MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Localized Balanced Truncation

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joint work with Xin Du, Guanghong Yang, and Dan Ye

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Overview

Outline

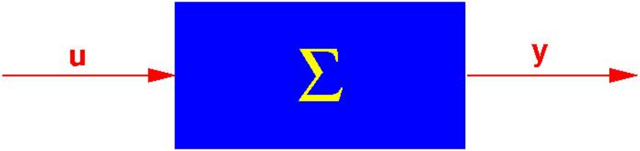
Nonlinear Control Systems

$$\Sigma : \begin{cases} E\dot{x}(t) &= f(t, x(t), u(t)), \quad Ex(t_0) = Ex_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- (generalized) states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

If E singular $\rightsquigarrow$ descriptor system. Here, $E = I_n$ for simplicity.



Original System ($E = I_n$)

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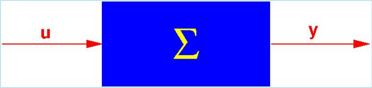

Goals:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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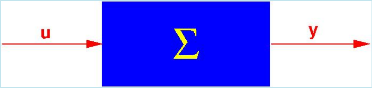

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{A}\hat{x}(t) + \widehat{B}u(t), \\ \hat{y}(t) = \widehat{C}\hat{x}(t) + \widehat{D}u(t). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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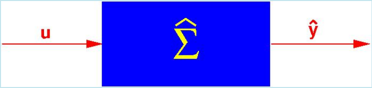

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Model Reduction for Control Systems
Linear Time-Invariant (LTI) Systems

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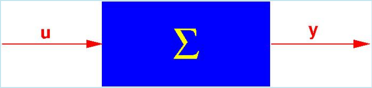

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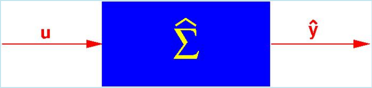

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Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s) - x(0))$ to LTI system

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$\Longrightarrow$ I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)} \Big) u(s).$$

$G(s)$ is the **transfer function** of Σ.

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Model reduction in frequency domain: Fast evaluation of mapping $u \rightarrow y$.

Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{p\times r},\ \hat{D} \in \mathbb{R}^{p\times m} \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \left\|Gu - \hat{G}u\right\| \le \left\|G - \hat{G}\right\| \cdot \|u\| < \text{tolerance} \cdot \|u\|\,.$$

$\Longrightarrow$ Approximation problem: $\displaystyle\min_{\text{order}\,(\hat{G})\le r} \left\|G - \hat{G}\right\|,$

where, mostly, $\|\,.\,\| = \|.\,\|_{\mathcal{H}_\infty}$ or $\|\,.\,\| = \|\,.\,\|_{\mathcal{H}_2}$.

- Often in engineering, only a certain frequency range, or an operating frequency is of interest.
- Local good approximation can be achieved by moment matching/rational interpolation or frequency-limited/-weighted balanced truncation, but with little theoretical support (stability preservation, error bound).

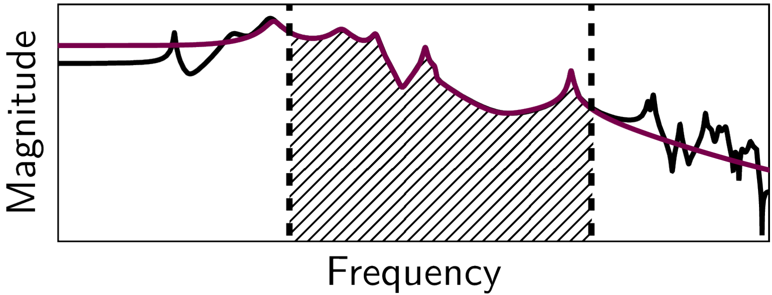

Image by Steffen Werner, produced using MORLAB.

Balanced Truncation for LTI Systems

Idea

- $\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$ with A stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$,

  is balanced, if system Gramians, i.e., solutions P, Q of Lyapunov equations

  $$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C) &\mapsto (TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right). \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}) = (A_{11}, B_1, C_1)$.

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- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

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- Adaptive choice of r via computable error bound:

$$\| G - \hat{G} \|_{\mathcal{H}_\infty} \leq 2 \sum_{k=r+1}^{n} \sigma_k =: \delta$$

$$\Rightarrow \quad \|y - \hat{y}\|_2 \leq \delta \, \|u\|_2 \, .$$

Definition (Popov function)

Given a realization (A, B, C) of an LTI system, matrices $Q = Q^T$, $R = R^T$ and S of appropriate size, the associated Popov function is

$$\Phi(s) = \begin{bmatrix} (sI_n - A)^{-1}B \\ I_m \end{bmatrix}^H \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI_n - A)^{-1}B \\ I_m \end{bmatrix}.$$

