MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

SYSTEM-THEORETIC METHODS FOR LINEAR AND NONLINEAR MOR

Part I: Linear Systems

Peter Benner

Model Order Reduction Summer School 2020
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1. Introduction
2. Balanced Truncation
3. Balancing-Related Model Reduction

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
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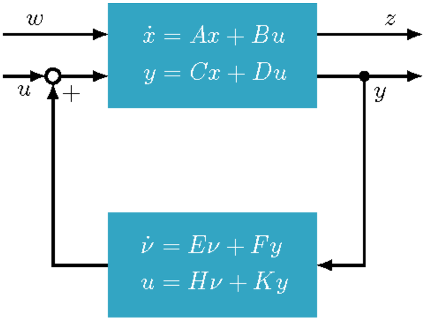
Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



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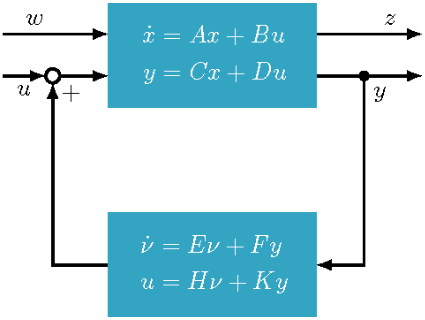

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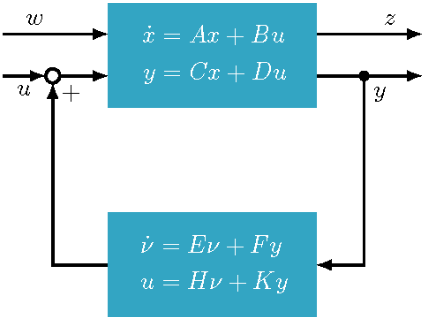

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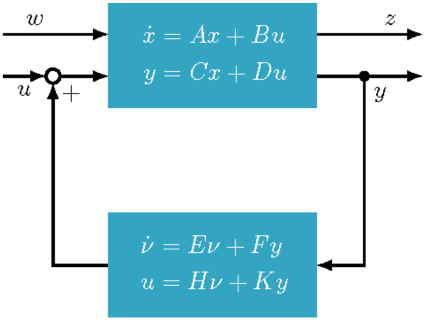

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Standard MOR techniques in systems and control: balanced truncation and related methods.

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Application Areas

Micro Electronics/Circuit Simulation

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- Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when
  - decoupling large linear subcircuits,
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  - modeling pin packages in VLSI chips,
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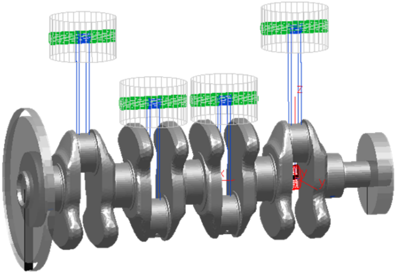
Standard MOR techniques in circuit simulation: Krylov subspace / Padé approximation / rational interpolation methods.

Application Areas

Structural Mechanics / Finite Element Modeling



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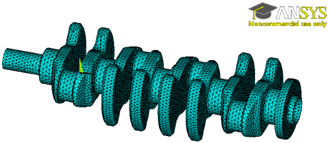

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- Analysis of elastic deformations requires many simulation runs for varying external forces.

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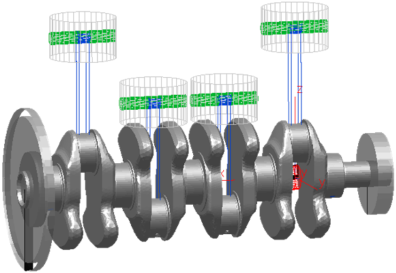

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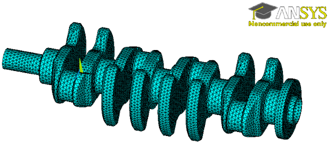

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- Analysis of elastic deformations requires many simulation runs for varying external forces.

Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) $\rightsquigarrow$ Craig-Bampton method.

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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Model Reduction for Dynamical Systems

Original System

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Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear, Time-Invariant (LTI) Systems

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State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t \in \mathbb{R}.$$

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- Problem: in general, $\mathcal{S}$ does not have a discrete SVD and can therefore not be approximated as in the matrix case!

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Alternative to State-Space Operator: Hankel operator

Instead of

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$\Longrightarrow$ SVD-type approximation of $\mathcal{H}$ possible!

