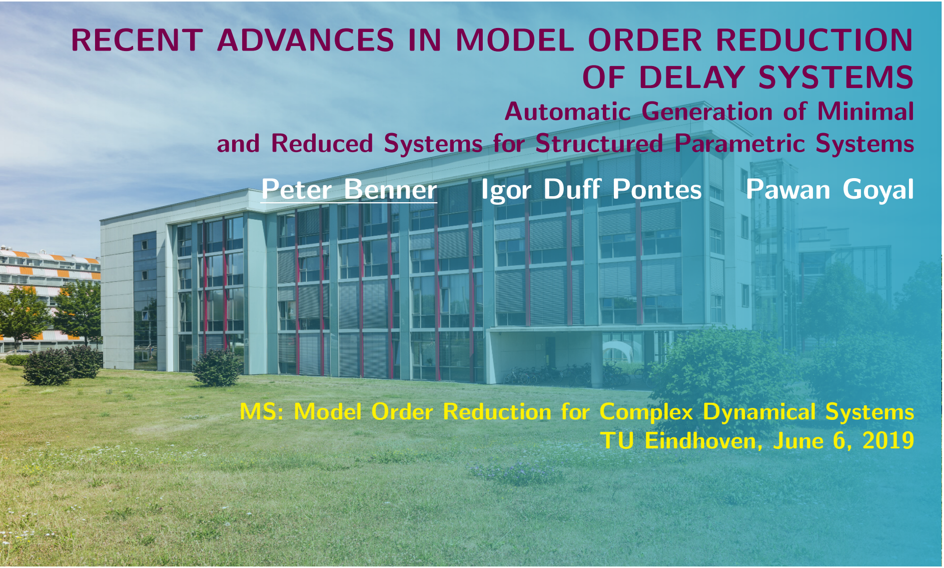
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY



RECENT ADVANCES IN MODEL ORDER REDUCTION OF DELAY SYSTEMS

Automatic Generation of Minimal and Reduced Systems for Structured Parametric Systems

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MS: Model Order Reduction for Complex Dynamical Systems
TU Eindhoven, June 6, 2019

Overview

1. Introduction

2. Minimal Realization

3. Reachability and Observability for SLS

4. Model Order Reduction

5. Numerical Results

6. Outlook and Conclusions

Outline

1. Introduction
   - Structered Linear Systems
   - Projection-based Framework
   - Existing Approaches
2. Minimal Realization
3. Reachability and Observability for SLS
4. Model Order Reduction
5. Numerical Results
6. Outlook and Conclusions

We consider the class of **Structured Linear System (SLS)**

$$\boxed{\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s),} \tag{1}$$

where

$$\mathcal{C}(s) = \sum_{i=1}^{k} \alpha_i(s)\mathbf{C}_i, \quad \mathcal{K}(s) = s\mathbf{E} - \sum_{i=1}^{l} \beta_i(s)\mathbf{A}_i, \quad \mathcal{B}(s) = \sum_{i=1}^{m} \gamma_i(s)\mathbf{B}_i,$$

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- with $\mathbf{E}, \mathbf{A}_i \in \mathbb{R}^{n \times n}, \mathbf{B}_i \in \mathbb{R}^{n \times m}$, and $\mathbf{C}_i \in \mathbb{R}^{p \times n}$, and $\alpha_i(s), \beta_i(s)$ and $\gamma_i(s)$ are meromorphic functions.

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3) **Time delay systems :** $\mathcal{C}(s) = \mathbf{C}$, $\mathcal{B}(s) = \mathbf{B}$, and $\mathcal{K}(s) = (s\mathbf{E} - \mathbf{A}_1 - \mathbf{A}_2e^{-s\tau})^{-1}$.

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4) **Integro-differential Volterra systems**, **input delays**, **fractional order systems** . . .

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find projection matrices

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(with $r \ll n$), such that

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- Note $\hat{\mathbf{A}}_i = \mathbf{W}^T\mathbf{A}_i\mathbf{V}$, $\hat{\mathbf{E}} = \mathbf{W}^T\mathbf{E}\mathbf{V}$, $\hat{\mathbf{C}}_i = \mathbf{C}_i\mathbf{V}$ and $\hat{\mathbf{B}}_i = \mathbf{W}^T\mathbf{B}_i$.
- The ROM preserves $\alpha_i(s), \beta_i(s)$ and $\gamma_i(s)$ functions.

