MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

LEARNING COMPACT DYNAMICAL MODELS FROM DATA

From Projection-based to Data-driven Model Order Reduction

Peter Benner

MathCoRe Seminar
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Supported by:

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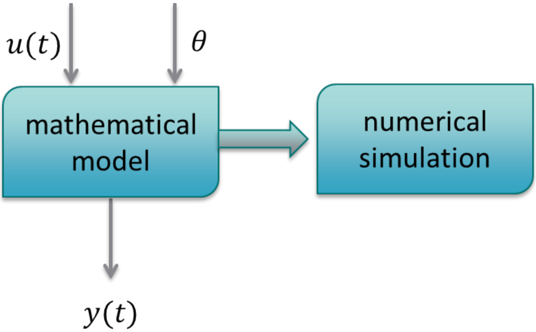
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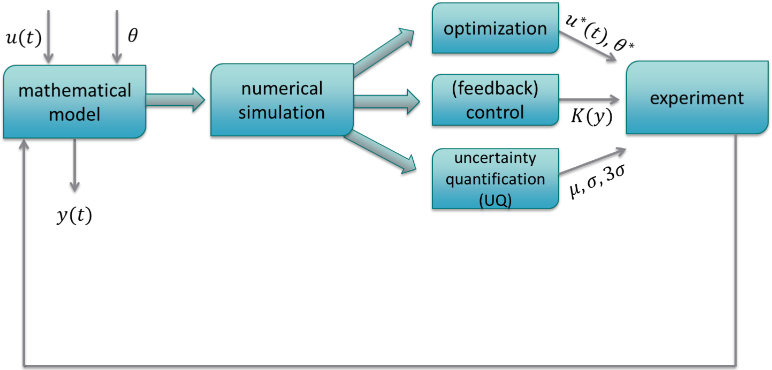
θ
mathematical model
numerical simulation
$y(t)$