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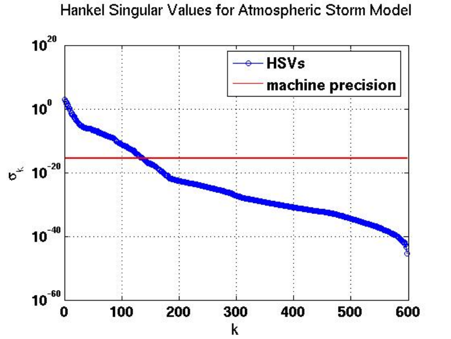

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But: computationally challenging for large-scale systems.

Recent progress in [B./Werner 2020].

Linear Systems in Frequency Domain

Linear, Time-Invariant (LTI) Systems

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Assumptions: $t_0 = 0$, $x_0 = x(0) = 0$.

Laplace Transform / Frequency Domain

Application of Laplace transform

$$\mathcal{L} : x(t) \mapsto x(s) = \int_0^\infty e^{-st} x(t)\, dt \quad (\Rightarrow \; \dot{x}(t) \mapsto sx(s))$$

with $s \in \mathbb{C}$ leads to linear system of equations:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s).$$

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Laplace Transform / Frequency Domain

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s)$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=: G(s)} \Big) u(s) = G(s)u(s).$$

G is the transfer function of Σ, $G : \mathcal{L}_2^m \to \mathcal{L}_2^p \quad (\mathcal{L}_2 := \mathcal{L}(L_2(-\infty, \infty)))$.

Approximation Problem

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \qquad \hat{A} \in \mathbb{R}^{r\times r}, \quad \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \qquad \hat{C} \in \mathbb{R}^{p\times r}, \quad \hat{D} \in \mathbb{R}^{p\times m}. \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \left\|Gu - \hat{G}u\right\| \leq \left\|G - \hat{G}\right\| \|u\| \leq \text{tolerance} \cdot \|u\| \,.$$

Model Reduction as Approximation Problem

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$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G}) \leq r} \left\| G - \hat{G} \right\|$.

Definition

A linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \mid \Re(z) < 0\}$.

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Lemma

Sufficient for asymptotic stability is that A is asymptotically stable (or Hurwitz), i.e., the spectrum of A, denoted by $\Lambda(A)$, satisfies $\Lambda(A) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

Definition

$\dot{x} = Ax + Bu$ is controllable (reachable) iff for all $x_0, x_1 \in \mathbb{R}^n$ and $t_1 > 0$ there exists feasible control function u s.t.

$$x(t_1; u, x_0) := e^{At_1}x^0 + \int_0^{t_1} e^{A(t_1-t)}Bu(t)\,dt = x_1.$$

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Theorem

For an LTI system defined by $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, T.F.A.E.:

a) *The LTI system $\dot{x} = Ax + Bu$ is controllable (reachable).*

b) *The finite time Gramian $P(0, t_1) = \int_0^{t_1} e^{-At}BB^Te^{-A^Tt}dt$ is spd[a] $\forall\ t_1 > 0$.*

c) *The controllability matrix $K(A, B) := [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times n\cdot m}$ has full rank n.*

[a] spd = symmetric positive definite

The Gramian characterization of controllability for stable systems can be based on positive definiteness of the (infinite) controllability Gramian

$$P := \int_0^\infty e^{As} B B^T e^{A^T s} ds.$$

using congruence of $P(0, t_1)$ to $\int\limits_0^{t_1} e^{As} B B^T e^{A^T s} ds$ and taking the limit $t_1 \to \infty$.

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Theorem

For a stable LTI system defined by $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$, T.F.A.E.:

a) *The LTI system $\dot{x} = Ax + Bu$ is controllable.*

b) *The controllability Gramian P is positive definite.*

Definition (Observability)

An LTI system is reconstructible (observable) if for solution trajectories $x(t), \tilde{x}(t)$ obtained with the same input function u, we have

$$\begin{aligned} & y(t) = \tilde{y}(t) \quad \forall t \leq 0 \quad (\forall t \geq 0) \\ \Longrightarrow \quad & x(t) = \tilde{x}(t) \quad \forall t \leq 0 \quad (\forall t \geq 0). \end{aligned}$$

Characterization of observability/reconstructibility:

Theorem (Duality)

An LTI system is reconstructible if and only if the dual system $\dot{x}(t) = -A^T x(t) - C^T u(t)$ is controllable.