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Theorem (Kalman-Yakubovich-Popov (KYP) Lemma)

Let Σ be controllable ($\operatorname{rank}([A - \lambda I_n,\ B]) = n$ for all $\lambda \in \mathbb{C}$). Then

$$\Phi(\mathrm{i}\omega) \succcurlyeq 0 \ \forall \mathrm{i}\omega \in \mathrm{i}\mathbb{R} \setminus \Lambda(A)$$

if and only if there exists $X = X^T$ such that the linear matrix inequality (LMI)

$$\begin{bmatrix} A^T X + XA - Q & XB - S \\ B^T X - S^T & -R \end{bmatrix} \preccurlyeq 0$$

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History

- **1961:** Popov's criterion for stability of a feedback system with a memoryless nonlinearity.
- **1962/1963:** Original version of the lemma by Kalman and Yakubovich.
- **until today:** Many generalizations and extensions.

Relation to Kalman-Yakubovich-Popov (KYP) lemma

- Structural properties of reduced-order models can be proved using KYP.
- Error bound can be proved using KYP as follows:

$$E(s) = \begin{bmatrix} C & \hat{C} \end{bmatrix} \left(sI_{n+r} - \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ -\hat{B} \end{bmatrix} =: \tilde{C} \left(sI_{n+r} - \tilde{A} \right)^{-1} \tilde{B}$$

is a stable transfer function, i.e., $E \in \mathcal{H}_\infty$. Hence,

$$\|E\|_{\mathcal{H}_\infty} < \delta \quad \Longleftrightarrow \quad \Phi_\delta(\mathrm{i}\omega) \succcurlyeq 0 \ \forall \omega$$

for Popov function

$$\Phi_\delta(s) = \begin{bmatrix} (sI_{n+r} - \tilde{A})^{-1}\tilde{B} \\ I_m \end{bmatrix}^H \begin{bmatrix} -\tilde{C}^T\tilde{C} & 0 \\ 0 & \delta^2 I_m \end{bmatrix} \begin{bmatrix} (sI_{n+r} - \tilde{A})^{-1}\tilde{B} \\ I_m \end{bmatrix}.$$

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Using KYP and properties of Gramians of reduced-order model, one can prove existence of symmetric solution of corresponding LMI.

Outline

Motivation

Disadvantages of Balanced Truncation

Global error bound can be pessimistic in relevant frequency bands, e.g., in mechanical systems, often only frequencies $0 \leq 2\pi\omega \leq 1000$ (in Hz) are relevant, in VLSI design only an operating frequency, e.g., 2.6 GHz, may be of interest.

Remedies

1. Frequency-weighted BT (FWBT): aim at minimizing $\|G_o(G - \hat{G})G_i\|_{\mathcal{H}_\infty}$, where G_i, G_o are rational transfer functions, e.g., lowpass/highpass filters.

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Both approaches yield good local approximation properties, but error bounds are still global and stability preservation often requires some modifications!

The Frequency-dependent KYP Lemma

Theorem [IWASAKI/HARA '05]

Consider $G(\mathrm{i}\omega) = C(\mathrm{i}\omega I - A)^{-1}B + D$, $\varpi \in \mathbb{R}$ such that $\mathrm{i}\varpi$ is not a pole of G, and let $\Pi = \Pi^T \in \mathbb{R}^{n\times n}$. Then TFAE:

a) $\begin{bmatrix} G(\mathrm{i}\varpi) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(\mathrm{i}\varpi) \\ I \end{bmatrix} \succcurlyeq 0.$

b) There exist symmetric matrices P and $Q \succ 0$ of appropriate dimensions, satisfying

$$\begin{bmatrix} A & I \\ C & 0 \end{bmatrix} \begin{bmatrix} -Q & P + \mathrm{i}\varpi Q \\ P - \mathrm{i}\varpi Q & -\mathrm{i}\varpi^2 Q \end{bmatrix} \begin{bmatrix} A & I \\ C & 0 \end{bmatrix}^T - \begin{bmatrix} B & 0 \\ D & I \end{bmatrix} \Pi \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}^T \preccurlyeq 0.$$

Note: in standard KYP, we used $\Pi = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$.