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$\mu, \sigma, 3\sigma$
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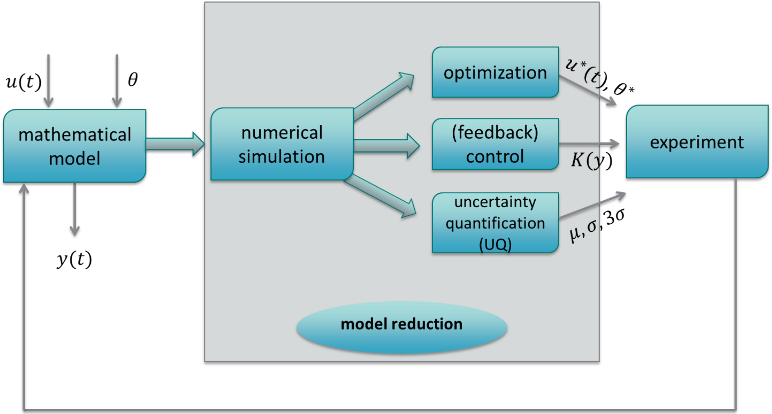
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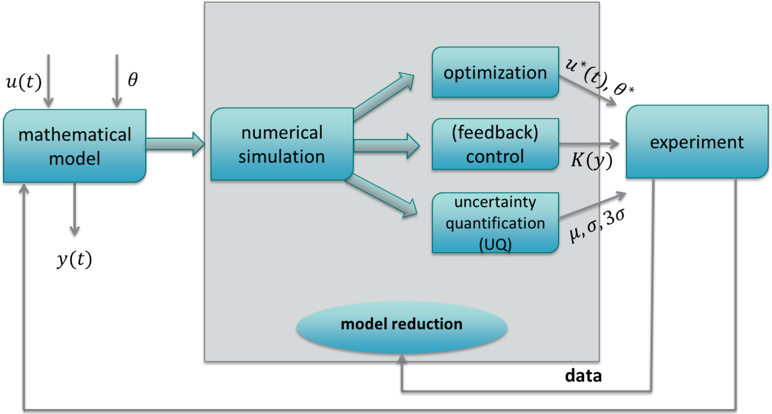
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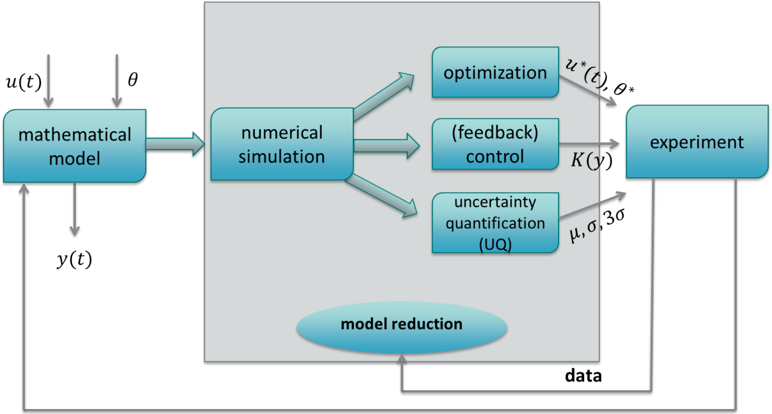
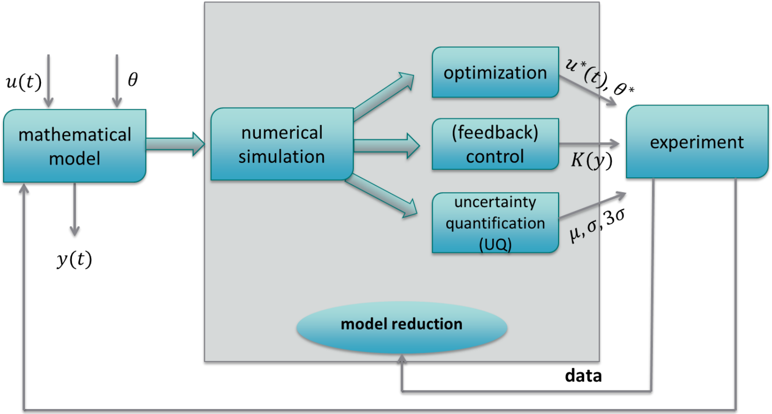


Goal: *Use all acquired knowledge about the model during the CSE process chain in the design of the reduced-order model, including experimental data.*



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$\rightsquigarrow$ *Data-enhanced model reduction methods.*

Outline

1. Model Order Reduction of Dynamical Systems

2. Data-driven/-enhanced Model Reduction

Original System

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(t, x(t), u(t)), \\ y(t) & = & g(t, x(t), u(t)), \end{array} \right.$$

- states $x(t) \in \mathbb{R}^n$,
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Reduced-Order Model (ROM)

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$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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Secondary goal: reconstruct approximation of x from $\hat{x}$.

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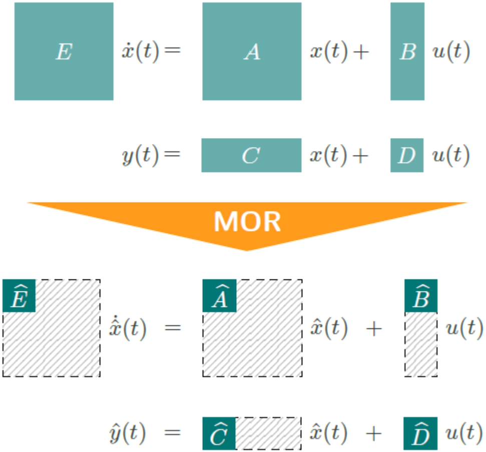

- $E, A \in \mathbb{R}^{n\times n}$
- $B \in \mathbb{R}^{n\times m}$
- $C \in \mathbb{R}^{p\times n}$
- $D \in \mathbb{R}^{p\times m}$

- $\hat{E}, \hat{A} \in \mathbb{R}^{r\times r}$
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Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sX(s) - x(0))$ to LTI system

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Model reduction in frequency domain: Fast evaluation of mapping $U \rightarrow Y$.