Interpolation-based methods

- Interpolatory projection methods for structure-preserving model reduction.

[Beattie/Gugercin '09]

Interpolation points $\sigma_k,\ \mu_j \Rightarrow$ $\boxed{\begin{array}{l} \mathcal{K}^{-1}(\sigma_k)\mathcal{B}(\sigma_k) \in \mathrm{range}\,(\mathbf{V}) \ \text{ and} \\ \mathcal{K}^{-T}(\mu_j)\mathcal{C}^T(\mu_j) \in \mathrm{range}\,(\mathbf{W})\,. \end{array}}$

Interpolation-based methods

- Interpolatory projection methods for structure-preserving model reduction.
  [Beattie/Gugercin '09]

Balancing truncation methods

- Structure-preserving model reduction for integro-differential equations.
  [Breiten '16]

$$\boxed{\begin{aligned}\mathbf{P} &= \frac{1}{2\pi}\int_{-i\infty}^{i\infty} \mathcal{K}_s(s)^{-1}\mathcal{B}(s)\mathcal{B}(s)^T\mathcal{K}(s)^{-T}ds,\\ \mathbf{Q} &= \frac{1}{2\pi}\int_{-i\infty}^{i\infty} \mathcal{K}_s(s)^{-T}\mathcal{C}(s)^T\mathcal{C}(s)\mathcal{K}(s)^{-1}ds.\end{aligned}} \Rightarrow \text{Find } \mathbf{V}, \mathbf{W} \text{ as in the BT procedure.}$$

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Data-driven methods

- Data-driven structured realization. [Schulze/Unger/Beattie/Gugercin '18]

Outline

1. Introduction

2. Minimal Realization
   Motivation
   ...of Structured Linear Systems
   Some Results

3. Reachability and Observability for SLS

4. Model Order Reduction

5. Numerical Results

6. Outlook and Conclusions

Let us consider the first-order system

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \text{ with } \mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{C}^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

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Minimal realization problem

Find an order r and matrices $\mathbf{V}$ and $\mathbf{W}$ such that the reduced-order model obtained by projection satisfies

$$\mathbf{H}(s) = \hat{\mathbf{H}}(s), \forall s.$$

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Solutions:

- Kalman reachability/observability criteria,
- Hankel matrix (Silverman method),
- Reachability and observability Gramians,
- **Loewner matrix.** [Mayo/Antoulas '07]

For illustration, consider the time-delay systems

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1}\mathbf{B}, \text{ with } \mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

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Minimal Realization

. . . of Structured Linear Systems

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- $\mathbf{H}(s) = \hat{\mathbf{H}}(s), \forall s.$
- $\mathbf{H}$ has order 3 and $\hat{\mathbf{H}}$ order 2.

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Minimal realization problem

Is there a way to find the order r and matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n\times r}$ such that the system $\hat{\mathbf{H}}(s)$ obtained by projection is "minimal", *i.e*

$$\mathbf{H}(s) = \hat{\mathbf{H}}(s), \forall s?$$

Given a first order system

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, \text{ with } \mathbf{E} \in \mathbb{R}^{n \times n} \text{ invertible.}$$

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Reachability characterization [Anderson/Antoulas '90]

If $(\mathbf{E}, \mathbf{A}, \mathbf{B})$ is R^n-reachable, $t \geq n$, $\sigma_i \neq \sigma_j$ for $i \neq j$, and

$$\mathbf{R} = \begin{bmatrix} (\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \dots & (\sigma_t\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}. \text{ Then } \operatorname{rank}(\mathbf{R}) = n.$$

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Observability characterization [Anderson/Antoulas '90]

If $(\mathbf{E}, \mathbf{A}, \mathbf{C})$ is R^n-observable, $t \geq n$, $\sigma_i \neq \sigma_j$ for $i \neq j$, and

$$\mathbf{O} = \begin{bmatrix}(\sigma_1\mathbf{E} - \mathbf{A})^{-T}\mathbf{C}^T & \dots & (\sigma_t\mathbf{E} - \mathbf{A})^{-T}\mathbf{C}^T\end{bmatrix}. \text{ Then } \operatorname{rank}(\mathbf{O}) = n.$$

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Observability characterization [Anderson/Antoulas '90]