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For an LTI system defined by $(A, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times n}$, *T.F.A.E.:*

a) *The LTI system is reconstructable.*

b) *The LTI system is observable.*

c) *The observability matrix*

$$\mathcal{O}(A,C) = \left[C^T, A^T C^T, (A^2)^T C, \ldots, (A^{n-1})^T C^T\right]^T \in \mathbb{R}^{np\times n} \text{ has } \mathrm{rank}\ n.$$

Characterization of observability/reconstructibility:

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Theorem

A stable LTI system is observable if and only if the observability Gramian

$$Q := \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

is symmetric positive definite.

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &=& Ax(t) + Bu(t), \\ y(t) &=& Cx(t) + Du(t), \end{cases} \quad \begin{array}{l} \text{with transfer function} \\ G(s) = C(sI - A)^{-1}B + D, \end{array}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ is called a realization of Σ.

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Realizations are not unique!

Transfer function is invariant under state-space transformations,

$$\mathcal{T} : \begin{cases} x & \rightarrow \quad Tx, \\ (A, B, C, D) & \rightarrow \quad (TAT^{-1}, TB, CT^{-1}, D), \end{cases}$$

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Realizations are not unique!

Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t),$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t),$$

for arbitrary $A_j \in \mathbb{R}^{n_j\times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1\times m}$, $C_2 \in \mathbb{R}^{q\times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

Definition

For a linear (time-invariant) system

$$\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with transfer function} \\ y(t) = Cx(t) + Du(t), & G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ is called a realization of Σ.

Realizations are not unique!

Hence,

$$(A, B, C, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ!

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the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ is called a realization of Σ.

Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

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A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Theorem

A realization (A, B, C, D) of a linear system is minimal $\Longleftrightarrow$
(A, B) is controllable and (A, C) is observable.

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1\text{).}$$

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When does a balanced realization exist?

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When does a balanced realization exist?

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$ (i.e, $\Lambda(A) \subset \mathbb{C}^-$), a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U\Sigma V^T$ is the SVD of SR^T.

Proof. Exercise.

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$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

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Theorem

The infinite controllability/observability Gramians P/Q satisfy the Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

Proof. Exercise!

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Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

Proof. Exercise.

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Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\operatorname{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u^*(\jmath\omega) u(\jmath\omega)\, d\omega.$$

Assume A is (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \operatorname{re} z < 0\}$.

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Then, the 2-induced operator norm

$$\|G\|_\infty := \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2}$$

is well defined. It can be shown that

$$\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\,(G(\jmath\omega))\,.$$

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Hardy space $\mathcal{H}_\infty$

Function space of analytic and bounded (in $\mathbb{C}^+$) matrix-/scalar-valued functions.
The $\mathcal{H}_\infty$-norm is

$$\|F\|_\infty := \sup_{\operatorname{re} s > 0} \sigma_{\max}\,(F(s)) = \sup_{\omega \in \mathbb{R}} \sigma_{max}\,(F(\jmath\omega))\,.$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = p = 1$);
- $\mathcal{H}_\infty^{p \times m}$ in the MIMO case (multi-input, multi-output, $m > 1, p > 1$).

Consider the transfer function

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Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty,\infty) \cong \mathcal{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

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$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_2 = \left\|Gu - \hat{G}u\right\|_2 \leq \left\|G - \hat{G}\right\|_\infty \|u\|_2\,.$$

$\Longrightarrow$ compute reduced-order model such that $\left\|G - \hat{G}\right\|_\infty < tol$!

Note: error bound holds in time- and frequency domain due to Paley-Wiener!

Consider transfer function $G(s) = C\,(sI - A)^{-1}\,B$, i.e. $D = 0$.

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic in $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\begin{aligned} \|F\|_2 \quad &:= \quad \left(\sup_{\mathrm{re}\,\sigma > 0} \int_{-\infty}^{\infty} \|F(\sigma + \jmath\omega)\|_F^2 \; d\omega \right)^{\frac{1}{2}} \\ &= \quad \left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2 \; d\omega \right)^{\frac{1}{2}}. \end{aligned}$$

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$$\|F\|_2 \quad = \quad \left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2 \; d\omega\right)^{\frac{1}{2}}.$$

$\mathcal{H}_2$ approximation error for impulse response $(u(t) = u_0\delta(t))$

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_2 = \left\|Gu_0\delta - \hat{G}\,u_0\delta\right\|_2 \leq \left\|G - \hat{G}\right\|_2 \|u_0\|.$$

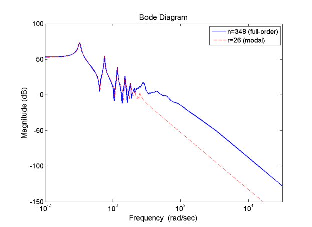
$\Longrightarrow$ compute reduced-order model such that $\left\|G - \hat{G}\right\|_2 < tol$!