Formulating model reduction in frequency domain

Approximate the time domain dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, & C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{p\times r},\ \hat{D} \in \mathbb{R}^{p\times m} \end{aligned}$$

of order $r \ll n$, such that

$$\begin{aligned} ||y - \hat{y}|| \simeq \left|\left|Y - \hat{Y}\right|\right| &= \left|\left|\mathbf{H}U - \hat{\mathbf{H}}U\right|\right| \\ &\leq \left|\left|\mathbf{H} - \hat{\mathbf{H}}\right|\right| \cdot ||U|| \simeq \left|\left|\mathbf{H} - \hat{\mathbf{H}}\right|\right| \cdot ||u|| \\ &\leq \text{tolerance} \cdot ||u||\,. \end{aligned}$$

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Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x} = VW^T x =: \tilde{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|\,.$$

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Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T\left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) \quad = \quad W^T\left(VW^T\dot{x} - AVW^T x - Bu\right)$$

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MOR Methods Based on Projection

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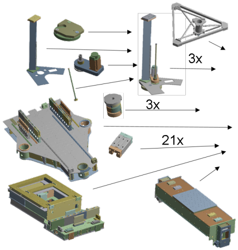
Classes of Projection-based MOR Methods

1. Modal Truncation
2. Rational Interpolation / Moment Matching
   (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. Proper Orthogonal Decomposition (POD) / Principal Component Analysis (PCA)
5. Reduced Basis Method
6. ...

MOR projects in Phase I of GRK 2297/1 "MathCoRe" are mostly based on projection:

Ph.D. projects of Shaimaa Monem, Steffen Werner, Jennifer Przybilla.

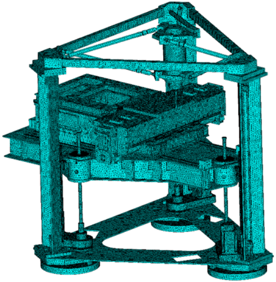
50 subassemblies



CAD model

$\overset{FEM}{\rightsquigarrow}$

FE-Model: 1.2M DOFs



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FE-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	196	6.5h
BTX0	174	4.5h

output-coupled

method	red. order tol 10^{-3}	t_{red}
2phase	3005	2h
BTX0	2515	1.8h

MAX: Results considering an inhomogeneous initial condition $T_0 \neq 0$

Results by Julia Vettermann (MiIT, TUC)

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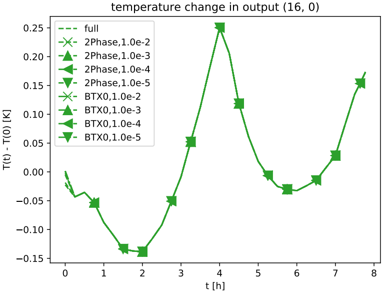
→ Required storage for reduced matrices just 1MB!



Vettermann, J., Sauerzapf, S., Naumann, A., Beitelschmidt, M., Herzog, R., Benner, P., Saak, J. (2020): Model order reduction methods for coupled machine tool models. Submitted.

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This is often impossible!

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$\rightsquigarrow$ New Ph.D. projects in Phase II of GRK 2297/1 "MathCoRe" (Yevgeniya Filanova, ...).

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Some methods:

- System identification (incl. ERA, N4SID, MOESP): frequency and time domain
  [Ho/Kalman 1966; Ljung 1987/1999; Van Overschee/De Moor 1994; Verhaegen 1994; De Wilde, Eykhoff, Moonen, Sima, …]

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- System identification (incl. ERA, N4SID, MOESP): frequency and time domain
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- System identification tries to infer discrete linear time-invariant (LTI) systems

$$x_{k+1} = Ax_k + Bu_k + Kw_k,$$
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- Continuous-time system can be identified, e.g., by "inverse" Euler method.
- Many extensions to nonlinear systems, imposing certain structural assumptions, including artificial neural networks...

A paper from 1990...