If $(\mathbf{E}, \mathbf{A}, \mathbf{C})$ is R^n-observable, $t \geq n$, $\sigma_i \neq \sigma_j$ for $i \neq j$, and

$$\mathbf{O} = \left[(\sigma_1\mathbf{E} - \mathbf{A})^{-T}\mathbf{C}^T \quad \dots \quad (\sigma_t\mathbf{E} - \mathbf{A})^{-T}\mathbf{C}^T\right]. \text{ Then } \operatorname{rank}(\mathbf{O}) = n.$$

Rank encodes minimality [Anderson/Antoulas '90]

$$\operatorname{rank}\left(\mathbf{O}^T\mathbf{E}\mathbf{R}\right) = \text{order of minimal realization} = \text{r}.$$

Outline

1. Introduction

2. Minimal Realization

3. Reachability and Observability for SLS
   An Illustrative Example

4. Model Order Reduction

5. Numerical Results

6. Outlook and Conclusions

For **SLS**, we use the notion of $\mathbb{R}^n$ reachability and observability. Let us consider the SLS

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s) \text{ of order n.}$$

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If $(\mathcal{K}(s), \mathcal{B}(s))$ is $\mathbb{R}^n$-reachable, $t \geq n$, $\sigma_i \neq \sigma_j$ for $i \neq j$, and

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Let's go back to the time-delay example

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1}\mathbf{B}, \text{ with } \mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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$$\mathbf{V} = \mathbf{R}\mathbf{X}(:, 1:2) \text{ and } \mathbf{W} = \mathbf{O}\mathbf{Y}(:, 1:2).$$

The $\hat{\mathbf{H}}$ obtained using $\mathbf{V}$ and $\mathbf{W}$ satisfies

$$\mathbf{H}(s) = \hat{\mathbf{H}}(s), \forall s.$$

1. Introduction

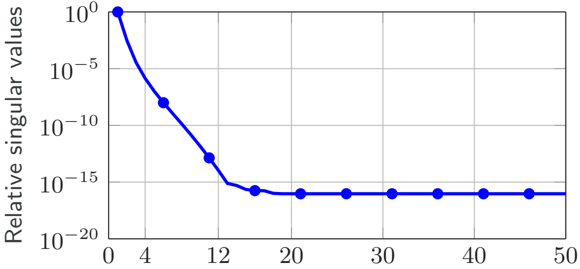
2. Minimal Realization

3. Reachability and Observability for SLS

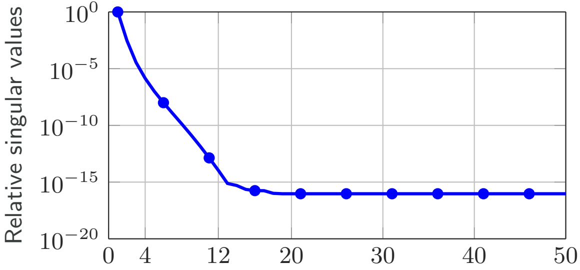
4. Model Order Reduction
   - The Basic Approach
   - Numerical Implementation
   - The Algorithm

5. Numerical Results

6. Outlook and Conclusions



- Figure represents the singular values of $\mathbf{O}^T\mathbf{E}\mathbf{R}$ for a large-scale time-delay example.



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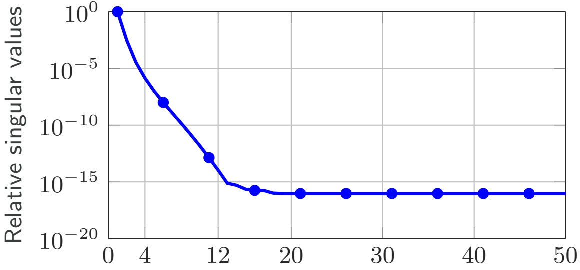
- For large-scale systems, often low-rank phenomena can be observed.



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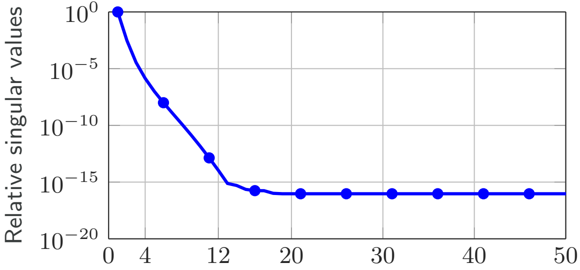

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- For large-scale systems, often low-rank phenomena can be observed.
- Numerical rank of $\mathbf{O}^T\mathbf{ER}$ generally small compared to n.
- We can cut off states that are related to very small singular value of $\mathbf{O}^T\mathbf{ER}$.