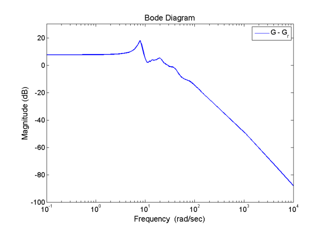
$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory)

Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\left\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right\|_2$, $\left\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right\|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\left\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right\|_2}{\|G(\jmath\omega_j)\|_2}$, $\frac{\left\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right\|_\infty}{\|G(\jmath\omega_j)\|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot:
  - for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}(\text{value})$;
  - for MIMO systems, $p \times m$ array of of plots G_{ij}.





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- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

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  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

Model Reduction by Projection

Methods using transfer function concepts:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. ...

Joint feature of system-theoretic methods:
computation of reduced-order model (ROM) by projection.

Model Reduction by Projection

**Joint feature of system-theoretic methods:
computation of reduced-order model (ROM) by projection.**

Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, where

$$\operatorname{range}(V) = \mathcal{V}, \quad \operatorname{range}(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T A V, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Model Reduction by Projection

Joint feature of system-theoretic methods:
computation of reduced-order model (ROM) by projection.

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Basic principle:

- Recall: LTI system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

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$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

Motivation:

- HSVs are system invariants: they are preserved under

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In balanced coordinates ... energy transfer from u_- to y_+:

$$E := \sup_{\substack{u \in L_2(-\infty, 0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\, dt}{\int_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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$\Longrightarrow$ Truncate states corresponding to "small" HSVs

$\Longrightarrow$ complete analogy to best approximation via SVD!

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$\Longrightarrow VW^T$ is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

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- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum_{k=r+1}^{n} \sigma_k\right) \|u\|_2 .$$

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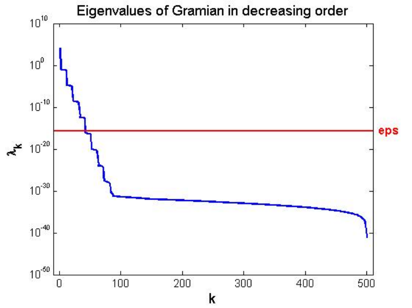
Use low-rank techniques ideas from numerical linear algebra:

- Instead of Gramians P, Q compute $S, R \in \mathbb{R}^{n\times k}$, $k \ll n$, such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

- Compute S, R with problem-specific Lyapunov solvers of "low" complexity directly.



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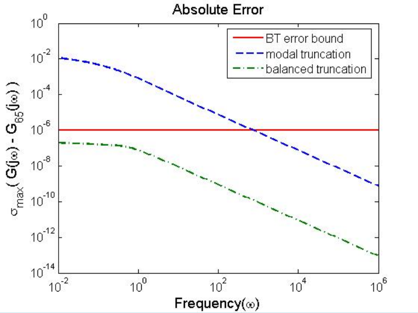
Use low-rank techniques ideas from numerical linear algebra:

Balanced Truncation Software:

- Implementation using sparse Lyapunov solver ($\rightarrow$ ADI+sparse LU).
- Complexity $\mathcal{O}(n(k^2 + r^2))$.
- Software:
  - + MORLAB (Model Reduction Laboratory) [B. 2006, B./Werner 2020],
  - + MATLAB toolbox LyaPack (Penzl 1999),
  - + Software library M.E.S.S. (Matrix Equation Sparse Solvers) in C/MATLAB [B./Saak/Köhler/uvm.],
  - + pyMOR [Milk/Mlinarić/Rave/Schindler/uvm.]