Given $\epsilon, \varpi \in \mathbb{R}$, we define

$$\begin{aligned} \dot{x}(t) &= A_\varpi x(t) + B_\varpi u(t), \\ y(t) &= C_\varpi x(t) + D_\varpi u(t), \end{aligned}$$

by

$$\begin{aligned} A_\varpi &:= \mathrm{i}\varpi I - \epsilon((\epsilon + \mathrm{i}\varpi)I - A)^{-1}(\mathrm{i}\varpi I - A), \\ B_\varpi &:= \epsilon((\epsilon + \mathrm{i}\varpi)I - A)^{-1}B, \\ C_\varpi &:= \epsilon C((\epsilon + \mathrm{i}\varpi)I - A)^{-1}, \\ D_\varpi &:= D + C((\epsilon + \mathrm{i}\varpi I) - A)^{-1}B. \end{aligned}$$

The associated transfer function is

$$G_\varpi(s) = C_\varpi(sI - A_\varpi)^{-1}B_\varpi + D_\varpi.$$

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e) *(A, B, C, D) is a minimal realization of G $\Longrightarrow$ $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ is a minimal realization of G_ϖ.*

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$$\hat{\epsilon}_\varpi = \min_{\lambda_u \in \Lambda(A) \cap \mathbb{C}_0^+} \left\{ \frac{(\varpi - \Im(\lambda_u))^2}{\Re(\lambda_u)} + \Re(\lambda_u) \right\}.$$

c) *(A, B) controllable $\Longrightarrow$ (A_ϖ, B_ϖ) controllable.*

d) *(A, C) observable $\Longrightarrow$ (A_ϖ, C_ϖ) observable.*

e) *(A, B, C, D) is a minimal realization of G $\Longrightarrow$ $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ is a minimal realization of G_ϖ.*

f) *$G_\varpi(\mathrm{i}\varpi) = G(\mathrm{i}\varpi)$, i.e., G_ϖ is a rational interpolant of G at $\mathrm{i}\varpi$!*

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h) *$\|G_\varpi\|_{\mathcal{H}_\infty} \le \gamma_\varpi \Longrightarrow \sigma_{\max}(G(\mathrm{i}\varpi)) \le \gamma_\varpi$.*

Theorem (B)

Suppose the LTI system (A, B, C, D) is Hurwitz and minimal, and denote its controllability, observability, and balanced Gramians as P, Q, Σ, then for any ϖ-dependent extended system $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ with Gramians $P_\varpi, Q_\varpi, \Sigma_\varpi$:

a) $P \succ P_\varpi, \quad Q \succ Q_\varpi, \quad \Sigma \succ \Sigma_\varpi.$

b) $\lim_{\varepsilon \to 0} P_\varpi = 0, \quad \lim_{\varepsilon \to 0} Q_\varpi = 0, \quad \lim_{\varepsilon \to 0} \Sigma_\varpi = 0.$

c) $\lim_{\varepsilon \to \infty} P_\varpi = P, \quad \lim_{\varepsilon \to \infty} Q_\varpi = Q, \quad \lim_{\varepsilon \to \infty} \Sigma_\varpi = \Sigma.$

Frequency-dependent Balanced Truncation (FDBT)

Apply the generic balancing procedure to $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$, i.e., solve

$$A_\varpi P_\varpi + P_\varpi A_\varpi^H + B_\varpi B_\varpi^H = 0, \quad A_\varpi^H Q_\varpi + Q_\varpi A_\varpi + C_\varpi^H C_\varpi = 0,$$

and compute the balancing transformation T_ϖ so that

$$T_\varpi P_\varpi T_\varpi^H = T_\varpi^{-H} Q_\varpi T_\varpi^{-1} = \Sigma_\varpi = \mathrm{diag}\left(\sigma_{\varpi,1}, \ldots, \sigma_{\varpi,n}\right), \quad \text{with } \sigma_{\varpi,k} \geq \sigma_{\varpi,k+1}.$$

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Balance the system:

$$\begin{aligned}(T_\varpi A_\varpi T_\varpi^{-1}, T_\varpi B_\varpi, C_\varpi T_\varpi^{-1}, D_\varpi) \\ = \left(\begin{bmatrix} A_{\varpi,11} & A_{\varpi,12} \\ A_{\varpi,21} & A_{\varpi,22} \end{bmatrix}, \begin{bmatrix} B_{\varpi,1} \\ B_{\varpi,2} \end{bmatrix}, \begin{bmatrix} C_{\varpi,1} & C_{\varpi,2} \end{bmatrix}, D_\varpi \right).\end{aligned}$$

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Reduced-order model is then obtained by truncation and back transformation: select r such that $\sigma_{\varpi,r} > \sigma_{\varpi,r+1}$ and set

$$
\begin{aligned}
\hat{A} &= \mathrm{i}\varpi I_r - \epsilon(\mathrm{i}\varpi I_r - A_{\varpi,11})\left((\epsilon - \mathrm{i}\varpi)I_r + A_{\varpi,11}\right)^{-1}, \\
\hat{B} &= \frac{1}{\epsilon}((\epsilon + \mathrm{i}\varpi)I_r - \hat{A})B_{\varpi,1}, \\
\hat{C} &= \frac{1}{\epsilon}C_{\varpi,1}((\epsilon + \mathrm{i}\varpi)I_r - \hat{A}), \\
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Theorem (Local Error Bound)

The reduced-order transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$ satisfies:

$$\sigma_{\max}\left(G(\mathrm{i}\varpi) - \hat{G}(\mathrm{i}\varpi)\right) \leq 2\sum_{k=r+1}^{n} \sigma_{\varpi,k}.$$

Proof: use proof for BT error bound based on FD-KYP instead of KYP.