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- we set

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- We use the truncated low-rank methods for generalized Sylvester equations from [Kressner/Sirkovic '15].

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Input: SLS $\mathcal{K}(s)$, $\mathcal{B}(s)$, $\mathcal{C}(s)$ and reduced order r.

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Output: Reduced-order model is given by

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Outline

1. Introduction

2. Minimal Realization

3. Reachability and Observability for SLS

4. Model Order Reduction

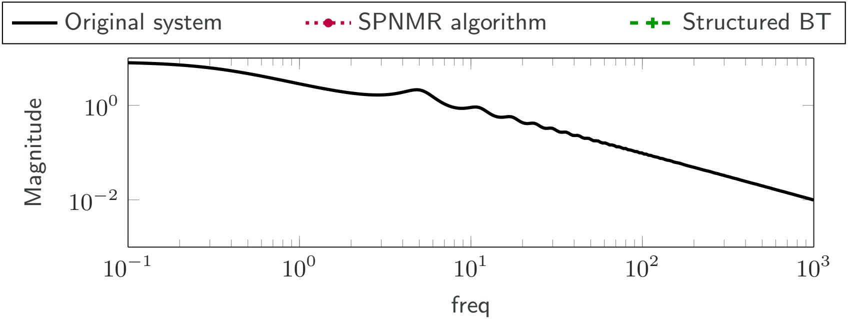
5. Numerical Results
   - A Time delay System
   - Second-order System
   - Parametric Systems

6. Outlook and Conclusions

Let us consider the time delay system

$$\dot{x}(t) = Ax(t) + A_\tau x(t-\tau) + Bu(t),$$
$$y(t) = Cx(t).$$

- Heated rod cooled using delayed feedback from [Breda/Maset/Vermiglio '09].
- Full order model $n = 120$ and $\tau = 1$.
- ROM obtained used SPNMR method (100 log. dist. points in $[1e^{-1}, 1e^{3}]i$) and Structured Balanced Truncation [Breiten '16].
- Reduced order $r = 4$.



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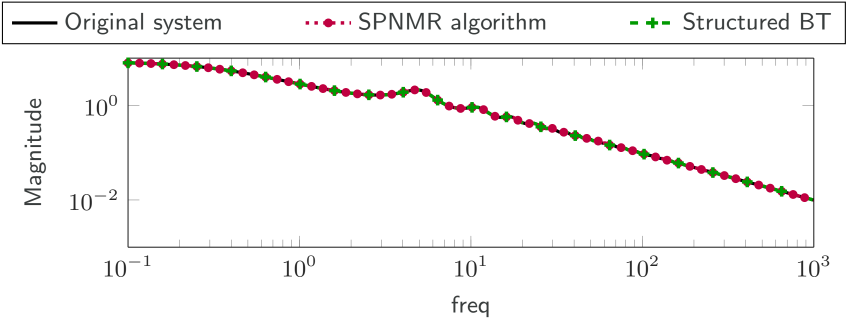

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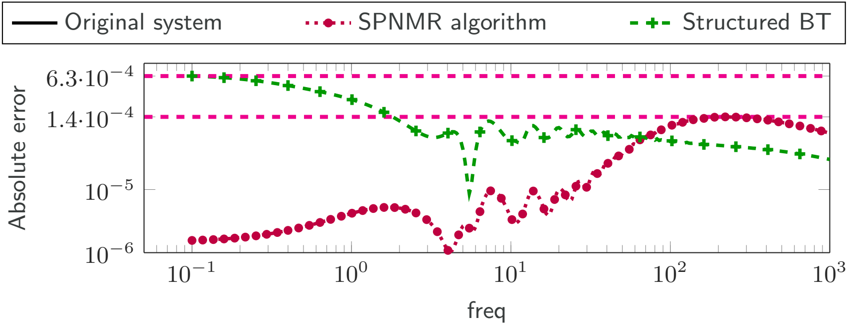