4 IEEE TRANSACTIONS ON NEURAL NETWORKS, VOL. 1, NO. 1, MARCH 1990

Identification and Control of Dynamical Systems Using Neural Networks

KUMPATI S. NARENDRA FELLOW, IEEE, AND KANNAN PARTHASARATHY

Abstract—The paper demonstrates that neural networks can be used effectively for the identification and control of nonlinear dynamical systems. The emphasis of the paper is on models for both identification and control. Static and dynamic back-propagation methods for the adjustment of parameters are discussed. In the models that are introduced, multilayer and recurrent networks are interconnected in novel configurations and hence there is a real need to study them in a unified fashion. Simulation results reveal that the identification and adaptive control schemes suggested are practically feasible. Basic concepts and definitions are introduced throughout the paper, and theoretical questions which have to be addressed are also described.

are well known for such systems [1]. In this paper our interest is in the identification and control of nonlinear dynamic plants using neural networks. Since very few results exist in nonlinear systems theory which can be directly applied, considerable care has to be exercised in the statement of the problems, the choice of the identifier and controller structures, as well as the generation of adaptive laws for the adjustment of the parameters.

Two classes of neural networks which have received considerable attention in the area of artificial neural net-

Narendra, K.S., Parthasarathy, K. (1990): Identification and control of dynamical systems using neural networks. IEEE Transactions on Neural Networks 1(1):4–27.

A paper from 1990...

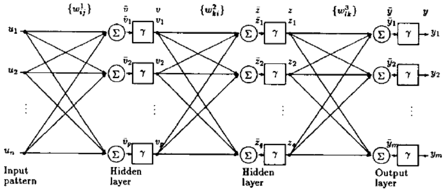

Fig. 2. A three layer neural network.

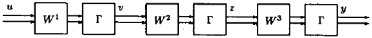

Fig. 3. A block diagram representation of a three layer network.

Narendra, K.S., Parthasarathy, K. (1990): Identification and control of dynamical systems using neural networks. IEEE Transactions on Neural Networks 1(1):4–27.

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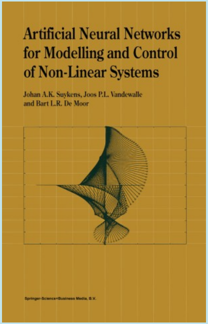

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Suykens, J.A.K., Vandewalle, J.P.L., de Moor, B.L. (1996): Artificial Neural Networks for Modelling and Control of Non-Linear Systems. Springer US.

Given a smooth dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n.$$

Take snapshots $x_k := x(t_k)$ on grid $t_k := kh$ for $k = 0, 1, \ldots, K$ and fixed $h > 0$ (using simulation software, or measurements from real life experiment $\rightsquigarrow$ nonintrusive!), and find "best possible" A_* such that

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Motivation: Koopman theory

- $\exists$ a linear, infinite-dimensional operator describing the evolution of $f(x(\cdot))$ in an appropriate function space setting.
- Can be considered as lifting of a finite-dimensional, nonlinear problem to a infinite-dimensional, linear problem.

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Basic DMD Algorithm

Set $X_0 := [\, x_0, x_1, \ldots, x_{K-1} \,] \in \mathbb{R}^{n \times K}$, $X_1 := [\, x_1, x_2, \ldots, x_K \,] \in \mathbb{R}^{n \times K}$ and note that $X_1 = AX_0$ is desired $\rightsquigarrow$ over-/underdetermined linear system, solved by linear least-squares problem (regression):

$$A_* := \arg\min_{A \in \mathbb{R}^{n \times n}} \|X_1 - AX_0\|_F + \beta \|A\|_q$$

with a potential regularization term choosing $\beta > 0$, $q = 0, 1, 2$.

Computation usually via singular value decomposition (SVD), many variants.

Given a smooth control system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n, \qquad y(t) = g(x(t), u(t)),$$

with control $u(t) \in \mathbb{R}^m$ and output $y(t) \in \mathbb{R}^p$.