Outline

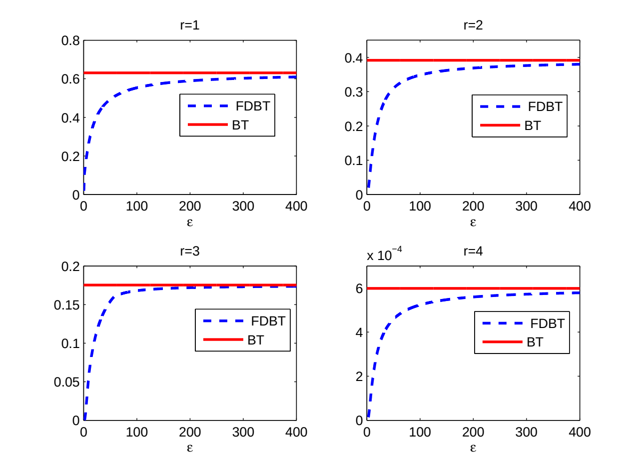
Simple example of electronic circuit from [Sorensen '05]

- input $\equiv$ voltage u, output $\equiv$ current y,
- scaled inductances, capacities, and resistance:
  $L_j = 1$, $C_j = 1$ for all j; $R_1 = 0.5$, $R_2 = 0.2$.
- $n = 5$, $m = p = 1$.

Comparison of FDBT and BT ($\varpi = 0,\ \varepsilon = 1$)

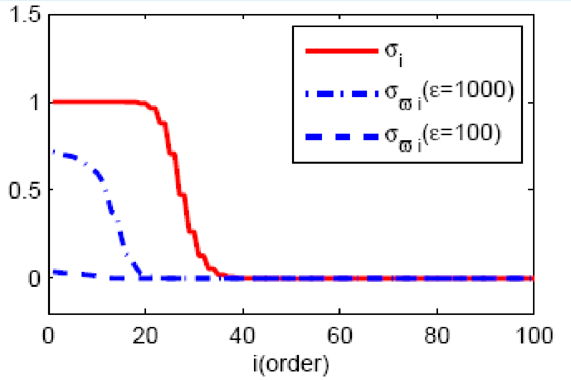
r	FDBT		BT	
	bound	true error	bound	true error
4	1.2201×10^{-7}	1.2201×10^{-7}	0.0006	0.0006
3	8.7426×10^{-5}	8.7182×10^{-5}	0.1752	0.1740
2	5.5028×10^{-4}	3.7568×10^{-4}	0.3914	0.0421
1	0.0584	0.0582	0.6311	0.1975

Comparison of FDBT and BT ($\bar{\omega} = 0$, varying ε)



Bandstop filter `[A,B,C,D]= butter(50,[90 110],'stop','s')`

Hankel singular values



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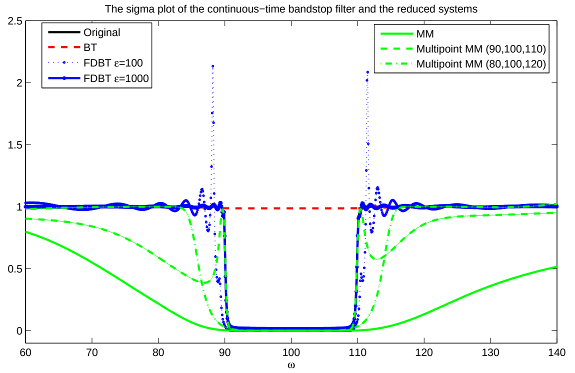
Transfer functions



Outline

Conclusions and Future Work

Summary:

- Relations of KYP lemma to balanced truncation.
- Frequency-dependent KYP lemma suggests new frequency-dependent balanced truncation (FDBT) method.
- FDBT offers alternative to interpolation-based method if good local approximation quality is desired.
- Continuous- and discrete-time FDBT derived.
- FDBT is stability preserving and has local error bound, which is often much better than global BT bound.
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Summary:

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- Continuous- and discrete-time FDBT derived.
- FDBT is stability preserving and has local error bound, which is often much better than global BT bound.
- Not shown: computational feasible method also for frequency bands.

Future work:

- MORLAB implementation.
- Details for non-minimal systems.
- Large-scale implementation and testing.
- Extension to descriptor systems.

References

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M R O LAB

https://www.mpi-magdeburg.mpg.de/projects/morlab

DOI: 10.5281/zenodo.1574083

See MORLAB poster by Steffen Werner!