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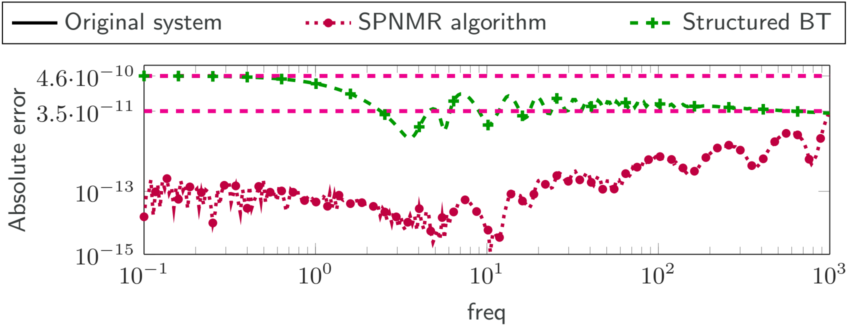

Let us consider the second-order system

$$\begin{aligned} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) &= Bu(t) \\ y(t) &= Cx(t). \end{aligned}$$

- Damped vibrational system.

- Full order model with $n = 301$.
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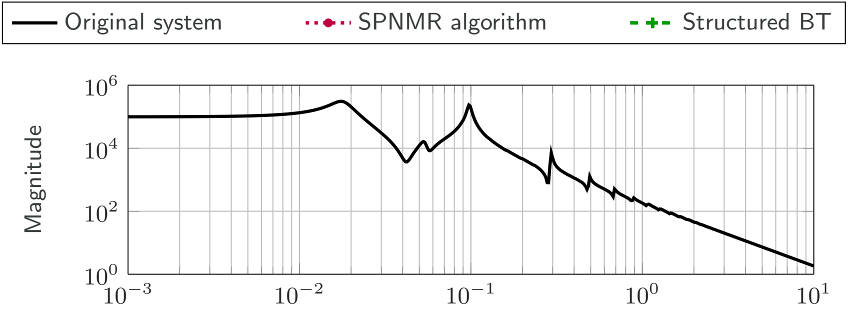

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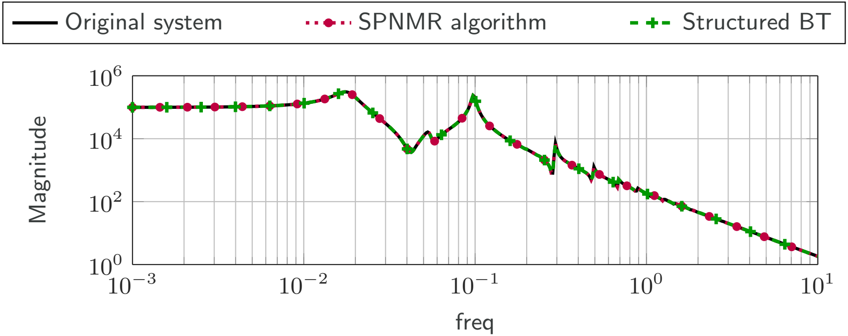

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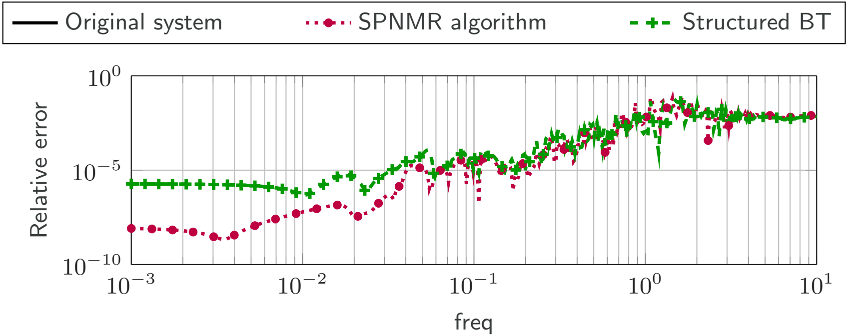

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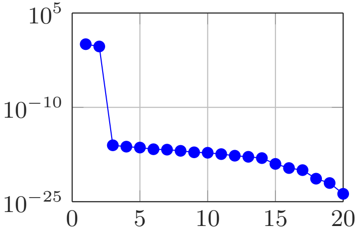
- Compute projectors $\mathbf{V}$ and $\mathbf{W}$ and $\hat{\mathbf{H}}(s,p)$.
- Then, $\mathbf{H}(s,p) = \hat{\mathbf{H}}(s,p)$.