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Take state, control, and output snapshots

$$x_k := x(t_k), \quad u_k := u(t_k), \quad y_k := y(t_k), \qquad k = 0, 1, \ldots, K$$

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Basic ioDMD Algorithm ($\equiv$ N4SID)

Let $\mathbb{S} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$. Set X_0, X_1 as before and

$$U_0 := [\, u_0, u_1, \ldots, u_{K-1} \,] \in \mathbb{R}^{m\times K}, \qquad Y_0 := [\, y_0, y_1, \ldots, y_{K-1} \,] \in \mathbb{R}^{p\times K}.$$

Solve the linear least-squares problem (regression):

$$(A_*, B_*, C_*, D_*) := \arg\min_{(A,B,C,D)\in\mathbb{S}} \left\| \begin{bmatrix} X_1 \\ Y_0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} \right\|_F + \beta \| [\, A \; B \; C \; D \,] \|_q$$

with a potential regularization term choosing $\beta > 0$, $q = 0, 1, 2$.

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Gosea, I.V., Pontes Duff, I. (2020): Toward fitting structured nonlinear systems by means of dynamic mode decomposition. arXiv:2003.06484.

Mauroy, A., Mezić, I., Susuki, Y., eds., (2020): The Koopman Operator in Systems and Control. Concepts, Methodologies, and Applications. LNCIS 484, Springer, Cham.

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Can be combined with ioDMD to obtain reduced-order LTI system.

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where $P \otimes Q := [p_{ij}Q]_{ij}$ denotes the Kronecker (tensor) product, from data

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- Use compressed trajectories (via POD / PCA) $\rightsquigarrow \hat{X}$.
- Compress snapshot matrix of time derivatives: if residuals $f(t_j, u_j)$ are available

  $$\dot{\hat{X}} := [\, \dot{x}(0), \dot{x}(t_1), \ldots, \dot{x}(t_K) \,] \approx [\, f(x_0, u_0), f(x_1, u_1), \ldots, f(x_K, u_K) \,] \in \mathbb{R}^{n \times (K+1)},$$

  otherwise, approximate time-derivatives by finite differences $\rightsquigarrow \dot{\hat{X}}$.

Basic idea: apply compressive ioDMD in continuous-time setting,

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n,$$

and impose a nonlinear structure.

Here: try to infer quadratic system

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{H}\left(\hat{x}(t) \otimes \hat{x}(t)\right) + \hat{B}u(t),$$

where $P \otimes Q := \left[p_{ij}Q\right]_{ij}$ denotes the Kronecker (tensor) product, from data

$$X := [\, x_0, x_1, \ldots, x_K \,] \in \mathbb{R}^{n \times (K+1)}, \quad U := [\, u_0, u_1, \ldots, u_K \,] \in \mathbb{R}^{m \times (K+1)}.$$

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  otherwise, approximate time-derivatives by finite differences $\rightsquigarrow \dot{\hat{X}}$.
- Solve the linear least-squares problem (regression):

  $$(\hat{A}_*, \hat{H}_*, \hat{B}_*) := \arg\min_{(\hat{A},\hat{H},\hat{B})} \left\| \dot{\hat{X}} - \begin{bmatrix} \hat{A} & \hat{H} & \hat{B} \end{bmatrix} \begin{bmatrix} \hat{X} \\ \widehat{X^2} \\ U \end{bmatrix} \right\|_F + \beta \left\| \begin{bmatrix} \hat{A}\ \hat{H}\ \hat{B} \end{bmatrix} \right\|_q$$

  with potential regularization as before and $\widehat{X^2} := [x_0 \otimes x_0, \ldots, x_K \otimes x_K]$.

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Conclusions

- DMD and operator inference are regression-based powerful methods to infer linear and certain nonlinear system from data.
- Both look simple, but the devil is in the details.
- Choice of good observables? (Learning to learn?)
- Statistical aspects are not to well understood: impact of noisy data on inferred system matrices?
- Combination with neural networks to solve nonlinear regression problems?
- Relation to physics-informed neural networks?
- Error bounds for non-intrusive MOR not well developed yet.