$n = 1357$, Absolute Error



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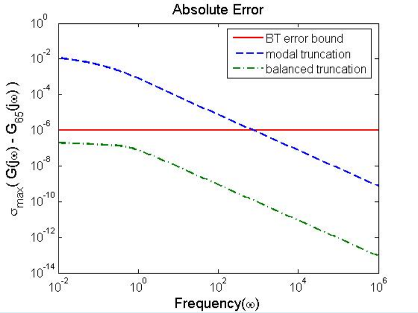

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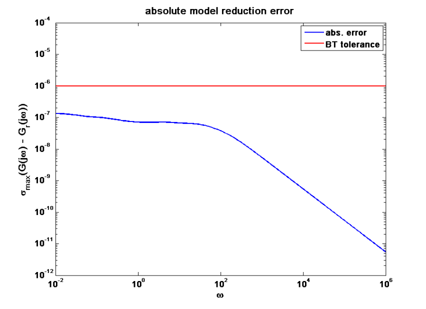
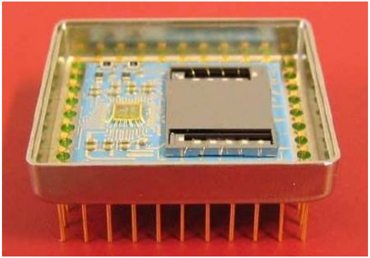
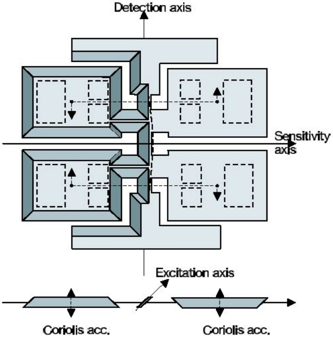
$n = 79841$, **Absolute Error**



- BT model computed using M.E.S.S. in MATLAB,
- computation time on current notebook: 35sec.



- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.

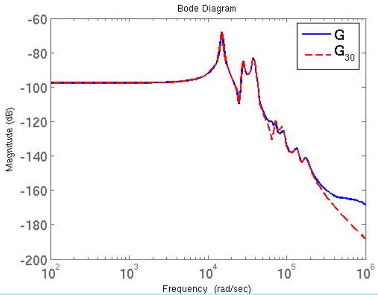


- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

Source: http://modelreduction.org/index.php/Modified_Gyroscope

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722,\ m = 1,\ p = 12.$
- Reduced-order model: $r = 30$.

Frequency Repsonse Analysis



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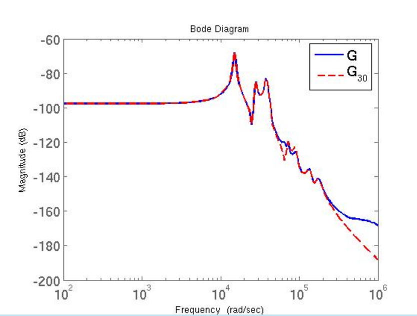
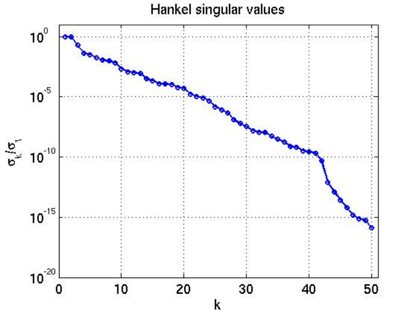


Hankel Singular Values



Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) [Mullis/Roberts 1976, Moore 1981]

- $P =$ controllability Gramian of system given by (A, B, C, D).
- $Q =$ observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

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LQG Balanced Truncation (LQGBT) [Jonckheere/Silverman 1983]

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

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Balanced Stochastic Truncation (BST)

[Desai/Pal 1984, Green 1988]

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT)

[Green '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^TQ + Q\bar{A} + QB\bar{R}^{-1}B^TQ + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

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Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

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- Computable error bounds, e.g.,

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\text{BT:} \quad \|G - G_r\|_\infty &\le 2\sum_{j=r+1}^{n} \sigma_j^{BT}, \\
\text{LQGBT:} \quad \|G - G_r\|_\infty &\le 2\sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}} \\
\text{BST:} \quad \|G - G_r\|_\infty &\le \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1\right)\|G\|_\infty ,
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- Can be combined with singular perturbation approximation for improved steady-state performance.
- Computations can be modularized $\rightsquigarrow$ software packages M-M.E.S.S., MORLAB, see http://www.mpi-magdeburg.mpg.de/823508/software.

Related Topics

- Special methods for second-order (mechanical) and delay systems.
- Extensions to bilinear, quadratic-bilinear, polynomial ($\rightsquigarrow$ Part II), and stochastic systems.
- Empirical variants using snapshots.
- MOR methods for discrete-time systems.
- Extensions to descriptor systems $E\dot{x} = Ax + Bu$, E singular.
- Parametric model reduction:

  $$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x,$$

  where $p \in \mathbb{R}^d$ is a free parameter vector; parameters should be preserved in the reduced-order model.

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