Decay of Singular values



- FOM example [MORWIKI][1] of order 1006 and $p \in [10, 100]$ of the form

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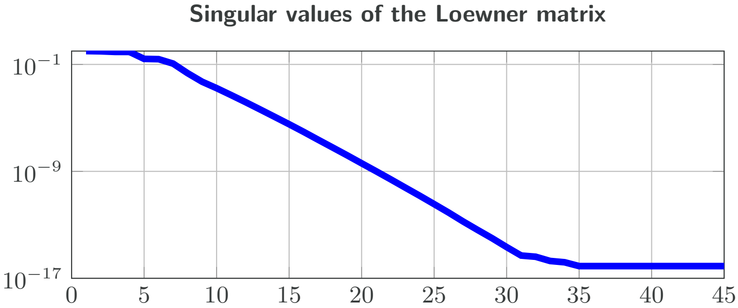

[1] morwiki.mpi-magdeburg.mpg.de/

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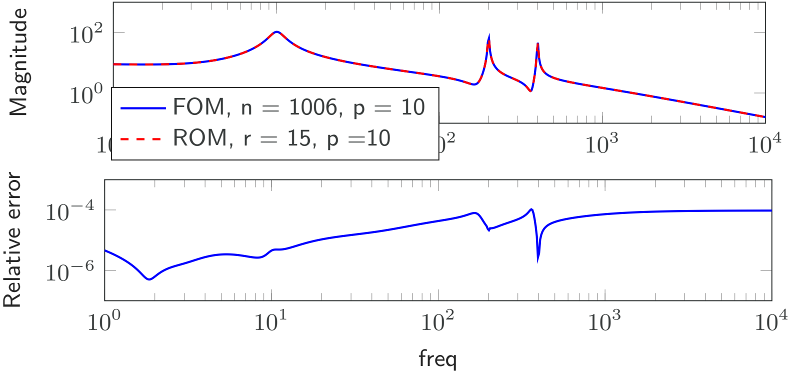

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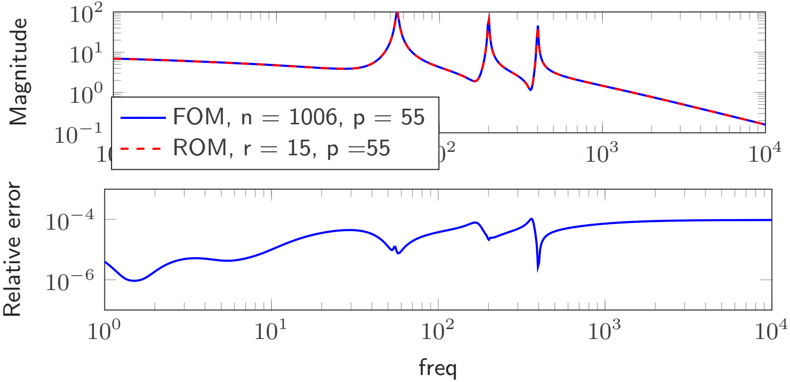

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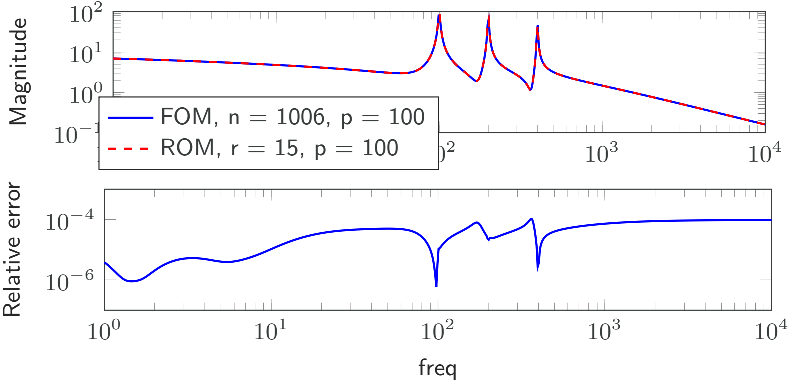

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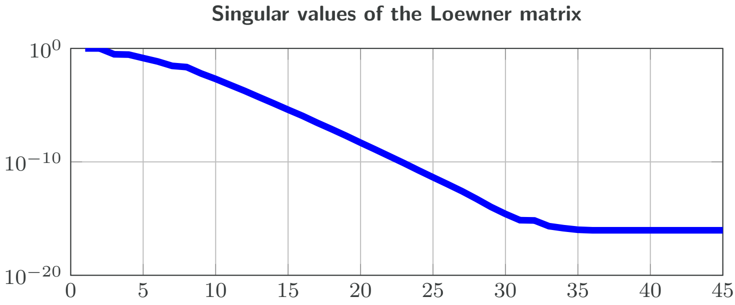

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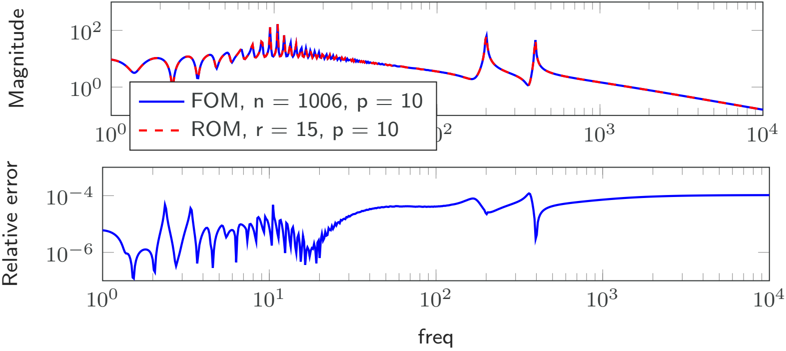

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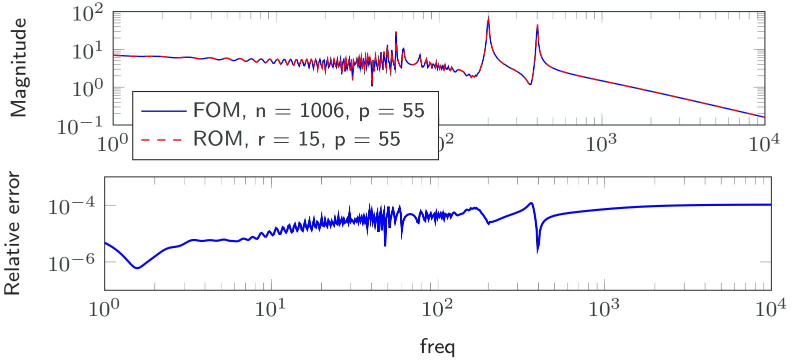

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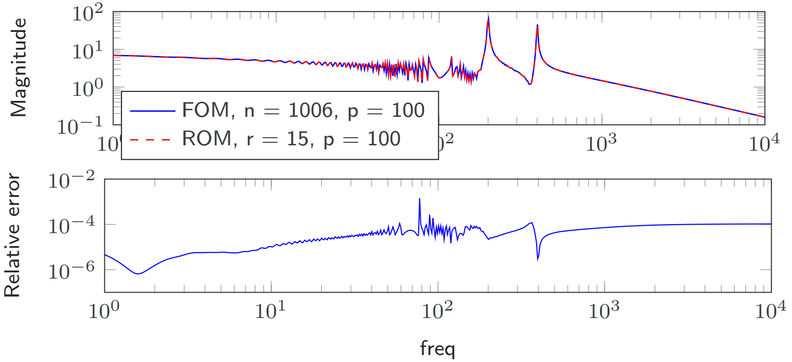

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Outline

1. Introduction
2. Minimal Realization
3. Reachability and Observability for SLS
4. Model Order Reduction
5. Numerical Results
6. Outlook and Conclusions

Contribution of this talk

- Minimal realization by projection of **SLS**.
- Model reduction technique inspired by numerical rank of matrix $\mathbf{O}^T\mathbf{E}\mathbf{R}$.
- Projector computation solving generalized Sylvester equation (low-rank methods).
- Performance illustrated by numerical examples for several system classes.
- Extended results to parametric **SLS**.

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Open questions and future work

- Stability preservation and error bounds.
- Application to real-world problems.
- Extension to nonlinear systems, first results in [Benner/Goyal '19, arXiv:1904.11891.]